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A SEVENTH–ORDER COMPUTATIONAL ALGORITHM FOR THE SOLUTION OF STIFF SYSTEMS OF DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we present a computationally cheap second derivative block hybrid method for the numerical solution of systems of stiff initial valued ordinary differential equations. Results of numerical experiments which validates our theoretical results are presented by figures and tables.

1. INTRODUCTION

A stiff system of differential equations refers to a set of differential equations in which the solution changes rapidly over a specific region of the problem domain. Such systems often arise in various scientific and engineering fields, including physics, chemistry, and mechanics. Solving stiff systems of differential equations accurately and efficiently is a challenging task, requiring the use of specialized numerical methods.

In recent years, researchers have developed a variety of numerical methods to tackle the difficulties associated with stiff systems of differential equations. One such method is the Seventh Order Second Derivative Block Hybrid Method, which combines the advantages of higher-order accuracy with the simplicity and efficiency of block methods. This method has shown promising results in solving stiff systems by providing accurate solutions while reducing computational cost.

Several researches have focused on the implementation and numerical algorithms of A-stable second derivative block hybrid method. Ramos et al, [1] proposed a new 2-step hybrid block method with 10th order convergence for numerical integration of initial value systems. Their method is A-stable and outperforms similar methods in solving such

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problems, as shown by numerical experiments. Besides, Skwame et al., [2] introduced a new A-stable backward difference second derivative linear multistep method for solving stiff ordinary differential equations. The method was derived using power series as basis functions and involves multi-step interpolation and collocation techniques. Their analysis confirmed that the method is consistent, zero-stable, and has a uniform order of twelve. Additionally, the region of absolute stability was plotted, demonstrating the A-stability of the method. Singh et al, [3] introduced an optimized hybrid block method for numerical integration of initial value ordinary differential systems. The method bypassed the first Dahlquist's barrier on linear multi-step methods. The scheme achieved fifth-order accuracy and A-stability by optimizing off-grid points to minimize local truncation errors.

Akinfenwa et al., [4] proposed a hybrid second derivative three-step method of order seven for solving first order stiff differential equations. The hybrid second derivative block backward differentiation formula is concurrently applied to the first order stiff systems to generate the numerical solution that do not coincide in time over a given interval.

Although there is no paucity of research on this method, there is however, further potential areas of improvement regarding the A-stable second derivative block hybrid method including investigating adaptive step size selection strategies and higher-order accuracy. The lack of such stability properties makes the continuous solution not reliable Yakubu et al., [5]. Hence, the need for a modification that involves incorporating the A-stable second derivative of the solution into the integration scheme, resulting in the enhanced stability property. This study builds on the existing works to derive block hybrid methods that show a high order of accuracy with very low error constants. In all the above mentioned literatures, our proposed method uses different off-grid points which were well selected for better accuracy.

2. THE HEART OF THE MATTER

In this section, we show how the new method is derived using an interpolation and collocation approach, presented a result with proof which shows a condition on the step size h under which the corresponding matrix is nonsingular as well as the final presentation of the new second derivative block hybrid method. The general form of a k-step second derivative block hybrid method for the numerical solution of a system of ordinary differential equation is [5]

$$y(x)=\sum_{j=0}^{k}\alpha_j y_{n+j}+h\sum_{j=0}^{k}\beta_j f_{n+j}+h^2\sum_{j=0}^{k}\beta_j g_{n+j}, \tag{2.1}$$

where the $\alpha_j' s, \beta_j' s$ and γ_j's are unknown coefficients to be found,

$$y_{n+j}=y(x_n+jh),$$

is the numerical approximation to the exact solution

$$y^{'}(x_{n+j})=f_{n+j}=f(x_n+jh,y(x_n+jh)),$$

and

$$y^{''}(x_{n+j})=f_{n+j}^{'}=f^{'}(x_n+jh,y(x_n+jh))=g_{n+j}.$$

In addition,

$$\alpha_j(x)=\sum_{j=0}^{k}\alpha_{j,j+1}x^j, \quad \text{for} \quad j=0,1,\cdots,k-1,$$

$$\beta_j(x)=\sum_{j=0}^{k}\beta_{j,j+1}x^j, \quad \text{for} \quad j=0,1,\cdots,k-1,$$

and

$$\gamma_j(x)=\sum_{j=0}^{k}\gamma_{j,j+1}x^j, \quad \text{for} \quad j=0,1,\cdots,k-1.$$

To get $\alpha_j(x),\beta_j(x)$ and $\gamma_j(x)$, Sirisena et al (2004) arrived at a matrix equation of the form

$$DC=I,$$

where I is an identity matrix of dimension $(t+m)\times(t+m)$, D and C are of the same dimensions

$$D=\begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & \cdots & x_n^{t+m-1}\\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \cdots & x_{n+1}^{t+m-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
1 & x_{n+t-1} & x_{n+t-1}^2 & x_{n+t-1}^3 & x_{n+t-1}^4 & \cdots & x_{n+t-1}^{t+m-1}\\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots & (t+m-1)x_n^{t+m+2}\\
0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & \cdots & (t+m-1)x_{n+1}^{t+m-2}\\
\vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots\\
0 & 1 & 2x_{n+t-1} & 3x_{n+t-1}^2 & 4x_{n+t-1}^3 & \cdots & (t+m-1)x_{n+t-1}^{t+m-2}\\
0 & 0 & 2 & 6x_n & 12x_n^2 & \cdots & (t+m-1)(t+2-m)x_n^{t+m-3}\\
0 & 0 & 2 & 6x_{n+1} & 12x_{n+1}^2 & \cdots & (t+m-1)(t+m-2)x_{n+1}^{t+m-3}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots\\
0 & 0 & 2 & 6x_{n+t-1} & 12x_{n+t-1}^2 & \cdots & (t+m-1)(t+m-2)x_{n+t-1}^{t+m-3}
\end{bmatrix}. \tag{2.2}$$

