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ANALYTICAL SOLUTIONS FOR THE FLOW OF A REACTIVE THIRD-GRADE FLUID WITH TEMPERATURE DEPENDENT VISCOSITY MODELS IN A PIPE

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ABSTRACT. This study focuses on the heat generation and various viscosity variational parameters of a third grade fluid flow in a circular pipe. The governing equations in cylindrical coordinates under quadratic heat generation is considered for three different viscosity models: constant viscosity, Reynold's and Vogel viscosity models. Perturbation technique is adopted and explicit, analytical expressions are derived for the dimensionless velocity and temperature of the fluid. These new analytical solutions are validated against the results obtained using numerical integration. Graphical results are presented and discussed quantitatively with respect to the competing effects of quadratic heat generation, viscous heating, pressure gradient and non-Newtonian material parameters. Thereafter, the analysis is focused on thermal criticality of physical parameters. The operating conditions of emerging parameters are discussed for the viscosity models

Key Words and Phrases: Third grade fluid, Quadratic heat generation term, Temperature dependent viscosity models, Perturbation method

2010 Mathematical Subject Classification: 76A05, 80A20, 80A35

1. INTRODUCTION

The heat released for a material undergoing a one-step exothermic reaction in an enclosure is described by the generalized Arrhenius kinetics. Many chemical reactions involve simple Arrhenius law [27], bimolecular law [3], sensitized law [5] while the Frank-Kamenetskii's approximation [10], quintic approximation in chemical kinetics [4] and Okoya [21] as well as the quadratic expression [11] also subsists. The first evidence of the occurrence of quadratic expression to the Arrhenius term as heat generation term in a closed

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vessel was in the article by Gray and Harper [11]. The quadratic heat released model has enhanced the investigation and understanding of a physical phenomenon called thermal explosion i.e. a jump from slow reaction to fast reaction. The main assumption during the highly exothermic reaction was that reactant consumption is negligible. In fact, this assumption implies absence of the conservation of energy in the model while the merits and demerits are established in Gray and Harper [11].

In addition to the explanation above, the heat generation term modeled by quadratic expression has been employed by few researchers. Boddington et al. [2] proved the existence of a number of vital quadratic expressions (that introduced by Gray and Harper [11] being one of them) to the Arrhenius term which enabled an induction time to be defined and calculated. Dik and Zurer [5], obtained the conditions for thermal explosion of a turbulent reactive stream in a pipe by method of integral relations. Ayeni [1] established the induction time which signals the onset of explosion for a chain reaction of oxygen (reactant) and hydrogen (active intermediary) with catalyst in the form of nitrosyl chloride (sensitizer) as a non-isothermal branched - chain reaction using standard techniques.

In the preceding paragraph, it is assumed that the material is motionless. In the event that the material is being transported in a flow field such as a third-grade fluid with nonlinear viscosities (see for example, [6], [9], [12], [13], [16], [17], [20], [25], [26] and [31]), then the hydrodynamic factors which control the dissipation of energy, viscosities of the fluid, nature of the fluid under consideration and conduction of matter, influence the reaction history. This is basically a one-dimensional problem and the reaction is assumed to take place entirely in the stream. To the best of our knowledge, analytical studies or numerical analysis of quadratic heat generation term in a non-Newtonian flow systems have not been studied in a pipe. Our aim is to construct analytical solutions using perturbation technique [30] to investigate the sensitivity analysis of emerging physical parameters of interest. The critical values of the temperature distribution and of the Frank- Kamenestkii parameter are obtained using standard Maple solver [23].

2. DEVELOPMENT OF THE PROBLEM

We shall consider in the following a class of steady one-dimensional flow of non-Newtonian fluid in an infinitely long cylinder of radius R_0 as depicted in Figure 1.

We choose cylindrical polar coordinate $(\bar{r}, \phi, \bar{z})$ with $\bar{z}$-axis along the axis of the cylinder, $\bar{r}$ normal to it with no flow in the ϕ-direction. The flows are induced by constant applied pressure gradient in the $\bar{z}$ direction. We seek velocity field of the form $\mathbf{V} = (0, 0, \bar{w}(\bar{r}))$ where $\bar{w}$ is the axial component of the velocity. The continuity equation is automatically satisfied with the above assumptions. The analyses performed is for a laminar, incompressible fluid flowing in a pipe with a uniformly reactive homogeneous third grade fluid in the absence of a body force. The definition of the exothermic reaction with reaction rate expression, k_0 obeying the generalized Arrhenius law is given by

$$k_0 = \bar{A}\left(\frac{k\bar{T}}{\nu\hbar}\right)^m \exp\left(-\frac{E}{R\bar{T}}\right), \tag{1}$$

where $\bar{A}$ is the rate constant, R is the universal gas constant, $\hbar$ is the Planck's number, E is the activation energy, ν is the vibration frequency k is the Boltzmann's constant, m is a numerical exponent and $\bar{T}$ is the temperature of the system. It is further assumed that there is negligible electrical heating as well as reactant consumption. The heat flux vector is represented by Fourier law with constant thermal conductivity and viscosity is assumed to be temperature dependent.

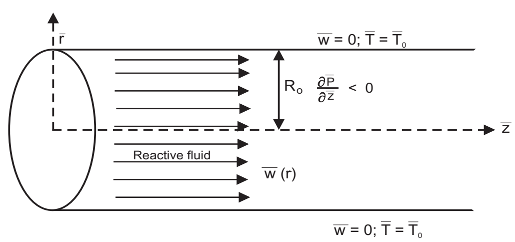

FIGURE 1. Schematic diagram of the flow domain.

Based on the above assumptions and consideration, the dimensional momentum balance and energy equation that govern the flow in the pipe as an extension of [18], [23] and [30] are non-dimensional by using the following characteristic parameters:

$$r = \frac{\bar{r}}{R_0},\ w = \frac{\bar{w}}{\bar{w}_0},\ \mu = \frac{\bar{\mu}}{\bar{\mu}_0^e},\ \bar{\mu}_0^e \in \{\bar{\mu}_0, \bar{\mu}_*\},\ \bar{\mu}_* = \bar{\mu}_0 \exp(\bar{T}_0),$$

$$\beta = \frac{R\bar{T}_0}{E},\ \Gamma = \frac{4\bar{\mu}_0^e \bar{w}_0^2}{k\bar{T}_0\beta},\ \theta = \frac{(\bar{T} - \bar{T}_0)E}{R\bar{T}_0^2},\ \Lambda = \frac{\beta_3 \bar{w}_0^2}{\bar{\mu}_0^e R_0^2},\ C = \frac{R_0^2}{\bar{\mu}_0^e \bar{w}_0}\frac{\partial \bar{p}}{\partial \bar{z}},$$

and $\delta = \dfrac{\bar{Q}E\bar{A}R_0^2C_0k^m\bar{T}_0^{m-2}}{RK\nu^m\hbar^m}\exp\left(-\dfrac{E}{R\bar{T}_0}\right)$,

where $\bar{w}_0$ is a reference velocity, Q is the heat of reaction, C_0 is the initial concentration and $\bar{T}_0$ is the initial temperature. Here r is the dimensionless perpendicular distance from the pipe axis, w is the dimensionless velocity, Γ is the viscous heating parameter which is related to the Prandtl and the Eckert numbers, δ is the modified Frank-Kamenetskii parameter, θ is the dimensionless temperature excess, β is the activation energy, μ is the dimensionless viscosity, C is the pressure gradient parameter, Λ is the non-Newtonian material parameter of the fluid and $\bar{\mu}_*$ is for the Vogel's viscosity model. Although the material constant β_3 and hence Λ may also depend on temperature, they are taken as constants for simplicity in this study.

