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DEMIDOVIC'S LIMITING REGIME TO A CERTAIN FOURTH ORDER NONLINEAR DIFFERENTIAL EQUATION: ANOTHER RESULT

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ABSTRACT. In this paper, we use the Lyapunov's second method to solve an age long problem on limiting regime in the sense of Demidovic for a general fourth order nonlinear differential equation. Sufficient conditions for the existence of a unique solution to be a limiting regime, periodic (or almost periodic) according as the forcing term is periodic (or almost periodic) are established for the concerned equation. Our results improve and generalize earlier ideas of Afuwape [A.U. Afuwape, On the existence of a limiting regime in the sense of Demidovic for a certain fourth order nonlinear differential equation, JMAA, 129, (1988) 389-393].

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1. INTRODUCTION

In 1961, Demidovic [8] considered a nonlinear system

$$\dot{x} = F(x) + G(t), \tag{1.1}$$

where F and G are continuous functions of x and t respectively. Sufficient conditions on all solutions of the system 1.1 to converge to a periodic solution (limiting regime) for $t \to +\infty$ or $t \to -\infty$ are obtained. In 1965, Ezeilo [9] extended the result of Demidovic [8] to a more general n-vector system

$$\dot{X} = f(X,t) + g(X,t), \tag{1.2}$$

where g satisfied a certain Lipschitz condition and $g(0,t) \equiv 0$ with

$$\|f(0,t)\| \leq m < \infty, \ -\infty < t < +\infty, \tag{1.3}$$

for constant $m (0 < m < \infty)$. Much later on in 1973, Ezeilo [10] used the ideas of Demidovic [8] and Ezeilo [9] to obtain sufficient

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conditions on the existence of limiting regime to the third order nonlinear differential equation

$$x''' + ax'' + bx' + h(x) = p(t, x, x', x''), \tag{1.4}$$

with a and b as constants. In addition, it was assumed that functions h and p are continuous in their respective arguments. Furthermore, with $h(0) = 0$, the function h was considered not to be differentiable but only restricted to satisfy a certain incrementary ratio

$$\eta^{-1}\{h(\xi + \eta) - h(\xi)\} \in I_0, \eta \neq 0, \tag{1.5}$$

for some designated ξ where I_0 was chosen as a closed sub-interval of the Routh-Hurwitz interval. On using the ideas in the above mentioned papers, Afuwape [7] extended the result of Ezeilo [10] to the fourth order nonlinear differential equation

$$x^{(iv)} + ax''' + bx'' + cx' + h(x) = p(t, x, x', x'', x'''), \tag{1.6}$$

with constants a, b, c and continuous functions h (satisfying increment ratio (1.5)) and p.

Higher order nonlinear differential equations have proved to be valuable tools in the modeling of many physical phenomena in the sciences, social sciences and engineering. It is interesting to note that many authors have worked on the various qualitative behaviour of solutions of fourth order nonlinear differential equations. The papers of Abou-El-Ela and Sadek [1] and Sadek [15] discussed asymptotic behaviour of solutions; Adesina [2] worked on exponential stability and periodic solutions; Adesina and Ogundare [4], Tunç [17, 18, 19, 20, 21], Ogundare [11], Ogundare and Okecha [13] and Wu and Xiong [22] treated stability and boundedness of solutions. Convergence of solutions were considered in the works of Adesina and Ogundare [3], Afuwape [5, 6] and Ogundare and Okecha [12]. Further classical expository results on fourth order nonlinear differential equations can be found in Reissig *et. al* [14]. However, the corresponding situation where more than one nonlinear function is present remains an open problem on limiting regime in the sense of Demidovic.

Unfortunately, with respect to our observation in the relevant literature, ever since the work of Afuwape [7] appeared, there has been no further attempt to discuss the limiting regime in the sense of Demidovic for fourth order nonlinear differential equations. This might not be unconnected with the obvious difficulty in finding a complete Lyapunov function.

Hence, the object of this paper is to consider a more general fourth order nonlinear differential equation of the form:

$$x^{(iv)} + \phi(x''') + f(x'') + g(x') + h(x) = p(t, x, x', x'', x'''), \quad (1.7)$$

where nonlinear functions ϕ, f, g, h and p are continuous in the arguments displayed explicitly. It must be emphasized here that equation (1.7) has so far remained intractable due to (i) the number of the nonlinear terms ϕ, f, g and h simultaneously involved and (ii) the form of the functions ϕ and f (see for instance Tejumola [16]).