The matrix (2.2) is the multi-step collocation matrix of dimension $(t+m)\times(t+m)$. For C we also define a matrix of dimension $(t+m)\times(t+m)$ whose columns gives the continuous coefficient as:

$$C=\begin{bmatrix}
\alpha_{0,1} & \cdots & \alpha_{t-1,1} & . & h\beta_{0,1}\cdots h\beta_{m-1,1} & . & h^2\gamma_{0,1} & \cdots & h^2\gamma_{m-1,1}\\
\alpha_{0,2} & \cdots & \alpha_{t-1,2} & . & h\beta_{0,2}\cdots h\beta_{m-1,2} & . & h^2\gamma_{0,2} & \cdots & h^2\gamma_{m-1,2}\\
. & . & ; & . & . & . & . & . & \\
. & . & . & . & . & . & . & . & \\
. & & . & . & . & . & . & . & \\
\alpha_{0,j+m} & \cdots & \alpha_{t-1,j+m} & . & h\beta_{0,j+m}\cdots h\beta_{m-1,j+m} & . & h^2\gamma_{0,j+m} & \cdots & h^2\gamma_{m-1,j+m}
\end{bmatrix}, \tag{2.3}$$

where t is the number of interpolation points while m is the number of collocation points used respectively. The columns of the matrix
$C=D^{-1}$ gives the continuous coefficients $\alpha_j(x); j=0,1\cdots,k-1$, $\beta_j(x); j=0,1,\cdots,k-1$ and $\gamma_j(x); j=0,1,\cdots,k-1$.

Thus, the matrix D in (2.2) in our case becomes

$$\mathbf{D} = \begin{bmatrix} 1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & x_n^7 & x_n^8 \\ 0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & 7x_n^6 & 8x_n^7 \\ 0 & 1 & 2x_{n+\frac{1}{2}} & 3x^2_{n+\frac{1}{2}} & 4x^3_{n+\frac{1}{2}} & 5x^4_{n+\frac{1}{2}} & 6x^5_{n+\frac{1}{2}} & 7x^6_{n+\frac{1}{2}} & 8x^7_{n+\frac{1}{2}} \\ 0 & 1 & 2x_{n+1} & 3x^2_{n+1} & 4x^3_{n+1} & 5x^4_{n+1} & 6x^5_{n+1} & 7x^6_{n+1} & 8x^7_{n+1} \\ 0 & 1 & 2x_{n+\frac{3}{2}} & 3x^2_{n+\frac{3}{2}} & 4x^3_{n+\frac{3}{2}} & 5x^4_{n+\frac{3}{2}} & 6x^5_{n+\frac{3}{2}} & 7x^6_{n+\frac{3}{2}} & 8x^7_{n+\frac{3}{2}} \\ 0 & 1 & 2x_{n+2} & 3x^2_{n+2} & 4x^3_{n+2} & 5x^4_{n+2} & 6x^5_{n+2} & 7x^6_{n+2} & 8x^7_{n+2} \\ 0 & 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 & 30x_n^4 & 42x_n^5 & 56x_n^6 \\ 0 & 0 & 2 & 6x_{n+\frac{1}{2}} & 12x^2_{n+\frac{1}{2}} & 20x^3_{n+\frac{1}{2}} & 30x^4_{n+\frac{1}{2}} & 42x^5_{n+\frac{1}{2}} & 56x^6_{n+\frac{1}{2}} \\ 0 & 0 & 2 & 6x_{n+1} & 12x^2_{n+1} & 20x^3_{n+1} & 30x^4_{n+1} & 42x^5_{n+1} & 56x^6_{n+1} \end{bmatrix}.$$

By replacing $x_{n+\frac{1}{2}}$ with $x_{n+1} - \frac{h}{2}$, $x_n = x_{n+1} - h$, $x_{n+\frac{3}{2}} = x_{n+1} + \frac{h}{2}$ and $x_{n+2} = x_{n+1} + h$, we obtained the determinant of D as

$$\det D = -\frac{25515h^{25}}{64}.$$

This now leads to the following result.

Theorem 1: Given that $\varepsilon > 0$, then

$$h > \left[\frac{64\varepsilon}{25515}\right]^{\frac{1}{25}}.$$

If in addition $\varepsilon = 2^{-52}$, $h > 0.1861$, then the D matrix is nonsingular.

Proof: Since by assumption $\varepsilon > 0$, then let

$$\frac{25515h^{25}}{64} > \varepsilon.$$

This implies

$$h^{25} > \frac{64\varepsilon}{25515}, \quad \text{and} \quad h > \left[\frac{64\varepsilon}{25515}\right]^{\frac{1}{25}}.$$

Now, substituting $\varepsilon = 2^{-52}$ yields

$$h > \left[\frac{64\varepsilon}{25515}\right]^{\frac{1}{25}} = \left[\frac{64 \times 2^{-52}}{25515}\right]^{\frac{1}{25}} = 0.1861.$$

Therefore, if $h > 0.1861$ then the determinant of D will be nonzero, ensuring the nonsingularity of D. $\square$

We state categorically here that the above result is purely theoretical and does not influence the choice of step sizes in actual computations as shown in [6].
In our case, equation (2.1) the continuous formulation is

$$\begin{aligned} y(x) &= \alpha_0 y_n + h\sum_{j=0}^{5}\beta_j f_{n+j} + h^2\sum_{j=0}^{2}\beta_j g_{n+j} \\ &= \alpha_0 y_n + h\left[\beta_0 f_n + \beta_{\frac{1}{2}} f_{n+\frac{1}{2}} + \beta_1 f_{n+1} + \beta_{\frac{3}{2}} f_{n+\frac{3}{2}} + \beta_2 f_{n+2}\right] \\ &\quad + h^2\left[\gamma_0 g_n + \gamma_{\frac{1}{2}} g_{n+\frac{1}{2}} + \gamma_1 g_{n+1}\right]. \end{aligned} \tag{2.4}$$