In the new variable introduced, the reaction rate becomes $k_0 = (1+\beta\theta)^m\exp(\theta/[1+\beta\theta])$. It is supposed in the analysis that the heat generation term follows the quadratic form (i.e. $k_0 = \theta^2 + (e-2)\theta + 1$) introduced by Gray and Harper [11] corresponding to Frank-Kamenetskii approximation (FKA) ($\beta = 0$).

The non-dimensional governing equations are written as

$$\frac{d\mu}{dr}\frac{dw}{dr} + \frac{\mu}{r}\left(\frac{dw}{dr} + r\frac{d^2w}{dr^2}\right) + \frac{\Lambda}{r}\left(\frac{dw}{dr}\right)^2\left(\frac{dw}{dr} + 3r\frac{d^2w}{dr^2}\right) = C, \quad (2)$$

$$\frac{d^2\theta}{dr^2} + \frac{1}{r}\frac{d\theta}{dr} + \Gamma\left(\frac{dw}{dr}\right)^2\left(\mu + \Lambda\left(\frac{dw}{dr}\right)^2\right) + \delta(\theta^2 + (\mathrm{e}-2)\theta + 1) = 0, \quad (3)$$

The associated boundary conditions for equations (1) and (2) with respect to the pipe geometry are

$$w(1) = \theta(1) = 0, \text{ and } \frac{dw}{dr}(0) = \frac{d\theta}{dr}(0) = 0. \quad (4)$$

The viscosity μ is assumed to be a function of temperature and the form of the equations (2) and (3) depend on the viscosity model chosen to represent the fluid. Here we present the results for three common viscosity models (see e.g., Ellahi et al. [8], Massoudi and Christe [23], Nadeem and Ali [19], Okoya [22], Okoya [24] and Sobamowo et al. [28]:

$$\bar{\mu}(\bar{T}) = \begin{cases} \bar{\mu}_0, & \text{constant case,} \\ \bar{\mu}_0\exp(-\bar{M}(\bar{T}-\bar{T}_0)), & \text{Reynolds' model case,} \\ \bar{\mu}_0\exp(a/(b+\bar{T})), & \text{Vogel's model case.} \end{cases} \quad (5)$$

with a, b, and $\bar{M}$ being constants so that the corresponding non-dimensional form is

$$\mu = \begin{cases} 1, & \text{constant case,} \\ \exp(-\gamma\theta), & \text{Reynolds' model case,} \\ \exp(A/(B+\theta) - \bar{T}_0), & \text{Vogel's model case.} \end{cases} \tag{6}$$

where

$$A = \frac{a}{\bar{T}_0\beta},\ B = \frac{b + \bar{T}_0}{\bar{T}_0\beta},\ \text{and } \gamma = \bar{M}\beta\bar{T}_0.$$

It is clear that equations (2), (3) and (6) for our model problem are similar to those considered by Sobamowo et al. [29] except that the heat generation term for their model is the simple quadratic term (i.e. θ^2). Other researchers investigated heat source term that is linear in temperature (see Sobamowo and Akinshilo [28], Jayeoba and Okoya [14] and the references contained therein). The aforementioned articles do not admit criticality of pertinent parameters. In the absence of heat source term, Our model is similar to that studied by Massoudi and Christie [18].

Investigating the present problem we have employed two methods, namely, numerical method and approximate analytical technique, which are individually presented below.

3. ANALYTICAL RESULTS

The governing equations (2) - (4) and (6) are nonlinear and exhibit no closed-form solutions, hence, we employ approximate analytical methods. In general, approximate solutions can be very useful in implementing and validating computer routines of complicated problems.

3.1 *Constant viscosity case*

For constant viscosity $\mu = 1$. Let us suppose that the approximate velocity and temperature profiles are given as

$$w = w_0 + \epsilon w_1 \qquad \theta = \theta_0 + \epsilon\theta_1, \tag{7}$$

where ϵ is the artificial perturbation quantity, a small parameter whereby, the dimensionless non-Newtonian parameter, Λ, (or the generalized Frank-Kamenetskii parameter, δ) is small compared to the Newtonian term, M (or the classical Frank Kamenetskii number, N), such that

$$\Lambda = \epsilon M,\ \ \delta = \epsilon N. \tag{8}$$

In order to avoid repetition, the details of the calculations are omitted and the solution subject to (4) is directly given by

$$w = \frac{-C}{4}(1-r^2) + \frac{\Lambda}{32}C^3(1-r^4), \tag{9}$$

$$\theta = \frac{\Gamma C^2}{64}(1-r^4) - \frac{\Lambda\Gamma C^4}{576}(1-r^6) + \frac{\delta\Gamma^2 C^4}{3686400}(134-225r^2+100r^6-9r^8) + \frac{\delta\Gamma C^2}{2304}(e-2)[8-9r^2+r^6] + \frac{\delta}{4}(1-r^2), \tag{10}$$

after returning back to the original parameters.

3.2 *Reynolds' viscosity model*

In this case, using the Maclaurin's series we can express that

$$\mu = \exp(-\gamma\theta) = 1 - \gamma\theta + O(\theta^2). \tag{11}$$

Here, our interest is to retain the perturbation terms (i.e Λ and δ) for the constant viscosity model and it is logical to select

$$\gamma = \epsilon P. \tag{12}$$

Expansions of (7) are also assumed for this study of velocity and temperature profiles. We obtained after some algebra the final solutions subject to conditions (4) as

$$w = \frac{-C}{4}(1-r^2) + \frac{\Lambda}{32}C^3(1-r^4) - \frac{\gamma\Gamma C^3}{768}(2-3r^2+r^6), \tag{13}$$

$$\theta = \frac{\Gamma C^2}{64}(1-r^4) - \frac{\Gamma\Lambda C^4}{576}(1-r^6) + \frac{\gamma\Gamma^2 C^4}{16384}\left(3-4r^4+r^8\right) + \frac{\delta\Gamma^2 C^4}{3686400}(134-225r^2+100r^6-9r^{10}) + \frac{\delta\Gamma C^2}{2304}(e-2)[8-9r^2+r^6] + \frac{\delta}{4}(1-r^2), \tag{14}$$

on changing back to the original parameters.

