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# FIXED POINT AND CONVERGENCE THEOREMS FOR CERTAIN CLASSES OF MAPPINGS

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ABSTRACT. We prove fixed point theorems for nonspreading-type mappings and obtain convergence theorems for approximation of common fixed points of $k$-strictly pseudocontractive mappings of Browder-Petryshyn type and $\beta$-strictly pseudononspreading mappings.

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## 1. INTRODUCTION

Let $H$ be a real Hilbert space. A mapping $T : D(T) \subseteq H \to H$ is said to be *L-Lipschitzian* if there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|, \quad \forall x, y \in D(T). \tag{1}$$

If $L < 1$ in (1), $T$ is said to be *strictly contractive* , while $T$ is said to be *nonexpansive* if $L = 1$. $T$ is said to be *Quasi-nonexpansive* If $F(T) = \{x \in D(T) : Tx = x\} \neq \emptyset$ and

$$\| Tx - p\| \leq \|x - p\|, \quad \forall p \in F(T), x \in D(T). \tag{2}$$

$T$ is said to be *firmly nonexpansive* if

$$\|Tx - Ty\|^2 \leq \langle x - y, Tx - Ty\rangle, \quad \forall x, y \in D(T). \tag{3}$$

Every nonexpansive mapping with nonempty fixed point set F(T) is quasi-nonexpansive, and firmly nonexpansive mappings are important examples of nonexpansive mappings.

A mapping $T : D(T) \subseteq H \to H$ is called a *k-strictly pseudocontractive* mapping of *Browder-Petryshyn* type (see [2]) if there exists

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$k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2, \ \forall \ x, y \in D(T). \tag{4}$$

The class of nonexpansive mappings is properly contained in the class of $k$-strictly pseudocontractive mappings.

Recently, Kohsaka and Takahashi [6],[7] introduced an important class of mappings which they called the class of *nonspreading mappings*. Let $E$ be a real smooth, strictly convex and reflexive Banach space, and let $j : E \to 2^E$ denote the duality mapping of $E$. Let $C$ be a nonempty closed convex subset of $E$. They called a mapping $T : C \to C$ *nonspreading* if

$$\phi(Tx, Ty) + \phi(Ty, Tx) \leq \phi(Tx, y) + \phi(Ty, x),$$

for all $x, y \in C$ where $\phi(x, y) = \|x\|^2 - 2\langle x, j(y)\rangle + \|y\|^2$, for all $x, y \in E$. They considered the class of nonspreading mappings to study the resolvents of a maximal monotone operator in real smooth, strictly convex and reflexive Banach spaces. This class of mappings is deduced from the class of firmly nonexpansive mappings (see [4],[6]). Observe that if $E$ is a real Hilbert space, then $j$ is the identity and

$$\phi(x, y) = \|x\|^2 - 2\langle x, y\rangle + \|y\|^2.$$

Thus if $C$ is a nonempty closed convex subset of a real Hilbert space, then $T : C \to C$ is nonspreading if

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2 \quad \forall x, y \in C. \tag{5}$$

It is shown in [5] that (5) is equivalent to

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + 2\langle x - Tx, y - Ty\rangle, \quad \forall x, y \in C. \tag{6}$$

Observe that if $T$ is nonspreading and $F(T) \neq \emptyset$, then $T$ is quasi-nonexpansive. In [10], we introduced another important class of mappings more general than the class of nonspreading mappings which we called the class of *$k$-strictly pseudononspreading mapping of Browder-Petryshyn type*. We called a mapping $T : D(T) \subseteq H \to H$ a *$k$-strictly pseudononspreading* if there exists $k \in [0, 1)$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + k\|(x - Tx) - (y - Ty)\|^2 + 2\langle x - Tx, y - Ty\rangle, \tag{7}$$

for all $x, y \in D(T)$. We showed that the class of nonspreading mappings is properly contained in the class of $\beta$-strictly pseudononspreading mappings and obtained a weak mean convergence theorem of Baillon's type [1] which is similar to the one obtained in

[8]. Also using an idea of mean convergence, we proved a strong convergence theorem similar to that obtained in [8].

In [5], Iemoto and Takahashi studied the approximation of common fixed points of a nonexpansive mapping $T$ and a nonspreading mapping $S$ of $C$ into itself in a Hilbert space. They considered the iterative procedure similar to the one used in Moudafi [9].

If $T, S : C \to C$ are respectively nonexpansive and nonspreading mappings, they considered the iterative scheme $\{x_n\}_{n=1}^{\infty}$ generated from an arbitrary $x_1 \in C$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1-\beta_n)Tx_n], \quad n \geq 1$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are suitable sequences in $[0,1]$.

They proved the following main theorem:

**Theorem 1 [([5]) Theorem 4.1]:** Let $H$ be a real Hilbert space. Let $C$ be a nonempty closed and convex subset of $H$. Let $S$ be a nonspreading mapping of $C$ into itself and $T$ a nonexpansive mapping of $C$ into itself such that $F(T) \bigcap F(S) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in $C$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1-\beta_n)Tx_n], \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty} \subset [0,1]$
Then the following hold:
(i) If $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\sum_{n=1}^{\infty}(1-\beta_n) < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(S)$;
(ii) if $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n) = \infty$, and $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T)$;
(iii) if $\liminf_{n\to\infty} \alpha_n(1-\alpha_n) > 0$, $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, then $\{x_n\}_{n=1}^{\infty}$ converges weakly to $v \in F(T) \bigcap F(S)$.

It is our purpose in this paper to first prove fixed point theorems for $\beta$-strictly pseudononspreading mappings. We then extend the above theorem of Iemoto and Takahashi [5] to the more general case where $T$ is a $k$-strictly pseudocontractive mapping and $S$ is $\beta$-strictly pseudononspreading mapping.