On setting $x' = y$, $y' = z$ and $z' = w$ in equation (1.7), we have the equivalent system

$$\begin{aligned} x' &= y, \\ y' &= z, \\ z' &= w, \\ w' &= -\phi(w) - f(z) - g(y) - h(x) + r(t, x, y, z, w) + Q(t), \end{aligned} \quad (1.8)$$

where $p(t, x, x', x'', x''')$ is separable in the form $r(t, x, y, z, w) + q(t)$ and $Q(t) = \int_0^t q(\tau)d\tau$. By constructing a complete Lyapunov function, sufficient conditions that guarantee all solutions of the equation (1.7) to converge to a limiting regime are given. Furthermore, we prove that this limiting regime is periodic or almost periodic in t according as p is periodic or almost periodic in t, uniformly in x, x', x'', x'''. For the sake of completeness, we shall now give the following definition.

The unique solution $X(t)$ of the fourth order nonlinear differential equation (1.7) is said to be a limiting regime in the sense of Demidovic if

$$(X^2 + {X'}^2 + {X''}^2 + {X'''}^2)^{\frac{1}{2}} \leq D_1 \quad (1.9)$$

for a finite D_1 and all $t \in \mathbb{R}$, and if every other solution converges to X as $t \to \infty$.

Results obtained in this work generalize and extend the results of Afuwape [7], and also complement existing results on fourth order nonlinear differential equations. This paper is organized as follows. We begin, in Section 2, by giving some preliminaries and stating the main results of the paper while Section 3 offers the proof of the main results.

2. PRELIMINARIES AND MAIN RESULTS

The linear homogeneous part of the equation (1.7) is given as

$$x^{(iv)} + ax''' + bx'' + cx' + dx = 0,$$

where a, b c and d are constants, are such that all its solutions tend to the trivial solution, as $t \to \infty$, provided that the Routh-Hurwitz conditions

$$a > 0,\ (ab - c) > 0,\ (ab - c)c - a^2 d > 0,\ d > 0$$

are satisfied.

We shall assume throughout that $\phi(0) = f(0) = g(0) = h(0) = 0$. For the remaining part of the paper, all D's with subscript are positive constants.

We shall now state our main results on the existence of a limiting regime in the sense of Demidovic.

Theorem 2.1: *Suppose that*

(i) there are positive constants a, a_0, b and b_0 such that

$$a \leq \frac{\phi(w_2) - \phi(w_1)}{w_2 - w_1} \leq a_0, \quad w_2 \neq w_1, \tag{2.1}$$

$$b \leq \frac{f(z_2) - f(z_1)}{z_2 - z_1} \leq b_0, \quad z_2 \neq z_1, \tag{2.2}$$

(ii) for any ζ, η, $(\eta \neq 0)$, the incrementary ratios for h and g satisfy

$$\frac{h(\zeta + \eta) - h(\zeta)}{\eta} \in I_0, \tag{2.3}$$

$$\frac{g(\zeta + \eta) - g(\zeta)}{\eta} \in I_1; \tag{2.4}$$

where I_0 and I_1 are closed intervals defined respectively by

$$I_0 \equiv \Big[\Delta_0, K_0\big[\frac{[(ab - c)c]}{a^2}\big]\Big], \tag{2.5}$$

$$I_1 \equiv \Big[\Delta_1, K_1\big[\frac{[(a^2 d + c^2)]}{ac}\big]\Big] \tag{2.6}$$

with a, b, c, d, $\Delta_0 > 0$, $\Delta_1 > 0$, $0 < K_0 < 1$ and $0 < K_1 < 1$ as constants.

(iii) there exists a positive constant B_0 such that

$$|Q(t)| \leq B_0, \forall t \in I\!R, \tag{2.7}$$

where

$$Q(t) = \int_0^t q(\tau)d\tau;$$

(iv) for a continuous function $\vartheta(t)$, the inequality

$$\begin{aligned} &|r(t, x_2, y_2, z_2, w_2) - r(t, x_1, y_1, z_1, w_1)| \\ &\leq \vartheta(t)(|x_2 - x_1| + |y_2 - y_1| + |z_2 - z_1| + |w_2 - w_1|) \end{aligned} \tag{2.8}$$

for arbitrary $t, x_1, y_1, z_1, w_1, x_2, y_2, z_2, w_2$ holds and satisfies

$$\int_{-\infty}^{\infty} \vartheta^\beta dt < \infty, \tag{2.9}$$

for some constant β in the range $1 \leq \beta \leq 2$.

Then there exists a unique solution $X(t)$ of the equation (1.7) *which is a limiting regime in the sense of Demidovic.*

Theorem 2.2: *In addition to hypotheses (i)–(iii) of the Theorem 2.1, suppose that there is a solution $X(t)$ of the equation* (1.7) *satisfying the inequality* (1.9) *and that $Q(t)$ is uniformly almost periodic in t for $(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}} \leq D_1$. Then the solution $X(t)$ is uniformly almost periodic in t. Furthermore, if $Q(t)$ is periodic (respectively almost periodic) with a period ω say, and $r(t, x, y, z, w)$ is periodic (respectively almost periodic) in t with period ω, then the solution $X(t)$ is periodic (respectively almost periodic) in t with period ω.*

Remark 2.3: If in the equation (1.7), $\phi(x''') = ax'''$, $f(x'') = bx''$ and $g(x') = cx'$, then our results (Theorems 2.1 and 2.2) generalize earlier results of Afuwape [7]. Furthermore, the Theorem 2.1 of this paper generalizes the convergence theorem of Afuwape [5].