3.3 *Vogel's viscosity model*

Here, after the Maclaurin's series expansion, we obtain

$$\mu = \exp\left(\frac{A}{B+\theta} - \bar{T}_0\right) = \exp\left(\frac{A}{B} - \bar{T}_0\right)\left(1 - \epsilon\frac{A\theta_0}{B^2} + O(\epsilon^2)\right). \tag{15}$$

We still retain the perturbation quantities (i.e. Λ and δ) for the constant viscosity model and we impose a similar restriction on the viscous heating parameter Γ by setting

$$\Gamma = \epsilon Q, \tag{16}$$

and this choice dictates the pertinent expansion of the form

$$w = w_0 + \epsilon w_1, \quad \theta = \epsilon\theta_0 + \epsilon^2\theta_1. \tag{17}$$

The solutions of the problem subject to (4) in the original parameters after some manipulations are

$$w = -\frac{C^*}{4}(1-r^2) + \frac{\Lambda C^{*4}}{32C}(1-r^4) - \frac{A}{B^2}\frac{\Gamma C C^{*2}}{768}(2-3r^2+r^6), \quad (18)$$

$$\theta = \frac{\Gamma C C^*}{64}(1-r^4) - \frac{\Gamma\Lambda C^{*4}}{576}(1-r^6) + \frac{A}{B^2}\frac{\Gamma^2 C^2 C^{*2}}{16384}(3-4r^4+r^8)$$
$$+\frac{\delta\Gamma^2 C^2 C^{*2}}{3686400}(134-225r^2+100r^6-9r^{10}) + \frac{\delta\Gamma C C^*}{2304}(e-2)[8-9r^2+r^6]$$
$$+\frac{\delta}{4}(1-r^2). \quad (19)$$

where C^* as been defined as

$$C^* = \frac{C}{\exp(A/B - \bar{T}_0)} \quad (20)$$

The restriction to small δ is in line with limit of practical relevance as contained in Seddeek and Aboeldahab [23].

We observe that the dimensionless velocities are the same with Yurusoy and Pakdemirli [30]. Consequently, we will look only at the dimensionless mainstream temperature. The temperature equation is different from Yurusoy and Pakdemirli [30] due to the terms associated with heat generation including the parameter δ. It is also evident that the results of the dimensionless mainstream temperature for constant viscosity model is recovered when γ and A are zero for the Reynolds' and Vogel's viscosity models, respectively.

In the following section, equations (2) - (4) and (6) are solved numerically using standard Maple solver.

4. NUMERICAL SOLUTIONS

We now discuss the numerical solution of the non-dimensional forms of equations (2), (3) and (6) with the boundary conditions (4). In doing so, we first note that equations (2) and (3) contain a singularity for $r = 0$. To handle this problem numerically, it is necessary to develop equations that will not contain a singularity. On using L'Hospital rule, we have new equations

$$\frac{d\mu}{dr}\frac{dw}{dr} + 2\mu\left(\frac{d^2w}{dr^2}\right) + 6\Lambda\left(\frac{dw}{dr}\right)^2\frac{d^2w}{dr^2} = C, \quad (21)$$

$$2\frac{d^2\theta}{dr^2} + \Gamma\left(\frac{dw}{dr}\right)^2\left(\mu + \Lambda\left(\frac{dw}{dr}\right)^2\right) + \delta[\theta^2 + (\mathrm{e}-2)\theta + 1] = 0, \quad (22)$$

which are valid near $r = 0$. The boundary conditions at $r = 1$ are specified in a straightforward manner. Equations (2), (3) and (6) are augmented with (21) and (22) and solved numerically subject to (4) by numerical algorithm based on the inbuilt Maple 12 solver. In practice, the neighborhood of the axis of the pipe to start the integration is chosen sufficiently small to the order of 10^{-2}. To ascertain the accuracy of the numerical solutions obtained by the standard Maple solver employed in this study, the dimensionless mainstream temperatures were generated for the special case of $\delta = 0$ so that comparison can be made with published numerical results for the viscosity models at different parameter regime. For this particular case, the numerical solution agree very well with the finite-difference results presented in Massoudi and Christe [18] with the difference being less than 10^{-3}.

The numerical solutions developed in this section are used to determine the interval of accuracy and effectiveness of the perturbation technique of the preceding section.

5. COMPARISON BETWEEN NUMERICAL AND PERTURBATION SOLUTIONS

Here, the solution for the temperature distribution at the central axis of the pipe $\theta\ (0) = \theta_{max}$ are tabulated. The numerical solution (N) is also compared with the perturbation solution (P) while the corresponding relative error (Rel. E.) is computed. The θ_{max} for the constant model viscosity are depicted in Tables 1 and 2, Reynolds' viscosity model are displayed in Tables 3-5 while those for Vogel's viscosity model are in Tables 6 and 9. In these tables, the variation of the emerging parameters $\Lambda, \Gamma, \gamma, \delta, A, B, C$ and $\bar{T}_0$ are taken into account. It is evident that the approximate solutions reveal the characteristics of the problems and compared favorably well with the numerical integration of same.

Table 1: Comparison of perturbation solution and numerical result for constant viscosity model with $\Lambda = 1$ and $\delta = 0.2$.

C	$\Gamma = 1$			Γ	$C = -1$		
	θ_{max}(P)	θ_{max}(N)	Rel. E.		θ_{max}(P)	θ_{max}(N)	Rel. E.
-0.25	0.05100	0.05246	2.79%	0.0	0.05000	0.05146	2.84%
-0.5	0.05392	0.05541	2.68%	2.5	0.08601	0.08870	3.03%
-0.75	0.05852	0.06010	2.63%	5.0	0.12212	0.12606	3.13%
-1	0.06439	0.06634	2.94%	10	0.19460	0.20113	3.25%
-1.1	0.06698	0.06923	3.25%	20	0.34066	0.35269	3.41%

Table 2: Comparison of perturbation solution and numerical result for constant viscosity model with $\Gamma = -C = 1$.

Λ	$\delta = 0.2$			δ	$\Lambda = 1$		
	θ_{max} (P)	θ_{max} (N)	Rel. E.		θ_{max} (P)	θ_{max} (N)	Rel. E.
0	0.06613	0.06767	2.27%	0.0	0.01389	0.01435	3.23%
0.5	0.06526	0.06691	2.46%	0.1	0.03914	0.03995	2.02%
1	0.06439	0.06634	2.94%	0.2	0.06439	0.06634	2.94%
1.5	0.06353	0.06589	3.58%	0.25	0.07702	0.07987	3.57%
2	0.06266	0.06550	4.34%	0.3	0.08965	0.09364	4.26%

Table 3: Comparison of perturbation solution and numerical result for Reynolds' viscosity model with $\Lambda = \gamma = 1$ and $\delta = 0.2$.

C	$\Gamma = 1$			Γ	$C = -1$		
	θ_{max}(P)	θ_{max}(N)	Rel. E.		θ_{max}(P)	θ_{max}(N)	Rel. E.
-0.25	0.05100	0.052498	2.85%	0.0	0.05	0.05146	2.84%
-0.5	0.05393	0.055530	2.87%	2.5	0.08716	0.09051	3.70%
-0.75	0.05858	0.06038	2.98%	5.0	0.12669	0.13159	3.72%
-1	0.06458	0.06685	3.40%	10	0.21291	0.22067	3.52%
-1.1	0.06725	0.06984	3.71%	20	0.41390	0.43342	4.50%

Table 4: Comparison of perturbation solution and numerical result for Reynolds' viscosity model with $\Gamma = -C = \gamma = 1$.