## 2. PRELIMINARY

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers, and by $\Re$ the set of real numbers . Let $H$ be a real Hilbert space with inner product $\langle .,. \rangle$ and norm $\|.\|$, then the following well known results hold:

(i) $\|\alpha x+(1-\alpha)y\|^2 = \alpha\|x\|^2+(1-\alpha)\|y\|^2-\alpha(1-\alpha)\|x-y\|^2$, (8)

for all $x, y \in H$, $\alpha \in [0,1]$.

(ii) $2\langle x-y, u-v\rangle = \|x-v\|^2+\|y-v\|^2-\|x-u\|^2-\|y-u\|^2$, (9)

for all $x, y, u, v \in H$ (see [5]).

(iii) If $\{x_n\}_{n=1}^{\infty}$ is a sequence in $H$ which converges weakly to $z \in H$, then

$$\limsup_{n\to\infty} ||x_n - y|| = \limsup_{n\to\infty} ||x_n - z|| + ||z - y|| \ \forall y \in H. \qquad (10)$$

Let $C$ be a closed convex subset of $H$ and let $T$ be a mapping of $C$ into itself. We denote the set of fixed points of $T$ by $F(T)$. A point $p \in C$ is called *asymptotic fixed point of* $T$ if there exists a sequence $\{x_n\} \subset C$ such that $\{x_n\}_{n=1}^{\infty}$ converges weakly to $p$ and $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0. We denote the set of asymptotic fixed points of $T$ by $\widehat{F}(T)$.

**Theorem 2([10]):** Let $H$ be a real Hilbert space, let $C$ be a nonempty closed convex subset of $H$ and let $S$ be a $k$-strictly pseudononspreading mapping of $C$ into itself. Then $F(S)$ is closed and convex.

**Lemma 1 ([11,14]):** Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of nonnegative real numbers such that $a_{n+1} \leq a_n + b_n$ for all $n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

**Lemma 2 ([10]):** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $T : C \to C$ be a $\beta$-strictly pseudononspreading mapping of $C$ into itself, then $(I - T)$ is demiclosed at zero.

**Lemma 3 ([12]):** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. let $T : C \to C$ be a $k$-strictly pseudocontractive mapping of $C$ into itself, then $(I - T)$ is demiclosed at zero.
An example of a nonspreading mapping which is not nonexpansive is given in [4]. The following example shows that a nonexpansive mapping need not be nonspreading so that the class of nonspreading mappings and the class nonexpansive mappings are independent.

**Example 1:** Let $\Re$ denote the reals with the usual norm and define $T : \Re \to \Re$ by

$$Tx = -x.$$

Then $T$ is nonexpansive. Now for arbitrary $x \neq 0$, $y = -x$ we have $|Tx - Ty|^2 = 4x^2 > -4x^2 = |x - y|^2 + 2\langle x - Tx, y - Ty\rangle$. Hence, $T$ is not nonspreading.

The following examples show that the class of $k$-strictly pseudocontractive mappings and the class of $\beta$-strictly pseudononspreading mappings are independent.

**Example 2:** Let $\Re$ denote the reals with the usual norm and define $T : \Re \to \Re$ by

$$Tx = -3x.$$

Then for all $x, y \in \Re$ we have

$$|Tx - Ty|^2 = 9|x - y|^2 = |x - y|^2 + \frac{1}{2}|x - Tx - (y - Ty)|^2.$$

Thus $T$ is $\frac{1}{2}$-strictly pseudocontractive. However, if we take $x = \frac{1}{2}$, $y = -\frac{1}{2}$, we obtain

$$\begin{aligned} |Tx - Ty|^2 = 9 &= |x - y|^2 + |x - Tx - (y - Ty)|^2 \\ &\quad +2\langle x - Tx, y - Ty\rangle \\ &> |x - y|^2 + \beta|x - Tx - (y - Ty)|^2 \\ &\quad +2\langle x - Tx, y - Ty\rangle \end{aligned}$$

for all $\beta \in [0, 1)$. Thus $T$ is not $\beta$-strictly pseudononspreading.

**Example 3:** Let $\Re$ denote the reals with the usual norm, for each $x \in \Re$ define $T : \Re \to \Re$ by

$$Tx = \begin{cases} 0, & x \in (-\infty, 2] \\ 1, & x \in (2, \infty). \end{cases}$$

Then for all $x, y \in (-\infty, 2]$ and for all $\beta \in [0, 1)$, we have $|x - y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle = x^2 + y^2 + \beta|x - y|^2 \geq 0 = |Tx - Ty|^2$.
Furtheremore, for all $x, y \in (2, \infty)$ and for all $\beta \in [0, 1)$, we have $|x - y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y - Ty\rangle = |x - y|^2 + \beta|x - y|^2 + 2(x - 1)(y - 1) > 0 = |Tx - Ty|^2$.
Finally, if $x \in (-\infty, 2]$ and $y \in (2, \infty)$, then $|x-y|^2+\beta|x-Tx-(y-Ty)|^2+2\langle x-Tx, y-Ty\rangle \geq (x-1)^2+y^2-1 > 3 > 1 = |Tx-Ty|^2$.
Thus, for all $x, y \in \Re$ and for all $\beta \in [0, 1)$, we have $|Tx - Ty|^2 \leq |x - y|^2 + \beta|x - Tx - (y - Ty)|^2 + 2\langle x - Tx, y -$

$Ty\rangle$. Hence $T$ is $\beta$-strictly pseudononspreading. Since every $k$-strictly pseudocontractive mapping $T : D(T) \subseteq H \to H$ satisfies the Lipschitz condition

$$||Tx - Ty|| \leq \frac{1+\sqrt{k}}{1-\sqrt{k}}||x-y||,$$

it is clear that $T$ is not $k$-strictly pseudocontractive.

**Definition 1:** Let $C$ be a nonempty closed convex subset of a real Hilbert space. A mapping $T : C \to C$ is said to be a $\delta$-*type nonspreading* if there exists $\delta \in \Re$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \delta\langle x - Tx, y - Ty\rangle, \ \ \forall x, y \in C.$$

## 3. MAIN RESULTS

We now prove the following:

**Proposition 1:**
Let $C$ be a nonempty closed convex subset of of a real Hilbert space and let $T : C \to C$ be a $\delta$-type nonspreading mapping with nonempty fixed-point-set $F(T)$. Then $F(T)$ is closed and convex.