Remark 2.4: If in the equation (1.7), $\phi(x''') = ax'''$ and $f(x'') = bx''$, then earlier results of Afuwape [6] and Ogundare and Okecha [12] are also generalized and extended by the Theorem 2.1 of this paper.

2. PROOF OF MAIN RESULT

Proof of Theorem 2.1: The main tool in the proof of our main results will be the following quadratic Lyapunov function $V =$

$V(x, y, z, w)$ defined by

$$\begin{aligned}2V = \; & [\beta(1-\epsilon)x + \gamma y + \delta z + w]^2 + [(1-\epsilon)D - 1](\delta z + w)^2 \\ & + \beta D[\epsilon + (1-\epsilon)D - 1]y^2 + \gamma(D-1)z^2 + \epsilon D w^2 \\ & + \beta^2\epsilon(1-\epsilon)x^2 + 2\gamma\delta[(1-\epsilon)^2 D - 1]yz,\end{aligned} \tag{3.1}$$

where $0 < \epsilon < 1-\epsilon < 1$, $\frac{\gamma\delta}{\beta} > 1-\epsilon$, β, γ, δ are positive real numbers and for $\gamma\delta \neq \beta\epsilon$; $D = 1 + \frac{\beta(1-\epsilon)[\gamma\delta - \beta(1-\epsilon)]}{\gamma\delta - \beta\epsilon}$ with $D > \frac{1}{(1-\epsilon)^2}$ always. It can be verified just in the same way as in the papers of Afuwape ([5], [6]) and Ezeilo [10] and using the hypotheses (i)–(iii) of Theorem 2.1 that the function V is indeed a Lyapunov function with the following properties:

(i) V is positive definite;
(ii) there exist positive constants D_2 and D_3 such that

$$D_2(x^2 + y^2 + z^2 + w^2) \leq V \leq D_3(x^2 + y^2 + z^2 + w^2) \tag{3.2}$$

for all x, y, z, w;
(iii) the derivative of V is negative definite.

From equation (3.1), we note that

$$\begin{aligned}\gamma(D-1)z^2 + 2\gamma\delta[(1-\epsilon)^2 D - 1]yz \; & = \gamma(D-1)X \\ & \left\{ z + \frac{\delta}{(D-1)}[(1-\epsilon)^2 D - 1]y \right\}^2 \\ & - \frac{\gamma\delta^2}{(D-1)}[(1-\epsilon)^2 D - 1]^2 y^2\end{aligned}$$

and

$$\begin{aligned}\beta D\epsilon y^2 + 2\gamma\delta[(1-\epsilon)^2 D - 1]yz \; & = \beta D\epsilon \left\{ y + \frac{\gamma\delta}{\beta D\epsilon}[(1-\epsilon)^2 D - 1]z \right\}^2 \\ & - \frac{(\gamma\delta)^2}{\beta D\epsilon} \left\{ [(1-\epsilon)^2 D - 1]z \right\}^2.\end{aligned}$$

If we substitute the above equations into the equation (3.1), then

$$\begin{aligned}2V \; & = \{\beta(1-\epsilon)x + \gamma y + \delta z + w\}^2 + [(1-\epsilon)D - 1](\delta z + w)^2 \\ & + \beta D\epsilon \left\{ y + \frac{\gamma\delta}{\beta D\epsilon}[(1-\epsilon)^2 D - 1]z \right\}^2 + \beta^2\epsilon(1-\epsilon)x^2 \\ & + \beta D[(1-\epsilon)D - 1]y^2 + \gamma(D-1)z^2 + \epsilon D w^2 \\ & - \frac{(\gamma\delta)^2}{\beta D\epsilon} \left\{ [(1-\epsilon)^2 D - 1] \right\}^2 z^2.\end{aligned}$$

Indeed we can rearrange $2V$ as

$$\begin{aligned} 2V \;&= \{\beta(1-\epsilon)x+\gamma y+\delta z+w\}^2+[(1-\epsilon)D-1](\delta z+w)^2\\ &+\beta D\epsilon\left\{y+\tfrac{\gamma\delta}{\beta D\epsilon}[(1-\epsilon)^2D-1]z\right\}^2\\ &+\beta^2\epsilon(1-\epsilon)x^2+\beta D[(1-\epsilon)D-1]y^2\\ &+\tfrac{1}{\beta D\epsilon}\{\gamma\beta D\epsilon(D-1)+2(\gamma\delta)^2D(1-\epsilon)^2\\ &-(\gamma\delta)^2[D^2(1-\epsilon)^4+1]\}\,z^2+\epsilon D w^2. \end{aligned} \tag{3.3}$$