Observe that since we have proved that D is nonsingular under a certain condition on h, we are now at liberty to find the inverse. Upon inverting D and replacing a with $x_n + h$, the elements of the first row constitutes the following continuous coefficients:

$$\begin{aligned}
\alpha_0 &= 1, \\
\beta_0 &= \frac{759h^8 - 1715a^3h^5 + 2415a^4h^4 + 5271a^5h^3 - 7945a^6h^2 - 3300a^7h + 4515a^8}{3780h^7}, \\
\beta_{\frac{1}{2}} &= \frac{466h^8 - 3080a^3h^5 - 210a^4h^4 + 7224a^5h^3 - 2660a^6h^2 - 3840a^7h + 2100a^8}{945h^7}, \\
\beta_1 &= -\frac{-3h^8 + 10ah^7 - 40a^3h^5 + 15a^4h^4 + 78a^5h^3 - 50a^6h^2 - 40a^7h + 30a^8}{10h^7}, \\
\beta_{\frac{3}{2}} &= -\frac{-6h^8 + 280a^3h^5 - 1050a^4h^4 + 1176a^5h^3 + 140a^6h^2 - 960a^7h + 420a^8}{945h^7}, \\
\beta_2 &= \frac{-h^8 + 35a^3h^5 - 105a^4h^4 + 21a^5h^3 + 245a^6h^2 - 300a^7h + 105a^8}{3780h^7}, \\
\gamma_0 &= \frac{3h^8 - 14a^3h^5 + 21a^4h^4 + 42a^5h^3 - 70a^6h^2 - 24a^7h + 42a^8}{252h^6}, \\
\gamma_{\frac{1}{2}} &= \frac{-2h^8 - 56a^3h^5 + 42a^4h^4 + 168a^5h^3 - 140a^6h^2 - 96a^7h + 84a^8}{63h^6}, \\
\gamma_1 &= \frac{-h^8 + 14a^2h^6 - 28a^3h^5 - 21a^4h^4 + 84a^5h^3 - 28a^6h^2 - 48a^7h + 28a^8}{28h^6}.
\end{aligned}$$

The continuous formulation becomes

$$
\begin{aligned}
y(x) = y_n &+ \frac{\left[759h^8 - 1715a^3h^5 + 2415a^4h^4 + 5271a^5h^3 - 7945a^6h^2 - 3300a^7h + 4515a^8\right]}{3780h^7} f_n \\
&+ \frac{\left[466h^8 - 3080a^3h^5 - 210a^4h^4 + 7224a^5h^3 - 2660a^6h^2 - 3840a^7h + 2100a^8\right]}{945h^7} f_{n+\frac{1}{2}} \\
&- \frac{\left[-3h^8 + 10ah^7 - 40a^3h^5 + 15a^4h^4 + 78a^5h^3 - 50a^6h^2 - 40a^7h + 30a^8\right]}{10h^7} f_{n+1} \\
&- \frac{\left[-6h^8 + 280a^3h^5 - 1050a^4h^4 + 1176a^5h^3 + 140a^6h^2 - 960a^7h + 420a^8\right]}{945h^7} f_{n+\frac{3}{2}} \qquad (2.5) \\
&+ \frac{\left[-h^8 + 35a^3h^5 - 105a^4h^4 + 21a^5h^3 + 245a^6h^2 - 300a^7h + 105a^8\right]}{3780h^7} f_{n+2} \\
&+ \frac{\left[3h^8 - 14a^3h^5 + 21a^4h^4 + 42a^5h^3 - 70a^6h^2 - 24a^7h + 42a^8\right]}{252h^6} g_n \\
&+ \frac{\left[-2h^8 - 56a^3h^5 + 42a^4h^4 + 168a^5h^3 - 140a^6h^2 - 96a^7h + 84a^8\right]}{63h^6} g_{n+\frac{1}{2}} \\
&+ \frac{\left[-h^8 + 14a^2h^6 - 28a^3h^5 - 21a^4h^4 + 84a^5h^3 - 28a^6h^2 - 48a^7h + 28a^8\right]}{28h^6} g_{n+1}.
\end{aligned}
$$

If we substitute $a = 0$ into the continuous formulation above, then we have

$$
y_{n+1} = y_n + \frac{h\left[759f_n + 1864f_{n+\frac{1}{2}} + 1134f_{n+1} + 24f_{n+\frac{3}{2}} - f_{n+2}\right] + h^2\left[45g_n - 120g_{n+\frac{1}{2}} - 135g_{n+1}\right]}{3780}.
$$

Substituting $a = \frac{h}{2}$ into the continuous scheme above, then

$$
\begin{aligned}
y_{n+\frac{1}{2}} = y_n &+ \frac{h\left[186367f_n + 235792f_{n+\frac{1}{2}} + 58212f_{n+1} + 3632f_{n+\frac{3}{2}} - 163f_{n+2}\right]}{967680} \\
&+ \frac{h^2\left[10590g_n - 57360g_{n+\frac{1}{2}} - 13500g_{n+1}\right]}{967680}.
\end{aligned}
$$

Evaluate the continuous scheme at $a = -h/2$ to obtain the scheme

$$
\begin{aligned}
y_{n+\frac{3}{2}} = y_n &+ \frac{h\left[8333f_n + 23088f_{n+\frac{1}{2}} + 17388f_{n+1} + 5008f_{n+\frac{3}{2}} - 57f_{n+2}\right]}{35840} \\
&+ \frac{h^2\left[570g_n + 720g_{n+\frac{1}{2}} + 2700g_{n+1}\right]}{35840}.
\end{aligned}
$$