**Proof:**
Let $\{x_n\}_{n=1}^{\infty} \subseteq F(T)$ be a sequence which converges to $x$. We show that $x \in F(T)$. Observe that

$$\begin{aligned} \|Tx - x\| &\leq \|Tx - Tx_n\| + \|x_n - x\|, \\ &\leq 2\|x_n - x\| \to 0 \ as \ n \to \infty. \end{aligned}$$

It then follows that $x = Tx$. Hence $x \in F(T)$.

Next, let $p_1, p_2 \in F(T)$, we show that $\lambda p_1 + (1-\lambda)p_2 \in F(T)$. Let $z = \lambda p_1 + (1-\lambda)p_2$.Then $p_1 - z = (1-\lambda)(p_1 - p_2)$, $p_2 - z = \lambda(p_2 - p_1)$

$$\begin{aligned} \|z - Tz\|^2 &= \|\lambda p_1 + (1-\lambda)p_2 - Tz\|^2 \\ &= \|\lambda(p_1 - Tz) + (1-\lambda)(p_2 - Tz)\|^2 \\ &= \lambda\|p_1 - Tz\|^2 + (1-\lambda)\|p_2 - Tz\|^2 \\ &\quad -\lambda(1-\lambda)\|p_1 - p_2\|^2 \\ &\leq \lambda\Big[\|p_1 - z\|^2 + \delta\langle p_1 - Tp_1, z - Tz\rangle\Big] \\ &\quad +(1-\lambda)\Big[\|p_2 - z\|^2 + \delta\langle p_2 - Tp_2, z - Tz\rangle\Big] \\ &\quad -\lambda(1-\lambda)\|p_1 - p_2\|^2 \end{aligned}$$

$$= \lambda\|p_1 - z\|^2 + (1-\lambda)\|p_2 - z\|^2 - \lambda(1-\lambda)\|p_1 - p_2\|^2 = 0.$$

Hence, $z = Tz$ which implies that $z \in F(T)$. $\square$

**Proposition 2:** Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$, and let $T : C \to C$ be a $\delta$-type nonspreading mapping. Then $(I - T)$ is demiclosed at 0.

**Proof:** Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $C$ which converges weakly to $p$ and $\{x_n - Tx_n\}_{n=1}^{\infty}$ converges strongly to 0, we prove that $p \in F(T)$. Since $\{x_n\}_{n=1}^{\infty}$ converges weakly, it is bounded. For each $x \in H$ define $f : H \to [0, \infty)$ by

$$f(x) := \limsup_{n\to\infty} ||x_n - x||^2.$$

Then using (10) we obtain

$$f(x) = \limsup_{n\to\infty} ||x_n - p||^2 + ||p - x||^2 \; \forall x \in H.$$

Thus
$f(x) = f(p) + ||p - x||^2 \; \forall x \in H$ and

$$f(Tp) = f(p) + ||p - Tp||^2. \tag{11}$$

Observe that

$$\begin{aligned} f(Tp) &= \limsup_{n\to\infty} ||x_n - Tp||^2 \\ &= \limsup_{n\to\infty} ||x_n - Tx_n + Tx_n - Tp||^2 \\ &= \limsup_{n\to\infty} ||Tx_n - Tp||^2 \\ &\leq \limsup_{n\to\infty} \Big[||x_n - p||^2 + \delta\langle x_n - Tx_n, p - Tp\rangle\Big] \\ &= \limsup_{n\to\infty} ||x_n - p||^2 = f(p). \end{aligned} \tag{12}$$

Hence it follows from (11) and (12) that $||p - Tp|| = 0$. $\square$

**Proposition 3:** Let $H$ be a real Hilbert space, and $C$ a nonempty closed and convex subset of $H$. Let $T : C \to C$ be a $\delta$-type nonspreading mapping. Then $F(T) = \widehat{F}(T)$.

**Proof:** Let $p \in F(T)$ be arbitrary, from Proposition 1 $F(T)$ is closed therefore there exists a sequence $\{x_n\} \subseteq F(T) \subseteq C$ such that $x_n \to p$. Observe that $x_n - Tx_n = 0$ for all $n \in \mathbb{N}$ so that $x_n - Tx_n \to 0$. Hence, $p \in \widehat{F}(T)$. Next, let $p \in \widehat{F}(T)$ then there exists a sequence $\{x_n\} \subseteq C$ such that $x_n \rightharpoonup p$ and $x_n - Tx_n \to 0$. From Proposition 2 $(I - T)$ is demiclosed at zero, therefore $p \in F(T)$. $\Box$

**Theorem 3:** Let $H$ be a real Hilbert space and $C$ a nonempty closed and convex subset of $H$. Let $T : C \to C$ be a $\delta$-type nonspreading mapping. Then the following are equivalent:

(i). There exists $x \in C$ such that $\{T^n x\}_{n\geq 0}$ is bounded;
(ii). $F(T) \neq \emptyset$.