Since $0<\epsilon<1$, $\frac{\gamma\delta}{\beta}>1-\epsilon$, $\beta,\gamma,\delta>0$ and D can be chosen such that $D>\frac{1}{1-\epsilon}$, it follows that

$$\begin{aligned} 2V \;&\geq \beta^2\epsilon(1-\epsilon)x^2+\beta D[(1-\epsilon)D-1]y^2\\ &+\tfrac{1}{\beta D\epsilon}\{\gamma\beta D\epsilon(D-1)+2(\gamma\delta)^2D(1-\epsilon)^2-(\gamma\delta)^2[D^2(1-\epsilon)^4+1]\}\,z^2\\ &+\epsilon D w^2. \end{aligned}$$

Therefore, a constant $D_2>0$ can be found such that

$$V\geq D_2(x^2+y^2+z^2+w^2), \tag{3.4}$$

where

$$D_2=\frac{\min}{2}\left\{\beta^2\epsilon(1-\epsilon);\beta D[(1-\epsilon)D-1];M^*;\epsilon D\right\},$$

with

$$M^*=\frac{\gamma\beta\epsilon(D-1)+2(\gamma\delta)^2D(1-\epsilon)^2-(\gamma\delta)^2[D^2(1-\epsilon)^4+1]}{\beta D\epsilon}.$$

Furthermore, by using the Schwartz inequality $|y||z|\leq\dfrac{1}{2}(y^2+z^2)$, we note that

$$\begin{aligned} 2V\leq\;& \beta(1-\epsilon)[\beta+\gamma+\delta+1]x^2+\{\gamma\,(\gamma+\beta(1-\epsilon))\\ &+\delta[(1-\epsilon)^2D-1]+\beta D[\epsilon+(1-\epsilon)D-1]\}\,y^2\\ &+\{\delta^2(1-\epsilon)D+\delta[\beta(1-\epsilon)+(1-\epsilon)D-1]\\ &+\gamma([1-\epsilon]^2D-1)+(D-1)\}\,z^2\\ &+\{\beta(1-\epsilon)+(D-1)\delta+(1-\epsilon)D\}\,w^2. \end{aligned}$$

Thus there exists D_3 such that

$$V\leq D_3(x^2+y^2+z^2+w^2), \tag{3.5}$$

where

$$\begin{aligned} D_3=\frac{\max}{2}\;\; &\{\beta(1-\epsilon)[\beta+\gamma+\delta+1];\gamma\,(\gamma+\beta(1-\epsilon)\\ &+\;\delta[(1-\epsilon)^2D-1])+\beta D[\epsilon+(1-\epsilon)D-1];\\ &\;\;\delta^2(1-\epsilon)D+\delta[\beta(1-\epsilon)+(1-\epsilon)D-1]\\ &+\gamma([1-\epsilon]^2D-1)+(D-1);\beta(1-\epsilon)\\ &+(D-1)\delta+(1-\epsilon)D\}\,. \end{aligned}$$

On combining inequalities (3.4) and (3.5), we have

$$D_2(x^2+y^2+z^2+w^2) \le V \le D_3(x^2+y^2+z^2+w^2). \tag{3.6}$$

Now, to show that the derivative of V is negative definite, we proceed as follows. Let the hypotheses of the Theorem 2.1 hold, and let the function $W(t) = W(x_2-x_1, y_2-y_1, z_2-z_1, w_2-w_1)$ be defined by

$$\begin{aligned}
2W \; &= [\beta(1-\epsilon)(x_2-x_1)+\gamma(y_2-y_1)+\delta(z_2-z_1)+(w_2-w_1)]^2\\
&+[(1-\epsilon)D-1](\delta(z_2-z_1)+(w_2-w_1))^2\\
&+\beta D[\epsilon+(1-\epsilon)D-1](y_2-y_1)^2+\gamma(D-1)(z_2-z_1)^2\\
&+\epsilon D(w_2-w_1)^2+\beta^2\epsilon(1-\epsilon)(x_2-x_1)^2\\
&+2\gamma\delta[(1-\epsilon)^2D-1](y_2-y_1)(z_2-z_1),
\end{aligned}$$

then we can show that there exist positive constants D_4 and D_5 such that

$$\frac{dW}{dt} \le -2D_4S + D_5S^{1/2}|\theta|, \tag{3.7}$$

where $\theta = r(t,x_2,y_2,z_2,w_2) - r(t,x_1,y_1,z_1,w_1)$.