Evaluating the continuous scheme at $a = -h$ gives the discrete scheme

$$
\begin{aligned}
y_{n+2} = y_n &+ \frac{h\left[-128f_n + 1512f_{n+1} - 608f_{n+\frac{1}{2}} + 992f_{n+\frac{3}{2}} + 122f_{n+2}\right]}{945} \\
&+ \frac{h^2\left[-30g_n - 480g_{n+\frac{1}{2}} - 540g_{n+1}\right]}{945}.
\end{aligned}
$$

The seventh-order second derivative block hybrid method is now summarised below

$$
\begin{aligned}
y_{n+1} &= y_n + \frac{h\left[759f_n + 1864f_{n+\frac{1}{2}} + 1134f_{n+1} + 24f_{n+\frac{3}{2}} - f_{n+2}\right] + h^2\left[45g_n - 120g_{n+\frac{1}{2}} - 135g_{n+1}\right]}{3780}\\
y_{n+\frac{1}{2}} &= y_n + \frac{h\left[186367f_n + 235792f_{n+\frac{1}{2}} + 58212f_{n+1} + 3632f_{n+\frac{3}{2}} - 163f_{n+2}\right]}{967680}\\
&\quad + \frac{h^2\left[10590g_n - 57360g_{n+\frac{1}{2}} - 13500g_{n+1}\right]}{967680}\\
y_{n+\frac{3}{2}} &= y_n + \frac{h\left[8333f_n + 23088f_{n+\frac{1}{2}} + 17388f_{n+1} + 5008f_{n+\frac{3}{2}} - 57f_{n+2}\right]}{35840}\\
&\quad + \frac{h^2\left[570g_n + 720g_{n+\frac{1}{2}} + 2700g_{n+1}\right]}{35840}\\
y_{n+2} &= y_n + \frac{h\left[-128f_n + 1512f_{n+1} - 608f_{n+\frac{1}{2}} + 992f_{n+\frac{3}{2}} + 122f_{n+2}\right]}{945}\\
&\quad + \frac{h^2\left[-30g_n - 480g_{n+\frac{1}{2}} - 540g_{n+1}\right]}{945}.
\end{aligned}
\tag{2.6}
$$

2.1. **Convergence Analysis.** The crux of the matter in this section is to examine properties of the new second derivative block hybrid method where we calculated the order, examined its zero stability, consistency and convergence. method. In addition, we present the seventh order Newton based algorithm for the solution of linear and nonlinear stiff systems of differential equations. Since Newtons' method relies on the nonsingularity of the Jacobian matrix, we state a theorem with proof that the Jacobian obtained from the new method is nonsingular at the root using elementary row operations. While the results holds at the root, it is also holds when not at root.

To find the order of the new method, we use the second derivative block hybrid method (2.6) above to obtain

$$
\alpha_0 = -\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \quad \alpha_1 = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \quad \alpha_{\frac{1}{2}} = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \quad \alpha_{\frac{3}{2}} = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \quad \alpha_2 = \begin{bmatrix}0\\0\\0\\1\end{bmatrix};
$$

and

$$\beta_0 = \begin{bmatrix} \frac{253}{1260} \\ \frac{186367}{967680} \\ \frac{8333}{35840} \\ -\frac{128}{945} \end{bmatrix}, \quad \beta_1 = \begin{bmatrix} \frac{3}{10} \\ \frac{77}{1280} \\ \frac{621}{1280} \\ \frac{8}{5} \end{bmatrix}, \quad \beta_{\frac{1}{2}} = \begin{bmatrix} \frac{466}{945} \\ \frac{14737}{60480} \\ \frac{1443}{2240} \\ -\frac{608}{945} \end{bmatrix}, \quad \beta_{\frac{3}{2}} = \begin{bmatrix} \frac{2}{315} \\ \frac{227}{60480} \\ \frac{313}{2240} \\ \frac{992}{945} \end{bmatrix}, \quad \beta_2 = \begin{bmatrix} -\frac{1}{3780} \\ -\frac{163}{967680} \\ -\frac{57}{35840} \\ \frac{122}{945} \end{bmatrix};$$

and the second derivative countinuous coefficient are:

$$\gamma_0 = \begin{bmatrix} \frac{1}{84} \\ \frac{353}{32256} \\ \frac{57}{3584} \\ -\frac{2}{63} \end{bmatrix}, \quad \gamma_{\frac{1}{2}} = \begin{bmatrix} -\frac{2}{63} \\ -\frac{239}{4032} \\ \frac{9}{448} \\ -\frac{32}{63} \end{bmatrix}, \quad \gamma_1 = \begin{bmatrix} -\frac{1}{28} \\ -\frac{25}{1792} \\ \frac{135}{1792} \\ -\frac{4}{7} \end{bmatrix}.$$

To obtain the order, we substitute the above continuous coefficients into the following and after routine algebraic substitutions

$$C_0 = \alpha_0 + \alpha_1 + \alpha_{\frac{3}{2}} + \alpha_2 + \alpha_{\frac{1}{2}} = \mathbf{0};$$

$$C_1 = \left[\alpha_1 + \left(\tfrac{3}{2}\right)\alpha_{\frac{3}{2}} + 2\alpha_2 + \left(\tfrac{1}{2}\right)\alpha_{\frac{1}{2}}\right] - (\beta_0 + \beta_1 + \beta_{\frac{3}{2}} + \beta_2 + \beta_{\frac{1}{2}}) = \mathbf{0};$$