Λ	$\delta = 0.2$			δ	$\Lambda = 1$		
	θ_{max} (P)	θ_{max} (N)	Rel. E.		θ_{max} (P)	θ_{max} (N)	Rel. E.
0	0.06631	0.06835	2.98%	0.0	0.01407	0.01448	2.82%
0.5	0.06545	0.06748	3.01%	0.1	0.03932	0.04026	2.31%
1	0.06458	0.06684	3.39%	0.2	0.06458	0.06684	3.39%
1.5	0.06371	0.06634	3.96%	0.25	0.07720	0.08047	4.06%
2	0.06284	0.06592	4.67%	0.3	0.08983	0.09436	4.80%

Table 5: Comparison of perturbation solution and numerical result for Reynolds' viscosity model with $\Lambda = \Gamma = -C = 1$ and $\delta = 0.2$.

γ	θ_{max}(S)	θ_{max}(N)	Rel. E.
-3	0.06386	0.06495	1.70%
-2	0.06403	0.06540	2.09%
-1	0.06421	0.06586	2.50%
0	0.06439	0.06635	2.94%
1	0.06458	0.06684	3.39%
2	0.06476	0.06737	3.87%
3	0.06494	0.06790	4.36%

Table 6: Comparison of perturbation solution and numerical result for Vogel's viscosity model with $\Lambda = A = B = \bar{T}_0 = 1$ and $\delta = 0.2$.

C	$\Gamma = 1$			Γ	C = -1		
	θ_{max} (P)	θ_{max} (N)	Rel. E.		θ_{max} (P)	θ_{max} (N)	Rel. E.
-0.25	0.05100	0.05250	2.85%	0.0	0.05	0.05146	2.84%
-0.5	0.05393	0.05553	2.87%	2.5	0.08716	0.09039	3.58%
-0.75	0.05858	0.06037	2.97%	5.0	0.12670	0.13107	3.33%
-1	0.06458	0.06683	3.37%	10	0.21291	0.21754	2.13%
-1.1	0.06725	0.06982	3.68%	20	0.41390	0.40996	−0.96%

Table 7: Comparison of perturbation solution and numerical result for Vogel's viscosity model with $\Gamma = A = B = -C = \bar{T}_0 = 1$.

Λ	$\delta = 0.2$			δ	$\Lambda = 1$		
	θ_{max} (P)	θ_{max} (N)	Rel. E.		θ_{max} (P)	θ_{max} (N)	Rel. E.
0.0	0.06631	0.06832	2.94%	0.0	0.01407	0.01448	2.81%
0.5	0.06545	0.06746	2.98%	0.1	0.03932	0.04025	2.30%
1.0	0.06458	0.06683	3.37%	0.2	0.06458	0.06683	3.36%
1.5	0.06371	0.06632	3.94%	0.25	0.07720	0.08045	4.03%
2.0	0.06284	0.06590	4.65%	0.3	0.08983	0.09431	4.75%

Table 8: Comparison of perturbation solution and numerical result for Vogel's viscosity model with $\Lambda = \Gamma = B = -C = \bar{T}_0 = 1$ and $\delta = 0.2$.

A	$B = 1$			B	$A = 1$		
	θ_{max} (P)	θ_{max} (N)	Rel. E.		θ_{max} (P)	θ_{max} (N)	Rel. E.
1	0.06458	0.06683	3.37%	0.5	0.05600	0.05817	3.73%
2	0.05595	0.05778	3.16%	1.0	0.06458	0.06683	3.36%
3	0.05219	0.05386	3.09%	1.1	0.06535	0.06786	3.69%
4	0.05080	0.05237	2.98%	1.2	0.06585	0.06873	4.18%
5	0.05030	0.05181	2.92%	1.3	0.06612	0.06948	4.83%

Table 9: Comparison of perturbation solution and numerical result for Vogel's viscosity model with $\Lambda = \Gamma = A = B = -C = 1$ and $\delta = 0.2$.

T_0	$\delta = 0.2$		
	θ_{max} (PS)	θ_{max} (NS)	Rel. E.
0.5	0.05961	0.06139	2.89%
0.75	0.06203	0.06392	2.96%
1	0.06458	0.06683	3.37%
1.1	0.06546	0.06806	3.81%
1.2	0.06611	0.06930	4.60%

Tables 1 and 2 are for constant viscosity model. In Table 1, columns 1-4, the effect of pressure-gradient variation is depicted when $\Lambda = \Gamma = 1$ and $\delta = 0.2$. It is obvious that for $|C| \leq 1$ the perturbation solutions exhibit good agreement with relative error less than 5 %. For $|C| > 1$, deviations become substantial. In Table 1, columns 5-8, variation of Γ is considered. Analytical results are in very good agreement with the numerical results since the relative error is less than or equal to 5 % for $\Gamma \leq 20$. In Table 2, columns 1-4, the variation of the non-Newtonian parameter (Λ) is considered. It is worth noting that the case of $\Lambda = 0$ corresponds to a Newtonian fluid. For $\Lambda \leq 2$ the relative error is less than 5 % when $C = -1$, $\Gamma = 1$ and $\delta = 0.2$. In Table 2, columns 5-8, the range for 5 % relative error for the Frank-Kamenetskii parameter, δ is $\delta \leq 0.3$.

Tables 3-5 illustrate for the Reynolds' viscosity model. In Table 3, columns 1-4, the variation of parameter C is considered and found to be remarkably similar in error values to the constant case. For the variation of Γ in Table 3, columns 5-8, the relative error is less than 5% for the range of Γ values. In Table 4, columns 1-4, the variation of parameter (Λ) is considered. For $\Lambda \leq 2$ the relative error is less than 5 %. For the variation of δ in Table 4, columns 5-8, the results deviate from each other for $\delta > 0.3$ substantially since the relative error is greater than 5 %. For 5 % relative error, $\delta \leq 0.3$. For $\delta = 0$ results are the same up to three decimals. In Table 5, variation of the viscosity parameter γ is investigated. The perturbation solutions are in very good agreement to relative error of 5 % when $\gamma \leq 4$.

It should be noted that according to the Vogel's model, viscosity has two additional parameters namely A, B and the initial temperature $\bar{T}_0$ in the place of γ in the Reynold's case. Table 6, columns 1-4, C variations is considered. It is observed that C produces quite similar behavior as in the constant and Reynold's models. In Table 6, columns 5-8, variation of parameter Γ is considered. The relative errors are extremely small for these ranges of Γ values. The larger the viscous heating parameter, the smaller the relative errors between the numerical and the asymptotic solutions. Table 7, columns 1-4 shows the dependence of θ_{max} on parameter Λ. For $\Lambda < 2$, the relative error is less than 5% and results are in good agreement. The effect of heat parameter δ is shown in Table 7, column 5-8. For $\delta \leq 0.3$ the results are in good agreement with relative error of less than 5 %. For $\delta = 0$ results are the same up to three decimals. Variation of relative error as a function of A could be analyzed through Table 8, columns 1-4. It is seen from the table that there is a good agreement between perturbation and numerical results as the relative error decreases as A increases. For the variation of B in Table 8, columns 5-8, the relative error is greater than 5 % for $B > 1.2$. Finally, for $\bar{T}_0 \leq 1.2$, the relative error is less than 5 % in Table 9. For, $\bar{T}_0 > 1.2$ deviations become substantial. These comparisons lends confidence in the perturbation solutions and shows that the perturbation method is adequate for solutions of the present problem.