In fact, with respect to the system (1.8), a direct computation of $\frac{dW}{dt}$ gives after simplifications

$$\frac{dW}{dt} = -U_1 + U_2, \tag{3.8}$$

where

$$\begin{aligned}
U_1 \; &= \beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2+\gamma[G(y_2,y_1)\\
&-\beta(1-\epsilon)](y_2-y_1)^2+D\delta(1-\epsilon)[F(z_2,z_1)-\gamma(1-\epsilon)]\\
&(z_2-z_1)^2\\
&+D[\Phi(w_2,w_1)-\delta(1-\epsilon)](w_2-w_1)^2\\
&-\beta(1-\epsilon)[G(y_2,y_1)-\beta](x_2-x_1)(y_2-y_1)\\
&-\beta(1-\epsilon)[F(z_2,z_1)-\gamma](x_2-x_1)(z_2-z_1)\\
&-\beta(1-\epsilon)[\Phi(w_2,w_1)-\delta](x_2-x_1)(w_2-w_1)\\
&-[D\delta(1-\epsilon)G(y_2,y_1)+\gamma(F(z_2,z_1)-\gamma)-D\beta\gamma]\\
&(y_2-y_1)(z_2-z_1)\\
&-[DG(y_2,y_1)+\gamma\Phi(w_2,w_1)-\beta(1-\epsilon)-D\gamma\delta(1-\epsilon)^2]\\
&(y_2-y_1)(w_2-w_1)\\
&-[D(F(z_2,z_1)+D\delta\Phi(w_2,w_1)+D(\delta^2(1-\epsilon)+\gamma)]\\
&(z_2-z_1)(w_2-w_1)
\end{aligned} \tag{3.9}$$

and

$$U_2 = \theta[\beta(1-\epsilon)(x_2-x_1)+\gamma(y_2-y_1)+D\delta(1-\epsilon)(z_2-z_1)+D(w_2-w_1)], \tag{3.10}$$

with

$$\begin{array}{ll} H(x_2,x_1) & = \frac{h(x_2)-h(x_1)}{x_2-x_1},\ x_2 \neq x_1; \\ G(y_2,y_1) & = \frac{g(y_2)-g(y_1)}{y_2-y_1},\ y_2 \neq y_1; \\ F(z_2,z_1) & = \frac{f(z_2)-f(z_1)}{z_2-z_1},\ z_2 \neq z_1; \\ \Phi(w_2,w_1) & = \frac{\phi(w_2)-\phi(w_1)}{w_2-w_1},\ w_2 \neq w_1. \end{array}$$

Furthermore, let $\chi_1 = G(y_2,y_1) - \beta,\ y_2 \neq y_1,\ \chi_2 = F(z_2,z_1) - \gamma,\ z_2 \neq z_1$ and $\chi_3 = \Phi(w_2,w_1) - \delta,\ w_2 \neq w_1$. Also let $\lambda_1 = G(y_2,y_1) - \beta(1-\epsilon),\ y_2 \neq y_1,\ \lambda_2 = F(z_2,z_1) - \gamma(1-\epsilon),\ z_2 \neq z_1$ and $\lambda_3 = \Phi(w_2,w_1) - \delta(1-\epsilon),\ w_2 \neq w_1$. Define

$$\sum_{i=1}^{3} \mu_i = 1; \quad \sum_{i=1}^{7} \nu_i = 1; \quad \sum_{i=1}^{7} \kappa_i = 1; \quad \sum_{i=1}^{7} \tau_i = 1,$$

with $\mu_i > 0$, $\nu_i > 0$, $\kappa_i > 0$ and $\tau_i > 0$. Then we can rearrange U_1 as