$$C_2 = \frac{1}{2!}\left[\alpha_1 + \left(\frac{1}{2}\right)^2 \alpha_{\frac{1}{2}} + \left(\frac{3}{2}\right)^2 \alpha_{\frac{3}{2}} + 2^2\alpha_2\right] - \left[\beta_1 + \left(\frac{3}{2}\right)\beta_{\frac{3}{2}} + (2)\beta_2 + \left(\frac{1}{2}\right)\beta_{\frac{1}{2}}\right] - (\gamma_0 + \gamma_{\frac{1}{2}} + \gamma_1) = \mathbf{0};$$

$$C_3 = \frac{1}{3!}\left[\alpha_1 + \left(\frac{3}{2}\right)^3 \alpha_{\frac{3}{2}} + 2^3\alpha_2 + \left(\frac{1}{2}\right)^3 \alpha_{\frac{1}{2}}\right] - \left(\frac{1}{2}\right)\left[\beta_1 + \left(\frac{3}{2}\right)^2 \beta_{\frac{3}{2}} + (2)^2\beta_2 + \left(\frac{1}{2}\right)^2 \beta_{\frac{1}{2}}\right] - \left[\left(\frac{1}{2}\right)\gamma_{\frac{1}{2}} + \gamma_1\right] = \mathbf{0};$$

$$C_4 = \frac{1}{4!}\left[\alpha_1 + \left(\frac{3}{2}\right)^4 \alpha_{\frac{3}{2}} + 2^4\alpha_2 + \left(\frac{1}{2}\right)^4 \alpha_{\frac{1}{2}}\right] - \frac{1}{3!}\left[\beta_1 + \left(\frac{3}{2}\right)^3 \beta_{\frac{3}{2}} + (2)^3\beta_2 + \left(\frac{1}{2}\right)^3 \beta_{\frac{1}{2}}\right] - \frac{1}{2!}\left[\left(\frac{1}{2}\right)^2 \gamma_{\frac{1}{2}} + \gamma_1\right] = \mathbf{0};$$

$$C_5 = \frac{1}{5!}\left[\alpha_1 + \left(\frac{3}{2}\right)^5 \alpha_{\frac{3}{2}} + 2^5\alpha_2 + \left(\frac{1}{2}\right)^5 \alpha_{\frac{1}{2}}\right] - \frac{1}{4!}\left[\beta_1 + \left(\frac{3}{2}\right)^4 \beta_{\frac{3}{2}} + (2)^4\beta_2 + \left(\frac{1}{2}\right)^4 \beta_{\frac{1}{2}}\right] - \frac{1}{3!}\left[\left(\frac{1}{2}\right)^3 \gamma_{\frac{1}{2}} + \gamma_1\right] = \mathbf{0};$$

$$C_6 = \frac{1}{6!}\left[\alpha_1 + \left(\frac{3}{2}\right)^6 \alpha_{\frac{3}{2}} + 2^6\alpha_2 + \left(\frac{1}{2}\right)^6 \alpha_{\frac{1}{2}}\right] - \frac{1}{5!}\left[\beta_1 + \left(\frac{3}{2}\right)^5 \beta_{\frac{3}{2}} + (2)^5\beta_2 + \left(\frac{1}{2}\right)^5 \beta_{\frac{1}{2}}\right]$$
$$- \frac{1}{4!}\left[\left(\frac{1}{2}\right)^4 \gamma_{\frac{1}{2}} + \gamma_1\right] = \mathbf{0};$$

$$C_7 = \frac{1}{7!}\left[\alpha_1 + \left(\frac{3}{2}\right)^7 \alpha_{\frac{3}{2}} + 2^7\alpha_2 + \left(\frac{1}{2}\right)^7 \alpha_{\frac{1}{2}}\right] - \frac{1}{6!}\left[\beta_1 + \left(\frac{3}{2}\right)^6 \beta_{\frac{3}{2}} + (2)^6\beta_2 + \left(\frac{1}{2}\right)^6 \beta_{\frac{1}{2}}\right]$$
$$- \frac{1}{5!}\left[\left(\frac{1}{2}\right)^5 \gamma_{\frac{1}{2}} + \gamma_1\right] = \mathbf{0};$$

$$C_8 = \frac{1}{8!}\left[\alpha_1 + \left(\frac{3}{2}\right)^8 \alpha_{\frac{3}{2}} + 2^8\alpha_2 + \left(\frac{1}{2}\right)^8 \alpha_{\frac{1}{2}}\right] - \frac{1}{7!}\left[\beta_1 + \left(\frac{3}{2}\right)^7 \beta_{\frac{3}{2}} + (2)^7\beta_2 + \left(\frac{1}{2}\right)^7 \beta_{\frac{1}{2}}\right]$$
$$- \frac{1}{6!}\left[\left(\frac{1}{2}\right)^6 \gamma_{\frac{1}{2}} + \gamma_1\right] = \mathbf{0};$$

$$C_9 = \frac{1}{9!}\left[\alpha_1 + \left(\frac{3}{2}\right)^9 \alpha_{\frac{3}{2}} + 2^9\alpha_2 + \left(\frac{1}{2}\right)^9 \alpha_{\frac{1}{2}}\right] - \frac{1}{8!}\left[\beta_1 + \left(\frac{3}{2}\right)^8 \beta_{\frac{3}{2}} + (2)^8\beta_2 + \left(\frac{1}{2}\right)^8 \beta_{\frac{1}{2}}\right]$$
$$- \frac{1}{7!}\left[\left(\frac{1}{2}\right)^7 \gamma_{\frac{1}{2}} + \gamma_1\right] \neq \mathbf{0}.$$