Having confirm the validity of the perturbation solutions we are left to look only on the effect of embedding parameters on the solutions of the energy equations.

6. RESULTS AND DISCUSSION

6.1 The dependence of the physical parameters upon heat flow

In this section, the pair Figures. 2 and 3, 4 and 5, 6 and 7 as well as 8 and 9 present the constant, Reynold's and Vogel's viscosity models and the dimensionless radius versus the dimensionless temperature for different values of δ, Λ, Γ and C, respectively. Figures. 2 and 3 show the effect of a new heat generation parameter on the dimensionless temperature distributions. Since δ is the measure of the heat generation due to chemical reaction, in the absence of heat generation ($\delta = 0$) the temperature distribution depicts the least profile. For comparison, different values of the heat generation parameter δ ($= 0.0, 0.1, 0.3$) are also displayed. It is seen that as the amount of heat generated increases, the temperature distribution are noticeably higher than without the effect of heat generation (i.e. for $\delta = 0.0$) for the three viscosity models.

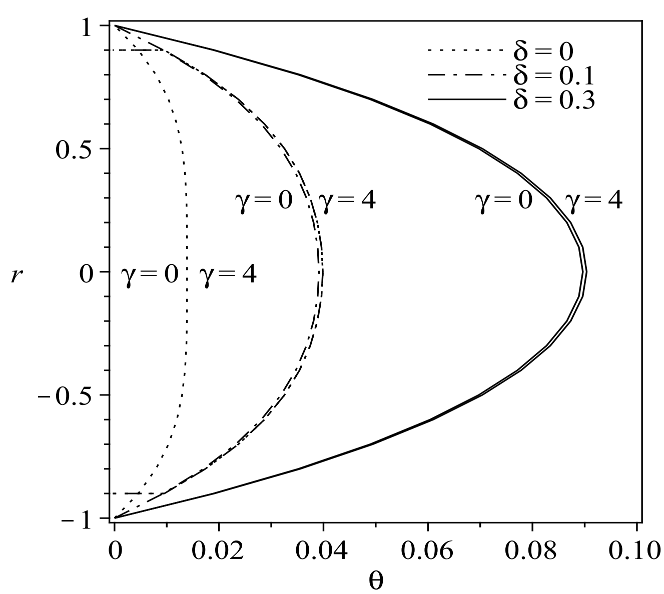

FIGURE 2. Effect of δ on the dimensionless temperature profiles of constant and Reynolds' viscosity models when $\Gamma = \Lambda = -C = 1$.

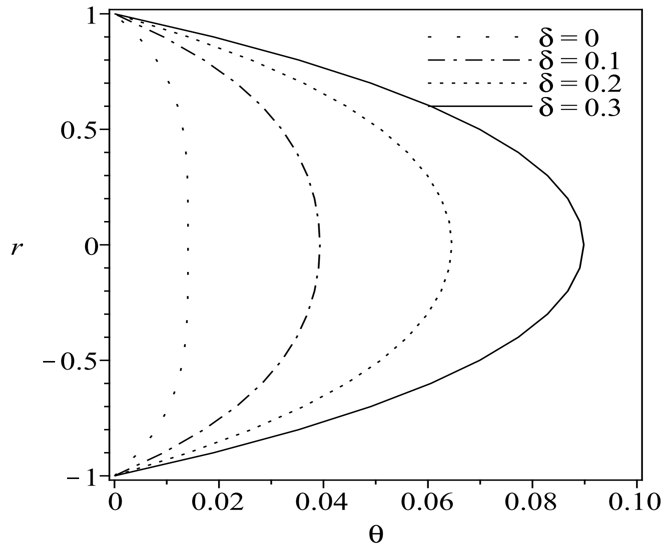

FIGURE 3. Effect of δ on the dimensionless temperature profiles of Vogel's viscosity model when $\Gamma = \Lambda = A = B = -C = \bar{T}_0 = 1$.

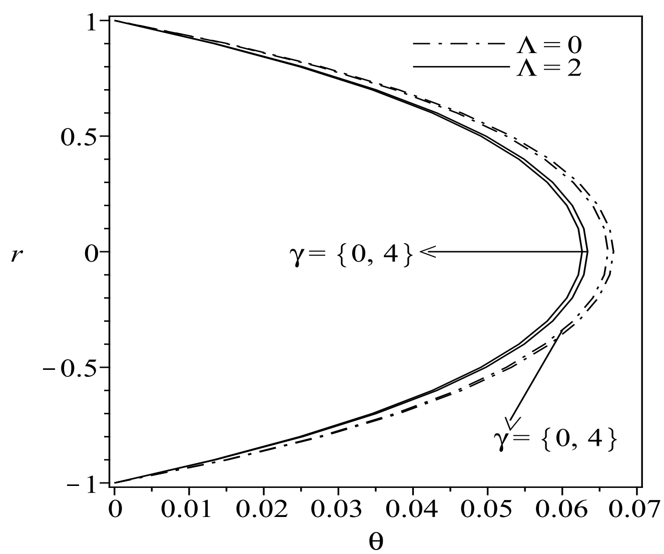

FIGURE 4. The graphs of radius r against temperature θ of constant and Reynolds' viscosity models for values of Λ when $\Gamma = -C = 1$ and $\delta = 0.2$.

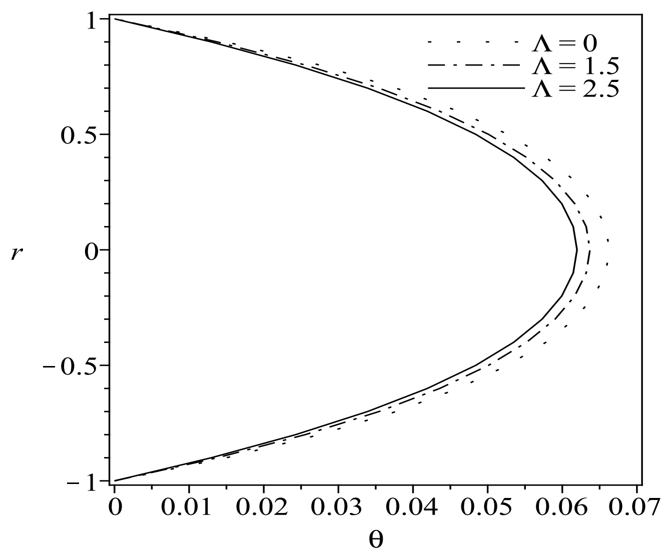

FIGURE 5. The plots of radius r versus the temperature θ of Vogel's viscosity model for various values of Λ when $\Gamma = A = B = -C = \bar{T}_0 = 1$ and $\delta = 0.2$.