**Proof:** (ii) implies (i) is obvious since $T^n p = p$ for all n. Now let us assume (i) and prove (ii).

Suppose there exists $x \in C$ such $\{T^n x\}_{n=1}^{\infty}$ is bounded.
Then for all $y \in C$ we have

$$\begin{aligned} \|T^{m+1}x - Ty\|^2 &= \|T(T^m x) - Ty\|^2 \\ &\leq \|T^m x - y||^2 + \delta\langle T^m x - T(T^m x), y - Ty\rangle. \end{aligned}$$

Thus

$$\begin{aligned} 0 &\leq \|T^m x - y\|^2 + \delta\langle T^m x - T^{m+1}x, y - Ty\rangle \\ &\quad -\|T^{m+1}x - Ty\|^2 \\ &= \|T^m x - y\|^2 + \delta\langle T^m x - T^{m+1}x, y - Ty\rangle \\ &\quad -\Big[\|T^{m+1}x - y\|^2 + \|y - Ty\|^2 + 2\langle T^{m+1}x - y, y - Ty\rangle\Big]. \end{aligned}$$

Summing this inequality with respect to $m = 0, 1, 2, 3, ..., n-1$, we have

$$\begin{aligned} 0 &\leq \|x - y\|^2 - \|T^n x - y\|^2 - n\|y - Ty\|^2 \\ &\quad +2\langle\sum_{m=0}^{n-1} T^{m+1}x - ny, Ty - y\rangle + \delta\langle x - T^n x, y - Ty\rangle. \end{aligned}$$

Dividing this inequality by $n$, we have

$$\begin{aligned} 0 &\leq \frac{1}{n}\Big[\|x - y\|^2 - \|T^n x - y\|^2\Big] - \|y - Ty\|^2 \\ &\quad +2\langle\frac{1}{n}\sum_{m=0}^{n-1} T^{m+1}x - y, Ty - y\rangle \\ &\quad +\frac{\delta}{n}\langle x - T^n x, y - Ty\rangle \end{aligned}$$

$$= \frac{1}{n}\Big[\|x-y\|^2 - \|T^n x - y\|^2\Big] - \|y - Ty\|^2$$
$$+2\langle S_n Tx - y, Ty - y\rangle + \frac{\delta}{n}\langle x - T^n x, y - Ty\rangle,$$

where $S_n v = \frac{1}{n}\sum_{m=0}^{n-1} T^m v$, for all $v \in C$. Since $\{T^n x\}_{n=1}^{\infty}$ in bounded by assumption, $\{S_n(Tx)\}_{n=1}^{\infty}$ is also bounded. Thus we have a subsequence $\{S_{n_i}(Tx)\}_{i=1}^{\infty}$ of $\{S_n(Tx)\}_{n=1}^{\infty}$ such that $\{S_{n_i}(Tx)\}_{=1}^{\infty}$ converges weakly to $p \in C$. Hence

$$0 \leq \frac{1}{n_j}\Big[\|x-y\|^2 - \|T_j^n x - y\|^2\Big] - \|y - Ty\|^2$$
$$+ 2\langle S_{n_j} Tx - y, Ty - y\rangle + \frac{\delta}{n_j}\langle x - T_j^n x, y - Ty\rangle. \tag{13}$$

Letting $i \to \infty$ in (13) we have

$$0 \leq -\|y - Ty\|^2 + 2\langle p - y, Ty - y\rangle. \tag{14}$$

Setting $y = p$ in (14) we have

$$\|p - Tp\|^2 \leq 0.$$

Hence $p = Tp$. Therefore $F(T)$ is nonempty. This completes the proof. □

**Corollary 1:** Let $H$ be a real Hilbert space and $C$ a nonempty closed convex subset of $H$. Let $T$ be a $k$-strictly pseudononspreading mapping of $C$ into itself. Define $T_\beta : C \to C$ by $T_\beta x = \beta x + (1-\beta)Tx$. Suppose there exists $x \in C$ such that $\{T_\beta^n x\}_{n=1}^{\infty}$ is bounded. Then $F(T_\beta) = F(T) \neq \emptyset$.

**Proof:**

$$\begin{aligned}\|T_\beta x - T_\beta y\|^2 &\leq \|x-y\|^2 + \frac{2}{(1-\beta)}\langle x - T_\beta x, y - T_\beta y\rangle \\ &= \|x-y\|^2 + \delta\langle x - T_\beta x, y - T_\beta y\rangle,\end{aligned}$$

where $\delta = \frac{2}{(1-\beta)}$. Hence the result follows from Theorem 3.2. □

**Corollary 2:** Every nonempty bounded closed convex subset of a Hilbert space $H$ has the fixed point property for $k$-strictly pseudononspreading self mappings.

To Prove a common fixed point theorem we need the following Lemma.

**Lemma 4:** Let $H$ be a real Hilbert space and $C$ a nonempty bounded closed convex subset of $H$. Let $\{T_1, T_2, T_3, ..., T_N\}$ be a commutative finite family of $k$-strictly Pseudononspreading mappings of $C$ into itself. Then $\{T_1, T_2, T_3, ..., T_N\}$ has a common fixed point.

**Proof:** We prove by induction. For $N = 2$ by Proposition 1 and Corollary 2, $F(T_1)$ is nonempty bounded closed convex. It follows from $T_1T_2 = T_2T_1$ that $F(T_1)$ is $T_2$-invariant. In fact, if $p \in F(T_1)$, then we have $T_1T_2p = T_2T_1p = T_2p$.

Thus we have $T_2p \in F(T_1)$. Hence the restriction of $T_2$ to $F(T_1)$ is $k$-strictly Pseudononspreading self mapping. By Corollary 2 $T_2$ has a fixed point in $F(T_1)$, thats is we have $p \in F(T_1)$ such that $T_2p = p$. Consequently $p \in F(T_1) \bigcap F(T_2)$.

Suppose for some $n \geq 2$, $\mathbb{P} = \bigcap_{k=1}^{n} F(T_k)$ is nonempty. Then $\mathbb{P}$ is nonempty bounded closed convex subset of $C$ and the restriction $T_{n+1}$ to $\mathbb{P}$ is $k$-strictly Pseudononspreading self mapping. By Corollary 2 $T_{n+1}$ has a fixed point in $\mathbb{P}$.This shows that $\mathbb{P} \bigcap F(T_{n+1})$ is nonempty, that is $\bigcap_{k=1}^{n+1} F(T_k)$ is nonempty. This completes the proof. □

**Theorem 4:** Let $H$ be a real Hilbert space and $C$ a nonempty bounded closed convex subset of $H$. Let $\{T_\alpha\}_{\alpha \in A}$ be a commutative family of $k$-strictly Pseudononspreading mappings of $C$ into itself. Then $\{T_\alpha\}_{\alpha \in A}\}$ has a common fixed point.