$$U_1 = W_1+W_2+W_3+W_4+W_5+W_6+W_7+W_8+W_9+W_{10}+W_{11}+W_{12},$$

where

$$\begin{array}{ll}
W_1 & = \mu_1\beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2 + \beta(1-\epsilon)\chi_1(x_2-x_1)(y_2-y_1) \\
 & +\nu_1\gamma\lambda_1(y_2-y_1)^2; \\
W_2 & = \mu_1\beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2 + \beta(1-\epsilon)\chi_2(x_2-x_1)(z_2-z_1) \\
 & +\kappa_1 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2; \\
W_3 & = \mu_1\beta(1-\epsilon)H(x_2,x_1)(x_2-x_1)^2 + \beta(1-\epsilon)\chi_3(x_2-x_1)(w_2-w_1) \\
 & +\tau_1 D\lambda_3(w_2-w_1)^2; \\
W_4 & = \nu_2\gamma\lambda_1(y_2-y_1)^2 + D\delta(1-\epsilon)\lambda_1(y_2-y_1)(z_2-z_1) \\
 & +\kappa_2 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2; \\
W_5 & = \nu_3\gamma\lambda_1(y_2-y_1)^2 + \gamma\chi_2(y_2-y_1)(z_2-z_1) \\
 & +\kappa_3 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2; \\
W_6 & = \nu_4\gamma\lambda_1(y_2-y_1)^2 + D\beta\delta((1-\epsilon)^2-1)(y_2-y_1)(z_2-z_1) \\
 & +\kappa_4 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2; \\
W_7 & = \nu_5\gamma\lambda_1(y_2-y_1)^2 + D\chi_1(y_2-y_1)(w_2-w_1) + \tau_2 D\lambda_3(w_2-w_1)^2; \\
W_8 & = \nu_6\gamma\lambda_1(y_2-y_1)^2 + \gamma\chi_3(y_2-y_1)(w_2-w_1) + \tau_3 D\lambda_3(w_2-w_1)^2; \\
W_9 & = \nu_7\gamma\lambda_1(y_2-y_1)^2 + [D(\beta-\gamma\delta(1-\epsilon)^2) \\
 & +\gamma\delta-\beta(1-\epsilon)](y_2-y_1)(w_2-w_1) + \tau_4 D\lambda_3(w_2-w_1)^2; \\
W_{10} & = \kappa_5 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2 + D\chi_2(z_2-z_1)(w_2-w_1) \\
 & +\tau_5 D\lambda_3(w_2-w_1)^2; \\
W_{11} & = \kappa_6 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2 + D\delta\chi_3(z_2-z_1)(w_2-w_1) \\
 & +\tau_6 D\lambda_3(w_2-w_1)^2; \\
W_{12} & = \kappa_7 D\delta(1-\epsilon)\lambda_2(z_2-z_1)^2 \\
 & +D[\delta^2(1-\epsilon)+2\gamma+\delta^2](z_2-z_1)(w_2-w_1) + \tau_7 D\lambda_3(w_2-w_1)^2.
\end{array}$$

It is not difficult to see that each of $W_1, W_2, ..., W_{12}$ is quadratic in its respective variables. Thus we can use the fact that any quadratic of the form $AX^2 + BXY + CY^2$ is non negative if $4AC - B^2 \geq 0$ to obtain the following inequalities:

$$\begin{array}{lll}
W_1 \geq 0, & \text{if } {\chi_1}^2 \leq \frac{4\mu_1\nu_1\gamma H(x_2,x_1)\lambda_1}{\beta(1-\epsilon)}, & \text{for } x_2 \neq x_1; \\
W_2 \geq 0, & \text{if } {\chi_2}^2 \leq \frac{4D\mu_2\kappa_1\delta H(x_2,x_1)\lambda_2}{\beta}, & \text{for } x_2 \neq x_1; \\
W_3 \geq 0, & \text{if } {\chi_3}^2 \leq \frac{4D\mu_3\tau_1 H(x_2,x_1)\lambda_3}{\beta(1-\epsilon)}, & \text{for } x_2 \neq x_1; \\
W_4 \geq 0, & \text{if } \lambda_1 \leq \frac{4\nu_2\kappa_2\gamma\lambda_2}{\beta(1-\epsilon)}, & \text{for } z_2 \neq z_1 \\
W_5 \geq 0, & \text{if } {\chi_2}^2 \leq 4\nu_3\kappa_3 D\delta(1-\epsilon)\lambda_1\lambda_2, & \text{for } y_2 \neq y_1,\ z_2 \neq z_1; \\
W_6 \geq 0, & \text{if } \lambda_2 \geq \frac{D\delta\beta^2((1-\epsilon)^2-1)^2}{4\nu_4\gamma\kappa_4(1-\epsilon)\lambda_1}, & \text{for } y_2 \neq y_1,\ z_2 \neq z_1 \\
W_7 \geq 0, & \text{if } {\chi_1}^2 \leq \frac{4\nu_5\tau_2\lambda_1\lambda_3}{D}, & \text{for } y_2 \neq y_1,\ w_2 \neq w_1; \\
W_8 \geq 0, & \text{if } {\chi_3}^2 \leq \frac{4D\nu_6\tau_3\lambda_1\lambda_3}{\gamma}, & \text{for } y_2 \neq y_1,\ w_2 \neq w_1; \\
W_9 \geq 0, & \text{if } \lambda_3 \geq \frac{[D(\beta-\gamma\delta(1-\epsilon)^2)+\gamma\delta-\beta(1-\epsilon)]^2}{4\nu_7\gamma\tau_4 D\lambda_1}, & \text{for } y_2 \neq y_1,\ w_2 \neq w_1; \\
W_{10} \geq 0, & \text{if } {\chi_2}^2 \leq 4\tau_5\kappa_5\delta(1-\epsilon)\lambda_2\lambda_3, & \text{for } z_2 \neq z_1,\ w_2 \neq w_1; \\
W_{11} \geq 0, & \text{if } {\chi_3}^2 \leq \frac{4\kappa_6\tau_6(1-\epsilon)\lambda_2\lambda_3}{\gamma}, & \text{for } z_2 \neq z_1,\ w_2 \neq w_1; \\
W_{12} \geq 0, & \text{if } \lambda_3 \geq \frac{[\delta^2(1-\epsilon)^2)+2\gamma+\delta^2]^2}{4\kappa_7\tau_7\delta(1-\epsilon)\lambda_2}, & \text{for } z_2 \neq z_1,\ w_2 \neq w_1.
\end{array}$$