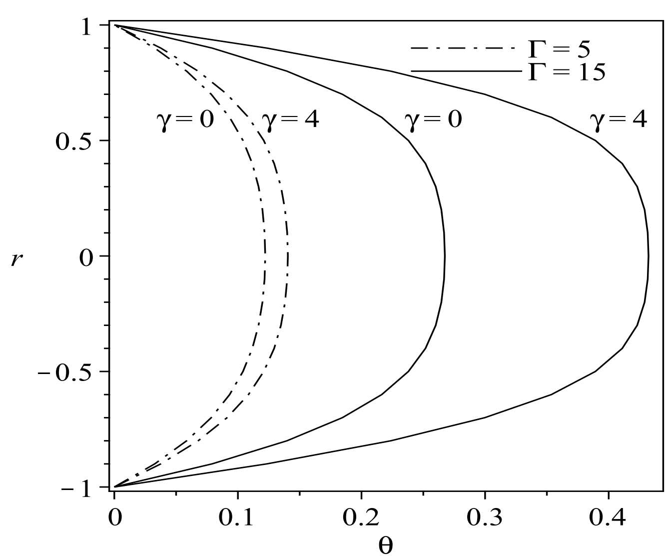

FIGURE 6. The variations of radius r with temperature θ of constant and Reynolds' viscosity models for values of Γ when $\Lambda = -C = 1$ and $\delta = 0.2$.

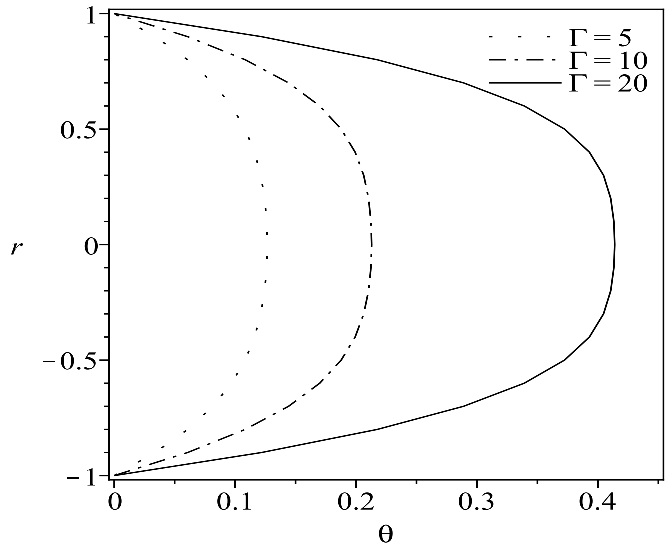

FIGURE 7. The dependence of radius r on the temperature θ of Vogel's viscosity model for various values of Γ when $\Lambda A = B = -C = \bar{T}_0 = 1$ and $\delta = 0.2$.

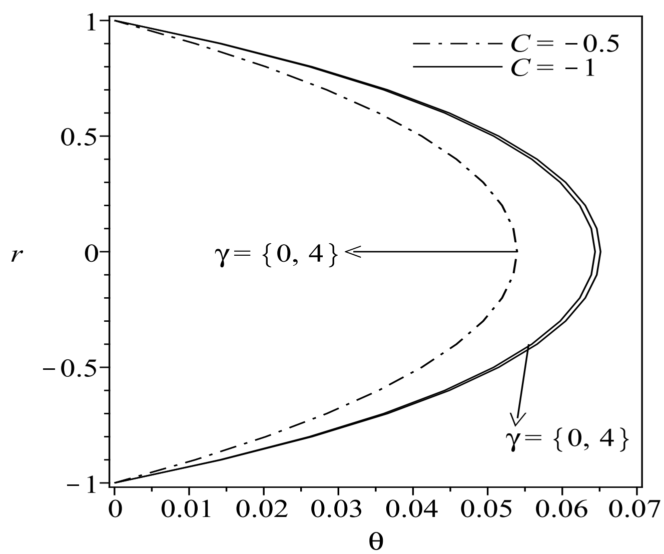

FIGURE 8. The profiles of radius r versus temperature θ of constant and Reynolds' viscosity models for values of C when $\Gamma = \Lambda = 1$ and $\delta = 0.2$.

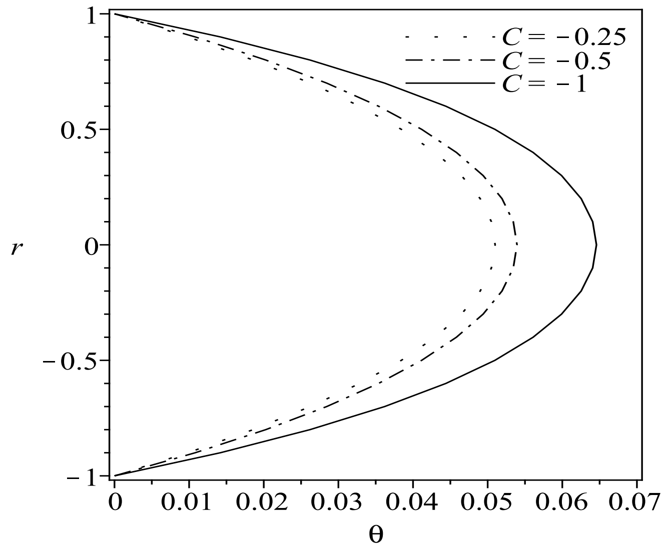

FIGURE 9. The curves of radius r with the temperature θ of Vogel's viscosity model for various values of C when $\Gamma = \Lambda = A = B = \bar{T}_0 = 1$ and $\delta = 0.2$.

In addition, the influences of other different values of the emerging parameters are also investigated for completeness.

We compare the temperature profiles for two kinds of fluids: the Newtonian fluid, for which $\Lambda = 0$ and the full third-grade non-Newtonian fluid, in which we choose arbitrary values without physical reasons. Figures. 4 and 5 are helpful for the determination of non-Newtonian behavior of the fluid under various viscosities. Within the permissible range of Λ, increasing Λ decreases the temperature for the flow in the three viscosity models.

Also, Figures. 6 and 7 show the importance of viscous heating parameter in the heat transport. The results show that, for a given non-Newtonian fluid which comply with any of the three viscosity models, as Γ increases, that is, as the effect of viscous dissipation becomes more significant, the temperature distribution increases.

Moreover, Figures. 8 and 9 present the variation of pressure gradient parameter which is controlled by external initiating device such as a pump far from the inlet of the pipe. For the three cases of viscosity models under consideration, it is seen that as C becomes more negative, the temperature distributions increase.

6.2 The dependence of the physical parameters upon thermal explosion

From the symmetry of the system it can easily be obtained that the central temperature results in $\theta(0) = \theta_{max}$. Consequently, the parameters for thermal explosion are θ_{max} and and the Frank-Kamenetskii parameter δ. The steady state response curve in the (θ, δ)-plane shows a rise to a maximum before it grows to zero as $\theta_{max} \to 0$. Thermal explosion (criticality) in the system is plainly determined as the turning point (point of uniqueness of solution) for certain parameter values in the bifurcation analysis and denoted by δ_{cr} and $\theta_{max\ cr}$. We accomplished this task mathematically by employing

$$\frac{d\delta}{d\theta_{max}} = 0 \text{ and } \frac{d^2\delta}{d\theta_{max}^2} < 0. \tag{21}$$

The steady state problem (1), (5), (7) subject to boundary conditions (3) is then solved numerically employing shooting method solver in Maple with δ taken to be the unknown parameter. For the problem considered here, it exhibits two solutions and with the conditions in equation (23), we obtained the unique solution at criticality. Table 10 shows the numerical values obtained from Makinde [15] and ours.