**Proof:** By Lemma 1 $F(T_\alpha)$ is closed convex subset of $C$ for each $\alpha \in A$. Since $H$ is reflexive and $C$ is bounded, closed and convex, $C$ is weakly compact. Therefore to show that $\bigcap_{\alpha \in A} F(T_\alpha)$ is nonempty, it suffices to show that $\{F(T_\alpha)\}_{\alpha \in A}$ has finite intersection property. By Lemma 4 $\{F(T_\alpha)\}_{\alpha \in A}$ has this property. This completes the proof. □

**Lemma 5:** Let $H$ be a real Hilbert space, and $C$ a nonempty closed and convex subset of $H$. Let $S$ be a $\beta$-strictly pseudononspreading mapping of $C$ into itself and $T$ a $k$-strictly pseudocontractive mapping of $C$ into itself such that $F(T) \bigcap F(S) \neq \emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in $C$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n[\beta_n S x_n + (1 - \beta_n)T x_n], \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in (0,1) satisfying $0 < \alpha_n \leq 1 - \max\{\beta, k\}$, $0 \leq \beta_n \leq 1$.
Then $\lim_{n\to\infty} ||x_n - u||$ exists for all $u \in F(T) \cap F(S)$, and hence $\{x_n\}, \{Tx_n\}$ and $\{Sx_n\}$ are bounded.

**Proof:** Observe: Put $u_n = \beta_n S + (1-\beta_n)T$. We show that $\{x_n\}_{n=1}^{\infty}$ is bounded. Now for each $u \in F(T) \bigcap F(S)$

$$\begin{aligned} \|x_{n+1} - u\|^2 &= \|(1-\alpha_n)x_n + \alpha_n(\beta_n Sx_n + (1-\beta_n)Tx_n) - u\|^2 \\ &= \|(1-\alpha_n)(x_n - u) + \alpha_n(u_n x_n - u)\|^2 \\ &= (1-\alpha_n)\|x_n - u\|^2 + \alpha_n\|u_n x_n - u\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n x_n\|^2. \end{aligned} \tag{15}$$

Observe that

$$\begin{aligned} \|u_n x_n - u\|^2 &= \|\beta_n(Sx_n - Su) + (1-\beta_n)(Tx_n - Tu)\|^2 \\ &= \beta_n\|Sx_n - Su\|^2 + (1-\beta_n)\|Tx_n - Tu\|^2 \\ &\quad -\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2 \\ &\leq \beta_n[\|x_n - u\|^2 + \beta\|x_n - Sx_n\|^2] \\ &\quad +(1-\beta_n)[\|x_n - u\|^2 + k\|x_n - Tx_n\|^2] \\ &\quad -\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2 \\ &= \|x_n - u\|^2 + \beta_n\beta\|x_n - Sx_n\|^2 \\ &\quad +(1-\beta_n)k\|x_n - Tx_n\|^2 \\ &\quad -\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2. \end{aligned} \tag{16}$$

(15) and (16) imply that

$$\begin{aligned} \|x_{n+1} - u\|^2 &\leq (1-\alpha_n)\|x_n - u\|^2 + \alpha_n\|x_n - u\|^2 \\ &\quad +\alpha_n\beta_n\beta\|x_n - Sx_n\|^2 \\ &\quad +\alpha_n(1-\beta_n)k\|x_n - Tx_n\|^2 \\ &\quad -\alpha_n\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n x_n\|^2 \\ &= \|x_n - u\|^2 + \alpha_n\beta_n\beta\|x_n - Sx_n\|^2 \\ &\quad +\alpha_n(1-\beta_n)k\|x_n - Tx_n\|^2 \\ &\quad -\alpha_n\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2 \\ &\quad -\alpha_n(1-\alpha_n)\|x_n - u_n x_n\|^2. \end{aligned} \tag{17}$$

Also

$$\begin{aligned}\|x_n - u_n x_n\|^2 &= \|\beta_n Sx_n + (1-\beta_n)Tx_n - x_n\|^2\\ &= \|\beta_n Sx_n - \beta_n x_n + \beta_n x_n - x_n + (1-\beta_n)Tx_n\|^2\\ &= \|\beta_n(Sx_n - x_n) + (1-\beta_n)(Tx_n - x_n)\|^2\\ &= \beta_n\|Sx_n - x_n\| + (1-\beta_n)\|Tx_n - x_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2. \qquad (18)\end{aligned}$$

It follows from (17) and (18) that

$$\begin{aligned}\|x_{n+1} - u\|^2 &\leq \|x_n - u\|^2 + \alpha_n\beta_n\beta\|x_n - Sx_n\|^2\\ &\quad +\alpha_n(1-\beta_n)k\|x_n - Tx_n\|^2\\ &\quad -\alpha_n\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2\\ &\quad -\alpha_n(1-\alpha_n)[\beta_n\|x_n - Sx_n\|\\ &\quad +(1-\beta_n)\|x_n - Tx_n\|^2\\ &\quad -\beta_n(1-\beta_n)\|Sx_n - Tx_n\|^2]\\ &= \|x_n - u\|^2 + [\alpha_n\beta_n\beta - \alpha_n(1-\alpha_n)\beta_n]\|x_n - Sx_n\|^2\\ &\quad +[\alpha_n(1-\beta_n)k - \alpha_n(1-\alpha_n)(1-\beta_n)]\|x_n - Tx_n\|^2\\ &\quad +[\alpha_n(1-\alpha_n)\beta_n(1-\beta_n)\\ &\quad -\alpha_n\beta_n(1-\beta_n)]\|Sx_n - Tx_n\|^2\\ &= \|x_n - u\|^2 - \alpha_n\beta_n[(1-\alpha_n) - \beta]\|x_n - Sx_n\|^2\\ &\quad -\alpha_n(1-\beta_n)[(1-\alpha_n) - k]\|x_n - Tx_n\|^2\\ &\quad -\alpha_n\beta_n(1-\beta_n)[1-(1-\alpha_n)]\|Sx_n - Tx_n\|^2. \qquad (19)\\ &\leq \|x_n - u\|^2. \qquad (20)\end{aligned}$$