2.2. **Region of Absolute Stability of the Block Hybrid Method.** Okuonghae and Ikhile [8] proposed the use of transforming the block method into general linear methods by expressing the new second derivative block hybrid method as

$$A\mathbf{u_1} = B\mathbf{u_2} + C\mathbf{u_3} + E\mathbf{u_4} + P\mathbf{u_5} + Q\mathbf{u_6},$$

where the matrices are as defined below

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}, \qquad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1k} \\ b_{21} & b_{22} & \cdots & b_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ b_{k1} & b_{k2} & \cdots & b_{kk} \end{bmatrix},$$

$$C = \begin{bmatrix} c_{11} & c_{12} & \cdots & c_{1k} \\ c_{21} & c_{22} & \cdots & c_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ c_{k1} & c_{k2} & \cdots & c_{kk} \end{bmatrix}, \qquad E = \begin{bmatrix} e_{11} & e_{12} & \cdots & e_{1k} \\ e_{21} & e_{22} & \cdots & e_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ e_{k1} & e_{k2} & \cdots & e_{kk} \end{bmatrix},$$

$$P = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1k} \\ p_{21} & p_{22} & \cdots & p_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ p_{k1} & p_{k2} & \cdots & p_{kk} \end{bmatrix}, \text{ and } Q = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1k} \\ q_{21} & q_{22} & \cdots & q_{2k} \\ \vdots & \vdots & \cdots & \vdots \\ q_{k1} & q_{k2} & \cdots & q_{kk} \end{bmatrix},$$

as well as the vectors

$$\mathbf{u_1} = \begin{bmatrix} y_{n+1} \\ y_{n+2} \\ \vdots \\ y_{n+k} \end{bmatrix}, \quad \mathbf{u_2} = \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ \vdots \\ y_{n} \end{bmatrix}, \quad \mathbf{u_3} = \begin{bmatrix} f_{n+1} \\ f_{n+2} \\ \vdots \\ f_{n+k} \end{bmatrix},$$

$$\mathbf{u_4} = \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ \vdots \\ f_{n} \end{bmatrix}, \quad \mathbf{u_5} = \begin{bmatrix} g_{n+1} \\ g_{n+2} \\ \vdots \\ g_{n+k} \end{bmatrix}, \text{ and } \mathbf{u_6} = \begin{bmatrix} g_{n-1} \\ g_{n-2} \\ \vdots \\ g_{n} \end{bmatrix}.$$

The second derivative block hybrid method (2.6) can be transformed into

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n+1} \\ y_{n+\frac{1}{2}} \\ y_{n+\frac{3}{2}} \\ y_{n+2} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_{n-1} \\ y_{n-2} \\ y_{n-3} \\ y_{n} \end{bmatrix} + \begin{bmatrix} \frac{3}{10} & \frac{466}{945} & \frac{2}{315} & -\frac{1}{3780} \\ \frac{77}{1280} & \frac{14737}{60480} & \frac{227}{60480} & -\frac{163}{967680} \\ \frac{621}{1280} & \frac{1443}{2240} & \frac{313}{2240} & -\frac{57}{35840} \\ \frac{8}{5} & -\frac{608}{945} & \frac{992}{945} & \frac{122}{945} \end{bmatrix} \begin{bmatrix} f_{n+1} \\ f_{n+\frac{1}{2}} \\ f_{n+\frac{3}{2}} \\ f_{n+2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \frac{253}{1260} \\ 0 & 0 & 0 & \frac{186367}{967680} \\ 0 & 0 & 0 & \frac{8333}{35840} \\ 0 & 0 & 0 & -\frac{128}{945} \end{bmatrix} \begin{bmatrix} f_{n-1} \\ f_{n-2} \\ f_{n-3} \\ f_{n} \end{bmatrix} + \begin{bmatrix} -\frac{1}{28} & -\frac{2}{63} & 0 & 0 \\ -\frac{25}{1792} & -\frac{239}{4032} & 0 & 0 \\ \frac{135}{1792} & \frac{9}{448} & 0 & 0 \\ -\frac{4}{7} & -\frac{32}{63} & 0 & 0 \end{bmatrix} \begin{bmatrix} g_{n+1} \\ g_{n+\frac{1}{2}} \\ g_{n+\frac{3}{2}} \\ g_{n+2} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \frac{1}{84} \\ 0 & 0 & 0 & \frac{353}{32256} \\ 0 & 0 & 0 & \frac{57}{3584} \\ 0 & 0 & 0 & -\frac{2}{63} \end{bmatrix} \begin{bmatrix} g_{n-1} \\ g_{n-2} \\ g_{n-3} \\ g_{n} \end{bmatrix}.$$

We substituted the above matrices into the characteristic equation

$$\det[r(A - Cz - Pz^2) - (B + Ez + Qz^2)] = 0, \tag{2.7}$$

where $y' = \lambda y, y'' = \lambda^2 y, z = \lambda h$ and $z^2 = \lambda^2 h^2$ are the usual test equations. Now, The characteristic equation reduces to

$$\rho(r,z) = -\frac{6r^3z^7 + \left(76r^3 - 2r^4\right)z^6 + \left(43r^4 + 589r^3\right)z^5 + \left(3248r^3 - 488r^4\right)z^4 + \left(3500r^4 + 12980r^3\right)z^3}{53760}$$
$$-\frac{\left(36240r^3 - 16080r^4\right)z^2 + \left(43680r^4 + 63840r^3\right)z - 53760r^4 + 53760r^3}{53760} = 0.$$

Differentiating the above stability polynomial with respect to z yields

$$\frac{\partial \rho(r,z)}{\partial z} = -\frac{42r^3z^6 + \left(456r^3 - 12r^4\right)z^5 + \left(215r^4 + 2945r^3\right)z^4 + \left(12992r^3 - 1952r^4\right)z^3}{53760}$$
$$-\frac{\left(10500r^4 + 38940r^3\right)z^2 + \left(72480r^3 - 32160r^4\right)z + 43680r^4 + 63840r^3}{53760}.$$

We plotted the region of absolute stability of the method using Newton's method for finding the roots of the stability polynomial above. The method is $A-$stable as shown in Figure 1. It is easily seen that the new method has a large stability region.