Table 10: Computational results of δ_{cr} for various values of Λ are compared with results by Makinde [15] when C = -1, $\gamma = 0$ and $\Gamma = \delta$.

	[15]	Ours	
Λ	δ_{cr} (FKA)	δ_{cr} (QE)	Rel. E.
0.0	1.945436	1.967296	1.12%
0.1	1.946071	1.967870	1.12%
0.2	1.946755	1.968402	1.11%
0.3	1.947482	1.968898	1.11%

The table indicates that the relative error of the result in Makinde [15] and ours is less than 1.15%. It is to be noted that the quadradic expression provides an upper solution for the model problem (2) - (4) and (6). We now proceed to investigate the parameter sensitivity to thermal explosion for the triplet viscosity models.

It is evident form Tables 3, 4, 6 and 7 that the effects of Γ, Λ and C on the dimensionless temperature distribution are similar for both exponential viscosity models. In addition, since we have examined the influence of the same physical parameters on the criticality of the Reynolds' viscosity model as contained in Table 10, we will look only at the effects of A, B and T_0 on the criticality of Vogel's viscosity model in Table 11.

TABLE 11: Computed critical values for the Reynolds' viscosity model problem.

			$\gamma=-0.4$		$\gamma=0$		$\gamma=0.4$	
Λ	C	Γ	δ_{cr}	$\theta_{max\ cr}$	δ_{cr}	$\theta_{max\ cr}$	δ_{cr}	$\theta_{max\ cr}$
0			2.005916	1.468974	1.999623	1.469541	1.990769	1.467398
1			2.007090	1.467364	2.001948	1.467604	1.995916	1.467048
2	1	1	2.007899	1.466288	2.003412	1.466382	1.998580	1.466193
4			2.009039	1.464812	2.005339	1.464766	2.001707	1.464804
7			2.010193	1.463367	2.007157	1.463236	2.004387	1.463358
	-0.25		2.025851	1.447594	2.025463	1.447622	2.024932	1.447498
	-1.0		2.007090	1.467363	2.001949	1.467604	1.995916	1.467048
1	-1.5	1	1.984689	1.490988	1.974875	1.491314	1.964458	1.491180
	-2.0		1.956508	1.521028	1.941682	1.521402	1.927148	1.522158
	-2.5		1.923960	1.556422	1.904083	1.556872	1.885794	1.558866
		0	2.027189	1.446178	2.027189	1.446178	2.027189	1.446178
		1	2.007090	1.467364	2.001948	1.467604	1.995916	1.467048
1	1	10	1.844051	1.658986	1.793184	1.668930	1.732059	1.671130
		30	1.572252	2.081102	1.430795	2.163102	1.254592	2.244634
		50	1.384127	2.484477	1.172990	2.705432	0.910247	3.035572

The numerical results in Table 11 for the Reynolds' model viscosity indicate the following observations on the parameter sensitivity on the bifurcation parameters (δ and θ_{max}) when other physical parameters are turned off. Notice that the non-Newtonian parameter is relatively insensitive to the bifurcation parameters; as it increases so does the critical Frank-Kamenetskii parameter increases while the critical central temperature decreases. It is evident that the Newtonian fluid represent an upper bound on the numerical results. Also, in decreasing the magnitude of the pressure gradient parameter, the same pattern of behaviour of the critical parameters subsists. We note, in contrast to the non-Newtonian parameter and the magnitude of the pressure gradient parameter, that for an increase in the viscous heating parameter there is a substantial increase (or decrease) in the the critical central temperature (or critical Frank-Kamenetskii) . From this viewpoint, high-heat dissipation materials pose a greater fire hazard. Moreover, the value of critical Frank-Kamenetskii parameter decreases for both gas and liquid states. The table shows that at low (or a higher) Λ, Γ and C there is a decrease (or increase) in the value of critical central temperature for both gas and liquid states.

Table 12 shows the effect of varying the parameters A and B as well as the initial temperature T_0 upon the critical Frank-Kamenetskii parameter and the critical central temperature for fixed values of other physical parameters under the Vogel model viscosity.

Table 12: Numerical result for Vogel's viscosity model with $\Lambda = -C = 1$.

			$\Gamma = 1$		$\Gamma = 10$		$\Gamma = 30$	
A	B	$\bar{T}_0$	δ_{cr}	$\theta_{max\ cr}$	δ_{cr}	$\theta_{max\ cr}$	δ_{cr}	$\theta_{max\ cr}$
0			1.983410	1.484038	1.643256	1.849578	1.147752	2.781780
1			1.993831	1.470509	1.717510	1.714988	1.251225	2.413840
1.5	1	1	1.999998	1.462918	1.766525	1.634230	1.331329	2.164757
2.0			2.005892	1.456496	1.817938	1.562056	1.433617	1.903230
3.0			2.014881	1.448948	1.903757	1.476221	1.655278	1.556470
	0.1		2.008668	1.446608	1.839344	1.455796	1.448439	1.574754
	0,3		2.003519	1.454634	1.791818	1.552864	1.353821	1.990034
1	0.5	1	1.999525	1.461228	1.759303	1.624170	1.305885	2.191988
	0.7		1.996671	1.465947	1.737874	1.671254	1.277401	2.308826
	1.0		1.993831	1.470509	1.717510	1.714988	1.251225	2.413840
		0.3	2.007717	1.459078	1.839107	1.584350	1.509612	1.925562
		1.0	1.993831	1.470509	1.717510	1.714988	1.251225	2.413840
1	1	1.5	1.984658	1.479606	1.645625	1.816166	1.131417	2.755056
		2.0	1.978092	1.486709	1.598227	1.892308	1.062626	2.993189
		2.5	1.973874	1.491449	1.569351	1.941944	1.023781	3.142820

It is apparent that as parameter A increases the critical Frank-Kamenetskii parameter increases while the critical central temperature decreases. Furthermore, it is seen that the critical Frank-Kamenetskii parameter increases with increasing parameter B but the increase is minor for the material. A similar pattern is observed with increase in the initial temperature of the material. Evidently, as the Viscous heating parameter, Γ, increases the critical central temperature increases while the critical Frank-Kamenetskii parameter decreases.

7. CONCLUDING REMARKS

We have investigated the flow and heat transfer for a reactive third-grade fluid. In nondimensionalizing the model problem, we have retained physical and material parameters of interest. Analytical solutions are presented for a pipe under constant, Reynolds' and Vogel's viscosity models using perturbation method. It is shown that for the choice of perturbation parameters and in the limit $\delta \to 0$, our new analytical solutions for the dimensionless mainstream temperature reduced to the well known cases in [30]. To validate the analytical solutions, we solved the equations numerically and the results are compared. Both results showed excellent agreement with relative error of less than 5 % . Further numerical computation on thermal explosion when compared with [15] reveals relative error of less than 1.15 % .