Therefore, from Lemma 1 and (20), we have that $\lim \|x_n - u\|$ exists so that $\{x_n\}_{n=1}^{\infty}$ is bounded and so are $\{Tx_n\}_{n=1}^{\infty}$ and $\{Sx_n\}_{n=1}^{\infty}$. $\square$

**Lemma 6:** Let $H$ be a real Hilbert space, and $C$ a nonempty closed and convex subset of $H$. Let $S$ be a $\beta$-strictly pseudononspreading mapping of $C$ into itself and $T$ a $k$-strictly pseudocontractive mapping of $C$ into itself. Let $\{x_n\}_{n=1}^{\infty}$ be a sequence in $C$ generated as follows:

$$\begin{cases} x_1 \in C,\\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n[\beta_n Sx_n + (1-\beta_n)Tx_n],\end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in $[0,1]$ satisfying the condition $0 < \alpha_n \leq 1-k$. Then

$$||x_{n+1} - Tx_{n+1}||^2 \leq ||x_n - Tx_n||^2 + D\beta_n$$

for some positive real number $D$.

**Proof:**

$$
\begin{aligned}
||x_{n+1} - Tx_{n+1}||^2 &= ||(1-\alpha_n)(x_n - Tx_{n+1}) \\
&\quad +\alpha_n(u_n x_n - Tx_{n+1})||^2 \\
&= (1-\alpha_n)||x_n - Tx_{n+1}||^2 \\
&\quad +\alpha_n||u_n x_n - Tx_{n+1}||^2 \\
&\quad -\alpha_n(1-\alpha_n)||x_n - u_n x_n||^2. \qquad (21)
\end{aligned}
$$

Observe that

$$
\begin{aligned}
||x_n - Tx_{n+1}||^2 &= ||x_n - x_{n+1} + x_{n+1} - Tx_{n+1}||^2 \\
&= ||x_n - x_{n+1}||^2 + ||x_{n+1} - Tx_{n+1}||^2 \\
&\quad +2\langle x_n - x_{n+1}, x_{n+1} - Tx_{n+1}\rangle \\
&= \alpha_n^2||x_n - u_n x_n||^2 + ||x_{n+1} - Tx_{n+1}||^2 \\
&\quad +2\alpha_n\langle x_n - u_n x_n, x_{n+1} - Tx_{n+1}\rangle \\
&= \alpha_n^2||x_n - Tx_n - \beta_n(Sx_n - Tx_n)||^2 \\
&\quad +||x_{n+1} - Tx_{n+1}||^2 \\
&\quad +2\alpha_n\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad -2\alpha_n\beta_n\langle Sx_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\leq \alpha_n^2||x_n - Tx_n||^2 \\
&\quad +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
&\quad +\alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 + ||x_{n+1} - Tx_{n+1}||^2 \\
&\quad +2\alpha_n\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad -2\alpha_n\beta_n\langle Sx_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle. \qquad (22)
\end{aligned}
$$

Also,

$$
\begin{aligned}
||u_n x_n - Tx_{n+1}||^2 &= ||Tx_n - Tx_{n+1} + \beta_n(Sx_n - Tx_n)||^2 \\
&\leq ||Tx_n - Tx_{n+1}||^2 + 2\beta_n||Tx_n - Tx_{n+1}|| \\
&\quad \times||Sx_n - Tx_n|| + \beta_n^2||Sx_n - Tx_n||^2 \\
&\leq ||x_n - x_{n+1}||^2 \\
&\quad +k||x_n - Tx_n - (x_{n+1} - Tx_{n+1})||^2 \\
&\quad +2\beta_n||Tx_n - Tx_{n+1}||||Sx_n - Tx_n|| \\
&\quad +\beta_n^2||Sx_n - Tx_n||^2 \\
&= \alpha_n^2||x_n - u_n x_n||^2 + k||x_n - Tx_n||^2 \\
&\quad -2k\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad +k||x_{n+1} - Tx_{n+1}||^2 + 2\beta_n||Tx_n - Tx_{n+1}|| \\
&\quad \times||Sx_n - Tx_n|| + \beta_n^2||Sx_n - Tx_n||^2
\end{aligned}
$$

$$
\begin{aligned}
&= \alpha_n^2||x_n - Tx_n - \beta_n(Sx_n - Tx_n)||^2 \\
&\quad +k||x_n - Tx_n||^2 + k||x_{n+1} - Tx_{n+1}||^2 \\
&\quad -2k\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad +2\beta_n||Tx_n - Tx_{n+1}||||Sx_n - Tx_n|| \\
&\quad +\beta_n^2||Sx_n - Tx_n||^2 \\
&\leq \alpha_n^2||x_n - Tx_n||^2 + \alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 \\
&\quad +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
&\quad +k||x_n - Tx_n||^2 + k||x_{n+1} - Tx_{n+1}||^2 \\
&\quad -2k\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad +2\beta_n||Tx_n - Tx_{n+1}||||Sx_n - Tx_n|| \\
&\quad +\beta_n^2||Sx_n - Tx_n||^2. \qquad (23)
\end{aligned}
$$

$$
\begin{aligned}
\|x_n - u_n x_n\|^2 &= ||x_n - Tx_n||^2 + \beta_n^2||Sx_n - Tx_n||^2 \\
&\quad -2\beta_n\langle x_n - Tx_n, Sx_n - Tx_n\rangle. \qquad (24)
\end{aligned}
$$