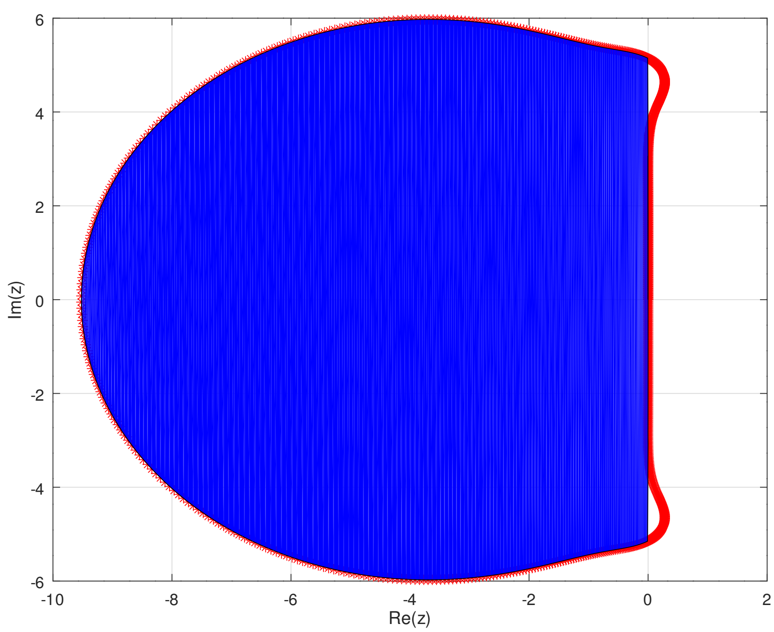

Fig. 1. Region of absolute stability of the second derivative block hybrid method.

2.3. **Zero Stability.** In order to examine the zero-stability of the methods, we let $z = 0$ in (2.7) such that

$$\begin{aligned}
\det[R(A - Cz - Pz^2) - (B + Ez + Qz^2)] &= \det[RA - B] \\
&= \begin{vmatrix} R & 0 & 0 & -1 \\ 0 & R & 0 & -1 \\ 0 & 0 & R & -1 \\ 0 & 0 & 0 & R-1 \end{vmatrix} \\
&= R^4 - R^3 \\
&= R^3(R-1) = 0.
\end{aligned}$$

The roots of the stability polynomial are $\{0,0,0,1\}$. Since the roots of stability polynomial did not exceed one in absolute value sense, the second derivative block hybrid method is zero stable. The properties of the new method are summarized in Table 1.

Table 1. Properties of the new second derivative block hybrid method.

y_{n+i}	Order	Error Constants $C_9 \neq 0$	Consistency?	Zero Stability?	Convergence?
y_{n+1}	7	4.6749023×10^{-8}	Yes	Yes	Yes
$y_{n+\frac{1}{2}}$	7	3.3466303×10^{-8}	Yes	Yes	Yes
$y_{n+\frac{3}{2}}$	7	1.4480279×10^{-7}	Yes	Yes	Yes
y_{n+2}	7	$-1.4566011 \times 10^{-6}$	Yes	Yes	Yes

So far, we have characterized the basic properties of the new block hybrid method. In the next brief but concise discussion, we give a guideline on how to find the Jacobian of the given system of differential equations and how to make them algorithm-ready for computation. The general form of a second order system of ordinary differential equation is

$$\mathbf{y}''(x) = \mathbf{f}(x, \mathbf{y}, \mathbf{y}'),$$

with initial conditions for $x \in [a,b]$

$$\mathbf{y}(x_0) = \mathbf{y}_0, \quad and \quad \mathbf{y}'(x_0) = \mathbf{y}'_0,$$

such that

$$\mathbf{y}(x) = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \quad \mathbf{f}(x, \mathbf{y}) = \begin{bmatrix} f_1(x, \mathbf{y}) \\ f_2(x, \mathbf{y}) \\ \vdots \\ f_n(x, \mathbf{y}) \end{bmatrix}.$$

Since $\mathbf{y}'(x,\mathbf{y}) = \mathbf{f}(x,\mathbf{y})$ is the general first order ordinary differential equation, then

$$\mathbf{y}''(x,\mathbf{y}) = \mathbf{f}'(x,\mathbf{y}) = \mathbf{g}(x,\mathbf{y}).$$

It is not difficult to see that the Jacobian of $\mathbf{g}(x,\mathbf{y})$ is

$$\begin{aligned} \mathbf{g}_{\mathbf{y}}(x,\mathbf{y}) &= \mathbf{f}_x(x,\mathbf{y}) + \mathbf{f}_{\mathbf{y}}(x,\mathbf{y})\mathbf{y}'(x) \\ &= \mathbf{f}_x(x,\mathbf{y}) + \mathbf{f}_{\mathbf{y}}(x,\mathbf{y})\mathbf{f}(x,\mathbf{y}), \end{aligned}$$

where $\mathbf{f}_{\mathbf{y}}(x,\mathbf{y})$ is the Jacobian of $\mathbf{f}(x,\mathbf{y})$.