Therefore, $U_1 \geq W_1$, provided that

$$\begin{array}{rl}
0 \leq {\chi_1}^2 \leq & 4\min\left\{\frac{\mu_1\nu_1\gamma H(x_2,x_1)\lambda_1}{\beta(1-\epsilon)}; \frac{\nu_5\tau_2\lambda_1\lambda_3}{D}\right\}; \\
0 \leq {\chi_2}^2 \leq & 4\min\left\{\frac{D\mu_2\kappa_1\delta H(x_2,x_1)\lambda_2}{\beta}; \nu_3\kappa_3 D\delta(1-\epsilon)\lambda_1\lambda_2; \tau_5\kappa_5\delta(1-\epsilon)\lambda_2\lambda_3\right\}; \\
0 \leq {\chi_3}^2 \leq & 4\min\left\{\frac{4D\mu_3\tau_1 H(x_2,x_1)\lambda_3}{\beta(1-\epsilon)}; \frac{4D\nu_6\tau_3\lambda_1\lambda_3}{\gamma}; \frac{4\kappa_6\tau_6(1-\epsilon)\lambda_2\lambda_3}{\gamma}\right\}; \\
& \lambda_1 \leq \frac{4\nu_2\kappa_2\gamma\lambda_2}{\beta(1-\epsilon)}; \\
& \lambda_2 \geq \frac{D\delta\beta^2((1-\epsilon)^2-1)^2}{4\nu_4\gamma\kappa_4(1-\epsilon)\lambda_1}; \\
0 \leq {\lambda_3}^2 \leq & \max\left\{\frac{[D(\beta-\gamma\delta(1-\epsilon)^2)+\gamma\delta-\beta(1-\epsilon)]^2}{4\nu_7\gamma\tau_4 D\lambda_1}; \frac{[\delta^2(1-\epsilon)^2)+2\gamma+\delta^2]^2}{4\kappa_7\tau_7\delta(1-\epsilon)\lambda_2}\right\},
\end{array}$$

where H and G lie respectively in

$$I_0 \equiv \Big[\Delta_0, K_0\big[\frac{[(ab-c)c]}{a^2}\big]\Big],$$

$$I_1 \equiv \Big[\Delta_1, K_1\big[\frac{[(a^2d+c^2)]}{ac}\big]\Big]$$

with $K_0 < 1$, $K_1 < 1$, $\Delta_0 > 0$ and $\Delta_1 > 0$ constants.

By choosing

$$2D_4 = \min\left\{\beta(1-\epsilon)\Delta_0; \gamma\lambda_1; D\delta(1-\epsilon)\lambda_2; D\lambda_3\right\},$$

it follows that

$$U_1 \geq W_1 \geq 2D_4 S. \tag{3.11}$$

Furthermore, if we choose

$$D_5 = \max\left\{\beta(1-\epsilon;\gamma;D\delta(1-\epsilon);D(w_2-w_1)\right\}, \tag{3.12}$$

we have

$$U_2 \leq D_5 S^{\frac{1}{2}}|\theta|.$$

On combining (3.11) and (3.12) in (3.10), we obtain (3.7).

At last, let (x_1, y_1, z_1, w_1) and (x_2, y_2, z_2, w_2) be any two distinct solutions of the system (1.8). Taking into account assumptions (i) and (ii) of the Theorem 2.1, it can be shown that

$$S(t_2) \leq D_6 S(t_1)\exp\left\{-D_7(t_2-t_1)\right\}, (t_2 \geq t_1), \tag{3.13}$$

where

$$S(t) = \left[(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 + (w_1-w_2)^2\right]^2.$$

Indeed, let $W(t)$ be defined by

$$W(t) = \left[(x_1-x_2), (y_1-y_2), (z_1-z_2), (w_1-w_2)\right], \tag{3.14}$$

then, for $t_1 \leq t_2$,

$$\begin{aligned} W(t_2) &\leq W(t_1)\exp\left\{-D_7(t_2-t_1) + D_8\int_{t_1}^{t_2}\vartheta^{\alpha}(\tau)d\tau\right\} \\ &\leq D_9 W(t_1)\exp\left\{-D_7(t_2-t_1)\right\}, (t_2 \geq t_1), \end{aligned} \tag{3.15}$$

where D_9 is a finite constant (see hypothesis (iv) of Theorem 2.1) given by

$$D_9 = \exp\{D_8\int_{-\infty}^{\infty}\vartheta^{\beta}(\tau)d\tau\},$$

with $1 \leq \beta \leq 2$. On using the inequalities (3.2), and in view of the fact that V is positive definite, we obtain

$$S(t_2) \leq D_{10} S(t_1)\exp\left\{-D_7(t_2-t_1)\right\}, (t_2 \geq t_1).$$

On setting $D_9 = D_{10}$, we have inequality (3.13).