Thus using (22), (23) and (24) in (21) we obtain

$$
\begin{aligned}
||x_{n+1} - Tx_{n+1}||^2 &\leq (1-\alpha_n)\Big[\alpha_n^2||x_n - Tx_n||^2 \\
&\quad +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
&\quad +\alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 + ||x_{n+1} - Tx_{n+1}||^2 \\
&\quad +2\alpha_n\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad -2\alpha_n\beta_n\langle Sx_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle\Big] \\
&\quad +\alpha_n\Big[\alpha_n^2||x_n - Tx_n||^2 + \alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 \\
&\quad +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
&\quad +k||x_n - Tx_n||^2 + k||x_{n+1} - Tx_{n+1}||^2 \\
&\quad -2k\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
&\quad +2\beta_n||Tx_n - Tx_{n+1}||||Sx_n - Tx_n|| \\
&\quad +\beta_n^2||Sx_n - Tx_n||^2\Big] - \alpha_n(1-\alpha_n) \\
&\quad \times\Big[||x_n - Tx_n||^2 + \beta_n^2||Sx_n - Tx_n||^2 \\
&\quad -2\beta_n\langle x_n - Tx_n, Sx_n - Tx_n\rangle\Big].
\end{aligned}
$$

Thus

$$
\begin{aligned}
||x_{n+1} - Tx_{n+1}||^2 \leq\ & [(1-\alpha_n) + \alpha_n k]||x_{n+1} - Tx_{n+1}||^2 \\
& +\alpha_n[(1-\alpha_n)\alpha_n + \alpha_n^2 + k - (1-\alpha_n)] \\
& \times||x_n - Tx_n||^2 + [2(1-\alpha_n)\alpha_n - 2\alpha_n k] \\
& \times\langle x_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle + [2(1-\alpha_n) \\
& \times\alpha_n^2\beta_n + 2\alpha_n^3\beta_n]||x_n - Tx_n||||Sx_n - Tx_n|| \\
& +[(1-\alpha_n)\alpha_n^2\beta_n^2 + \alpha_n^3\beta_n^2 + \alpha_n\beta_n^2 \\
& -\alpha_n(1-\alpha_n)\beta_n^2]||Sx_n - Tx_n||^2 \\
& -2\alpha_n\beta_n(1-\alpha_n)\langle Sx_n - Tx_n, x_{n+1} - Tx_{n+1}\rangle \\
& +2\alpha_n(1-\alpha_n)\beta_n\langle x_n - Tx_n, Sx_n - Tx_n\rangle \\
& +2\alpha_n\beta_n||Tx_n - Tx_{n+1}||||Sx_n - Tx_n|| \\
\leq\ & (1-\alpha_n^2)||x_{n+1} - Tx_{n+1}||^2 + \alpha_n^2||x_n - Tx_n||^2 \\
& +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
& +2\alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 + 2\alpha_n\beta_n(1-\alpha_n) \\
& \times\langle x_n - Tx_n - (x_{n+1} - Tx_{n+1}), Sx_n - Tx_n\rangle \\
& +2\alpha_n\beta_n||Tx_n - Tx_{n+1}||||Sx_n - Tx_n|| \\
\leq\ & (1-\alpha_n^2)||x_{n+1} - Tx_{n+1}||^2 + \alpha_n^2||x_n - Tx_n||^2 \\
& +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
& +2\alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 + 2\alpha_n\beta_n(1-\alpha_n) \\
& \times||x_n - x_{n+1} - (Tx_n - Tx_{n+1})||||Sx_n \\
& -Tx_n|| + 2\alpha_n\beta_n L||x_n - x_{n+1}||||Sx_n - Tx_n|| \\
\leq\ & (1-\alpha_n^2)||x_{n+1} - Tx_{n+1}||^2 + \alpha_n^2||x_n - Tx_n||^2 \\
& +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
& +2\alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 + 2\alpha_n\beta_n(1-\alpha_n) \\
& \times(1+L)||x_{n+1} - x_n||||Sx_n - Tx_n|| \\
& +2\alpha_n^2\beta_n L||x_n - u_n x_n||||Sx_n - Tx_n|| \\
\leq\ & (1-\alpha_n^2)||x_{n+1} - Tx_{n+1}||^2 + \alpha_n^2||x_n - Tx_n||^2 \\
& +2\alpha_n^2\beta_n||x_n - Tx_n||||Sx_n - Tx_n|| \\
& +2\alpha_n^2\beta_n^2||Sx_n - Tx_n||^2 + 2\alpha_n^2\beta_n(1-\alpha_n) \\
& \times(1+L)||x_n - u_n x_n||||Sx_n - Tx_n|| \\
& +2\alpha_n^2\beta_n L||x_n - u_n x_n||||Sx_n - Tx_n||.
\end{aligned}
$$

Hence

$$
\begin{aligned}
||x_{n+1}-Tx_{n+1}||^2 &\leq ||x_n-Tx_n||^2 \\
&\quad +2\beta_n||x_n-Tx_n||||Sx_n-Tx_n|| \\
&\quad +2\beta_n^2||Sx_n-Tx_n||^2+\big[2\beta_n(1-\alpha_n)(1+L) \\
&\quad +2\beta_nL\big]||x_n-u_nx_n||||Sx_n-Tx_n|| \\
&= ||x_n-Tx_n||^2+\beta_n\big[2||x_n-Tx_n|| \\
&\quad \times||Sx_n-Tx_n||+2\beta_n||Sx_n-Tx_n||^2 \\
&\quad +[2(1-\alpha_n)(1+L)+2L] \\
&\quad \times||x_n-u_nx_n||||Sx_n-Tx_n||\big] \\
&\leq ||x_n-Tx_n||^2+D\beta_n. \ \square
\end{aligned}
$$