The main features of the analytical solutions and numerical results for thermal explosion are as follows:
1. The contribution of heat due to reaction yields an increment in the dimensionless temperature of the fluid.

2. The increase of the non-Newtonian material parameter leads to the decrease of dimensionless temperature distribution.
3. The results pointed to significant differences in the solutions for the three viscosity models.
4. It has been shown through numerical studies that the thermal critical values for reactive fluid with the three viscosity models are particularly sensitive to viscous heating parameter.

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REFERENCES

[1] R. O. Ayeni, On the explosion of chain-thermal reactions. Journal of the Australian Mathematical Society (Series B) 24 (1982), 194-202.
[2] T. Boddington, P. Gray and G. C. Wake, Criteria for Thermal explosions with and without reactant consumption, Proc. R. Soc. Lond., A **357** (1977) 403 - 422.
[3] T. Boddington, P. Gray and C. Robinson, Thermal explosion and the disappearance of criticality at small activation energies: Exact results for slab, Proc. R. Soc. Lond., A **368** (1979) 441 - 461.
[4] T. Boddington, P. Gray and S. K. Scott, Thermal explosion and time-to-ignition in systems with uniform temperature: Analytical treatment for spherical reactant masses. Proc. R. Soc. Lond., A **378** (1981) 27 - 60.
[5] I. G. Dik and A. B. Zurer, Calculation of the critical conditions for a thermal explosion in an inhomogeneous medium by a method of integral relations. Inzhenerno-Fizicheskii Zhurnal, **38** (1)(1980), 95-98.
[6] J. E. Dunn and K. R. Rajagopal, Fluids of differential type: critical review and thermodynamic analysis. Int. J. Engineering Science, **33** (1995) 689-729.
[7] R. Ellahi and S. Afzal, Effects of variable viscosity in a third grade fluid with porous medium: An analytic solution, Communications in Nonlinear Science and Numerical Simulation, **14** (5) (2009) 2056-2072.
[8] R. Ellahi, M. Raza and K. Vafai, Series solutions of non-Newtonian nanofluids with Reynolds model and Vogels model by means of the homotopy analysis method, Mathematical and Computer Modelling **55**, Issues 78, (2012) 1876-1891.
[9] R. L. Fosdick and K. R. Rajagopal, Thermodynamics and stability of fluids of third grade. Proc. R. Soc. Lond. A **369**, (1980) 351-377.
[10] D. A. Frank-Kamenetskii, Diffusion and Heat Transfer in Chemical Kinetics, second ed., Plenum, New York, (1969).
[11] P. Gray and M. J. Harper, "Thermal explosions: part 1induction periods and temperature changes before spontaneous ignition", Trans. Faraday Soc. 55 (1959), 581-590.
[12] M. O. Iyoko, G. T. Okedoye, L. N. Ikpakyegh and J. O. Ode, Solving MHD Pipe Flow Of A Third Grade Fluid With Vogel Model Viscosity And Joule Heating Using The Adomian Decomposition Method. International Journal of Scientific and Technology Research 7 (2) (2018) 18-26.
[13] S. Jangili, S. O. Adesanya, H. A. Ogunseye and R. Lebelo, Couple stress fluid flow with variable properties: A second law analysis 42 (1) (2019) 85-98
[14] O. J. Jayeoba and S. S. Okoya, Approximate analytical solutions for pipe flow of a third grade fluid with variable models of viscosities and heat generation/absorption, J. Niger. Math. Soc. 31 (2012) 207-227.

[15] O. D. Makinde, Thermal stability of a reactive third grade fluid in a cylindrical pipe: An exploitation of Hermite-Pade approximation technique, Applied Mathematics and Computation **189** (1) (2007), 690-697.
[16] O. D. Makinde, Analysis of Non-Newtonian Reactive Flow in a Cylindrical Pipe, Journal of Applied Mechanics, **76** (2009) 034502-5.
[17] M. Massoudi and I. Christe, Heat transfer and flow of a third grade fluid in a pipe, Math. Modelling Sci. Comput. **2** (1993) 1273-
[18] M. Massoudi and I. Christe, Effects of variable viscosity and viscous dissipation on the flow of third grade fluid in a pipe, Int. J. Non-Linear Mech. **30** (5) (1995) 687-699.
[19] S. Nadeem and M. Ali, Analytical solutions for pipe flow of a fourth grade fluid with Reynold and Vogel's models of viscosities, Communications in Nonlinear Science and Numerical Simulation, **14** (5) (2009), 2073-2090.
[20] H. Ogunseye and S. S. Okoya, Criticality and thermal explosion in the flow of reactive viscous third grade fluid flow in a cylindrical pipe with surface cooling, Journal of the Nigerian Mathematical Society 36 (2) (2017) 399-418.
[21] S. S. Okoya, On the behaviour of solution to a system of ordinary differential equations modelling branched chain reaction, Int. Commu. Heat Mass Transfer 29 (8) (2002) 1169 - 1176.
[22] S. S. Okoya, Disappearance of criticality for reactive third-grade fluid with Reynold's model viscosity in a flat channel, Int. J. Non-Linear Mechanics **46** (9) (2011) 1110-1115.
[23] S. S. Okoya, Flow, thermal criticality and transition of a reactive third-grade fluid in a pipe with Reynolds model viscosity, J. Hydrodyn. 28 (1) (2016) 84-94
[24] S. S. Okoya, Computational study of thermal influence in axial annular flow of a reactive third grade fluid with non-linear viscosity, Alexandria Engineering Journal 58 (1)(2019) 401-411
[25] M. Pakdemirli and B. S. Yilbas, Entropy generation for pipe flow of a third grade fluid with Vogel model viscosity, International Journal of Non-Mechanics 41 (3)(2006) 431-437.
[26] S. O. Salawu, Analysis of third grade heat absorption hydromagnetic exothermic chemical reactive flow in a Darcy-Forchheimer porous medium with convective cooling.WSEAS Transaction on Mathematics, 17 (2018) 280-289.
[27] N. N. Semenov, Some Problems in Chemical Kinetics and Reactivity, Part 2, Pergamon Press, London, (1959).
[28] G. Sobamowo and A. Akinshilo, Analysis of flow, heat transfer and entropy generation in a pipe conveying fourth grade fluid with temperature dependent viscosities and internal heat generation Journal of Molecular Liquids 241 (2007) 188-198.
[29] G. Sobamowo, A. Akinshilo, A.Yinusa and O. Adedibu, Nonlinear slip effects on pipe flow and heat transfer of third grade fluid with nonlinear temperature-dependent viscosities and internal heat generation, Software Engineering 6(3) (2018) 69-88.
[30] M. Yürüsoy and M. Pakdemirli, Approximate analytical solutions for the flow of a third-grade fluid in a pipe, Int. J. Non-Linear Mechanics, **37** (2) (2002) 187-195.
[31] M. Yürüsoy, H. Bayrakceken, M. Kapucu and F. Aksoy, Entropy analysis for third grade fluid flow with Vogel model viscosity in annular pipe, Int. J. Non-Linear Mechanics, **43**, (2008) 588-599.

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