In addition to the above, on using the adaptation of the techniques in Afuwape[7], Demidovic[8] and Ezeilo[9], it is not difficult to utilize the property of $W(t)$ to show that there exists a unique solution of the equation (1.7) satisfying (1.9) for which every other solution distinct from it converges as $t \to \infty$. This fact completes the proof of the theorem.

Proof of Theorem 2.2: The case when $Q(t)$ is uniformly almost periodic and $r(x, y, z, w)$ is uniformly almost periodic in t for $(x^2 + y^2 + z^2 + w^2)^{\frac{1}{2}} \leq D_1$ will be considered. Let

$$\Psi(t) = V(X(t+\tau) - X(t), Y(t+\tau) - Y(t), Z(t+\tau) - Z(t)$$

$$, W^*(t+\tau) - W^*(t)), \tag{3.16}$$

where V is as defined in the equation (3.1), and τ is a constant. By uniform almost periodicity of $Q(t)$ and $r(x, y, z, w)$ in t, it follows that for an arbitrary $\epsilon > 0$, there exists a $\tau > 0$ such that

$$\begin{gathered} \|Q(t+\tau) - Q(t)\| \leq k\epsilon^2 \\ \|r(t+\tau, X(t), Y(t), Z(t), W^*) - r(t, X(t), Y(t), Z(t), W^*)\| \leq k\epsilon^2, \end{gathered} \tag{3.17}$$

where k is a constant whose exact value will be chosen to advantage later. On making use of the system (1.8) and inequalities (3.17), we have

$$\frac{d\Psi}{dt} \leq -\{D_{11} - D_{12}\vartheta^{\beta}(t+\tau)\}\Psi(t) + 4kD_{13}D_1\epsilon^2. \tag{3.18}$$

Now let

$$D_{14} = \exp\left\{D_{12}\int_{-\infty}^{\infty} \vartheta^{\beta}(t)\}dt, \right\}.$$

On integrating inequality (3.18) and using inequalities (3.17), we have

$$\Psi(t) \leq D_{14}\Psi(t_0)\exp\{-D_{11}(t-t_0)\} + kD_{15}\epsilon^2, \ (t \geq t_0) \tag{3.19}$$

where

$$D_{15} = \frac{4D_{13}D_{14}D_1}{D_{11}}.$$

Inequalities (3.19) hold for arbitrary t_0. In particular, on letting $t_0 \to -\infty$ in (3.17) and noting that $\Psi(t)$ is finite, we have

$$\Psi(t) \leq kD_{15}\epsilon^2$$

for arbitrary t. By inequalities (3.2) and using the definition of Ψ, these imply that

$$\|X(t+\tau) - X(t)\| \leq \left(k\frac{D_{15}}{D_1}\right)^{\frac{1}{2}}\epsilon. \tag{3.20}$$

Suppose that in the inequalities (3.17), the constant k had been defined by $k = \frac{D_1}{D_{15}}$. Then inequality (3.20) would read

$$\|X(t+\tau) - X(t)\| \leq \epsilon, \tag{3.21}$$

where τ is chosen to satisfy inequalities (3.17) with $k = \frac{D_1}{D_{15}}$. The set of all numbers τ satisfying inequalities (3.17) is relatively dense, and hence the inequality (3.21) implies that $X(t)$ is uniformly almost periodic. Hence the completion of the proof to the first part of the Theorem 2.2.

To prove the second part of the theorem, let us assume that

$$Q(t+\tau) = Q(t),$$
$$r(t+\tau, X(t), Y(t), Z(t), W^*) = r(t, X(t), Y(t), Z(t), W^*),$$

for

$$(X^2 + Y^2 + Z^2 + W^*)^{\frac{1}{2}} \leq D_1.$$

Fix τ in the equation (3.16). The terms on the left hand side of inequalities (3.17) are then both identically zero, and so, if we proceed as in the proof of the first part of the theorem, we shall have the following in place of inequality (3.20

$$\|X(t+\tau) - X(t)\| \leq 0. \tag{3.22}$$

Hence by the definition of $\|.\|$, we have

$$\|X(t+\tau) - X(t)\| = 0,$$

making $X(t)$ periodic with period τ.

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