**Theorem 5:** Let $H$ be a real Hilbert space, and $C$ a nonempty closed and convex subset of $H$. Let $S$ be a $\beta$-strictly pseudononspreading mapping of $C$ into itself and $T$ a $k$-strictly pseudocontractive mapping of $C$ into itself such that $F(T)\bigcap F(S)\neq\emptyset$. Define a sequence $\{x_n\}_{n=1}^{\infty}$ in $C$ as follows:

$$
\begin{cases}
x_1\in C, \\
x_{n+1}=(1-\alpha_n)x_n+\alpha_n[\beta_nSx_n+(1-\beta_n)Tx_n],
\end{cases}
$$

for all $n\in\mathbb{N}$, where $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ are sequences in (0,1) satisfying $0<\alpha_n\leq 1-\max\{\beta,k\}$, $0\leq\beta_n\leq 1$.
Then the following hold:
(i) If $\liminf\limits_{n\to\infty}\alpha_n(1-\alpha_n-\beta)>0$, $\liminf\limits_{n\to\infty}\beta_n>0$, then $\{x_n\}$ converges weakly to $p\in F(S)$.
(ii)If $\sum_{n=1}^{\infty}\alpha_n(1-\alpha_n-k)=\infty$, $\sum_{n=1}^{\infty}\beta_n<\infty$, then $\{x_n\}$ converges weakly to $p\in F(T)$.
(iii) If $\liminf\limits_{n\to\infty}\alpha_n(1-\alpha_n-\max\{\beta,k\})>0$, $\liminf\limits_{n\to\infty}\beta_n(1-\beta_n)>0$, then $\{x_n\}$ converges weakly to $p\in F(T)\bigcap F(S)$.

**Proof:** Since $0<\alpha_n\leq 1-\max\{\beta,k\}$, for all $n\in N$ then, we have from (19) that

$$\alpha_n\beta_n[(1-\alpha_n)-\beta]\|x_n-Sx_n\|^2\leq\|x_n-u\|^2-\|x_{n+1}-u\|^2.$$

Hence, $\sum\alpha_n\beta_n[(1-\alpha_n)-\beta]\|x_n-Sx_n\|^2<\infty$. Since $\liminf\alpha_n(1-\alpha_n-\beta)>0$, $\liminf\beta_n>0$ we have that $\lim\|x_n-Sx_n\|=0$. But $\{x_n\}_{n=1}^{\infty}$ is bounded therefore there exists a subsequence $\{x_{n_j}\}\subseteq\{x_n\}$ such that $\{x_{n_j}\}$ converges weakly $p$. Also, $\lim\|x_n-Sx_n\|=0$ implies that $\lim\|x_{n_j}-Sx_{n_j}\|=0$. From Lemma 2, $(I-S)$ is demiclosed at zero and hence we obtain $p\in F(S)$. To show our

conclusion, it suffices to show that for another subsequence $\{x_{n_i}\} \subseteq \{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to $q \in F(S)$, $p = q$.
Suppose $p \neq q$ we have from Opial's Theorem [13] that

$$\begin{aligned}\lim_{n\to\infty} \|x_n - p\| &= \lim_{j\to\infty} \|x_{n_j} - p\| < \lim_{j\to\infty} \|x_{n_j} - q\| \\ &= \lim_{n\to\infty} \|x_n - q\| = \lim_{i\to\infty} \|x_{n_i} - q\| \\ &< \lim_{i\to\infty} \|x_{n_i} - p\| = \lim_{n\to\infty} \|x_n - p\|.\end{aligned}$$

This is a contradiction. Therefore, $\{x_n\}$ converges weakly to $p \in F(S)$.
(ii). From (19) we have

$$\alpha_n(1-\beta_n)[(1-\alpha_n)-k]\|x_n - Tx_n\|^2 \leq \|x_n - u\|^2 - \|x_{n+1} - u\|^2.$$

Since $\lim_{n\to\infty}(1-\beta_n) = 1$, then there exists a positive integer $N$ such that $(1-\beta_n) \geq \frac{1}{2}\ \forall\ n \geq N$. Therefore $\frac{1}{2}\sum \alpha_n[(1-\alpha_n) - k]\|x_n - Tx_n\|^2 < \infty$. Since $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n-k) = \infty$, we have that $\liminf_{n\to\infty} \|x_n - Tx_n\| = 0$. It now follows from Lemma 5 that $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$. Using Lemma 3 and argument similar to the proof of (i) the conclusion follows.
(iii). Since $\liminf \alpha_n(1-\alpha_n-\max\{\beta,k\}) > 0$, $\liminf \beta_n(1-\beta_n) > 0$, we have from (19), (20), Lemma 2 and Lemma 3 that there exists a subsequence $\{x_{n_j}\} \subseteq \{x_n\}$ such that $\{x_{n_j}\}$ converges weakly $p \in F(S)\bigcap F(T)$. Using argument similar to ones in the proofs of (i) and (ii), the conclusion follows. □

As direct consequences of Theorem 3, we get the followings.

**Corollary 3:** Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $S$ be a $\beta$-strictly pseudononspreading mapping of $C$ into itself such that $F(S) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n S x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset [0,\ 1]$. If $\liminf \alpha_n(1-\alpha_n-\beta) > 0$, Then $\{x_n\}$ converges weakly to $p \in F(S)$.

**Proof:** Putting $\beta_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3, we get the conclusion. □

**Corollary 4:** Let $H$ be a real Hilbert space and let $C$ be a nonempty closed convex subset of $H$. Let $T$ be a $k$-strictly pseudocontractive mapping of $C$ into itself such that $F(T) \neq \emptyset$. Define a sequence $\{x_n\}$ as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\} \subset[0,1]$. If $0 \leq \alpha_n \leq 1-k$ and $\sum_{n=1}^{\infty} \alpha_n(1-\alpha_n - k) = \infty$, then $\{x_n\}$ converges weakly to $p \in F(T)$.

**Proof:** Putting $\beta_n = 0$ for all $n \in \mathbb{N}$ in Theorem 5, we get the conclusion. □

## 4. CONCLUDING REMARKS

**Remark 1:** Theorem 1 follows as a simple corollary of our Theorem 3 since every nonexpansive mapping is a special case of $k$-strictly pseudocontractive mapping for which $k = 0$ and every nonspreading mapping is a special case of $\beta$-strictly pseudononpreading mapping for which $\beta = 0$.

Corollary 4 is Theorem 3 of a popular result of Marino and Xu [3].

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