Next, we present the new block hybrid method as a system of non-linear equations as well as its corresponding Jacobian.

$$
\begin{aligned}
y_{n+1}-y_n-\frac{h\left[759f_n+1864f_{n+\frac{1}{2}}+1134f_{n+1}+24f_{n+\frac{3}{2}}-f_{n+2}\right]+h^2\left[45g_n-120g_{n+\frac{1}{2}}-135g_{n+1}\right]}{3780} &= 0\\
y_{n+\frac{1}{2}}-y_n-\frac{h\left[186367f_n+235792f_{n+\frac{1}{2}}+58212f_{n+1}+3632f_{n+\frac{3}{2}}-163f_{n+2}\right]+h^2\left[10590g_n-57360g_{n+\frac{1}{2}}-13500g_{n+1}\right]}{967680} &= 0\\
y_{n+\frac{3}{2}}-y_n-\frac{h\left[8333f_n+23088f_{n+\frac{1}{2}}+17388f_{n+1}+5008f_{n+\frac{3}{2}}-57f_{n+2}\right]+h^2\left[570g_n+720g_{n+\frac{1}{2}}+2700g_{n+1}\right]}{35840} &= 0\\
y_{n+2}-y_n-\frac{h\left[-128f_n+1512f_{n+1}-608f_{n+\frac{1}{2}}+992f_{n+\frac{3}{2}}+122f_{n+2}\right]+h^2\left[-30g_n-480g_{n+\frac{1}{2}}-540g_{n+1}\right]}{945} &= 0.
\end{aligned}
\tag{2.8}
$$

The Jacobian of the second derivative block hybrid method is

$$
J=\begin{bmatrix}
I-\frac{1134h}{3780}\frac{\partial f_{n+1}}{\partial y_{n+1}}-\frac{135h^2}{3780}\frac{\partial g_{n+1}}{\partial y_{n+1}} & -\frac{1864h}{3780}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}+\frac{120h^2}{3780}\frac{\partial g_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} & -\frac{24h}{3780}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} & \frac{h}{3780}\frac{\partial f_{n+2}}{\partial y_{n+2}}\\
-\frac{235792h}{967680}\frac{\partial f_{n+1}}{\partial y_{n+1}}+\frac{13500h^2}{967680}\frac{\partial g_{n+1}}{\partial y_{n+1}} & I-\frac{235792h}{967680}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}+\frac{57360h^2}{967680}\frac{\partial g_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} & -\frac{3632h}{967680}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} & \frac{163h}{967680}\frac{\partial f_{n+2}}{\partial y_{n+2}}\\
-\frac{17388h}{35840}\frac{\partial f_{n+1}}{\partial y_{n+1}}-\frac{2700h^2}{35840}\frac{\partial g_{n+1}}{\partial y_{n+1}} & -\frac{23088h}{35840}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}-\frac{720h^2}{35840}\frac{\partial g_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} & I-\frac{5008h}{35840}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} & \frac{57h}{35840}\frac{\partial f_{n+2}}{\partial y_{n+2}}\\
-\frac{1512h}{945}\frac{\partial f_{n+1}}{\partial y_{n+1}}+\frac{540h^2}{945}\frac{\partial g_{n+1}}{\partial y_{n+1}} & \frac{608h}{945}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}+\frac{480h^2}{945}\frac{\partial g_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} & -\frac{992h}{945}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} & I-\frac{122h}{945}\frac{\partial f_{n+2}}{\partial y_{n+2}}
\end{bmatrix}.
\tag{2.9}
$$

Since our algorithm relies on Newton's method, we need to show that the Jacobian of the second derivative block hybrid method is non-singular, which we state with a proof below.

Theorem 2: The Jacobian J in (2.9) is non-singular at the root.
Proof : We perform elementary row operations on J above and it reduces to

$$\sim \begin{bmatrix} 1 & \frac{120\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}h^2-1864\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}h}{135\frac{\partial g_{n+1}}{\partial g_{n+1}}h^2-1134\frac{\partial f_{n+1}}{\partial f_{n+1}}h+3780} & -\frac{8\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}h}{45\frac{\partial g_{n+1}}{\partial g_{n+1}}h^2-378\frac{\partial f_{n+1}}{\partial f_{n+1}}h+1260} & \kappa \\ 0 & 1 & \kappa & \mu \\ 0 & 0 & 1 & \nu_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (2.10)$$

where

$$\kappa = \frac{\frac{\partial f_{n+2}}{\partial y_{n+2}}h}{135\frac{\partial g_{n+1}}{\partial g_{n+1}}h^2 - 1134\frac{\partial f_{n+1}}{\partial f_{n+1}}h + 3780},$$

$$\omega=\frac{3024\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}h^2-28488\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}h+93744\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}}{\left(15432\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-20061\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h^2+\left(-45360\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-71198\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h+142884\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}},$$

$$\mu=\frac{3402\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+2}}{\partial y_{n+2}}h^3+\left(-27548\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+2}}{\partial y_{n+2}}-25515\frac{\partial g_{n+1}}{\partial g_{n+1}}\right)h^2+\left(92232\frac{\partial f_{n+2}}{\partial y_{n+2}}+214326\frac{\partial f_{n+1}}{\partial f_{n+1}}\right)h-714420}{\left(123456\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-160488\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h^3+\left(-362880\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-569584\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h^2+1143072\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}h},$$

and

$$\nu_1=\frac{20412\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}\frac{\partial f_{n+2}}{\partial y_{n+2}}h^4+\left(\left(233496\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-689337\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)\frac{\partial f_{n+2}}{\partial y_{n+2}}-153090\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}\right)h^3+a_1+a_2}{145152\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}h^4+\left(4129536\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-8361360\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}h^3+b_1+b_2},$$

such that

$$a_1=\left(\left(-163296\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-455706\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)\frac{\partial f_{n+2}}{\partial y_{n+2}}-1959552\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}+5501034\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h^2,$$

$$a_2=\left(-7837884\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\frac{\partial f_{n+2}}{\partial y_{n+2}}+1837080\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}+4204872\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h+58909032\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}},$$

for the numerator of ν_1 and

$$b_1=\left(\left(-8418816\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}-13910080\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)\frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}}-15802368\frac{\partial f_{n+1}}{\partial f_{n+1}}\frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}}+20542464\frac{\partial g_{n+1}}{\partial g_{n+1}}\frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}}\right)h^2,$$

$$b_2 = \left(-41394240 \frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} \frac{\partial f_{n+\frac{3}{2}}}{\partial y_{n+\frac{3}{2}}} + 46448640 \frac{\partial g_{n+\frac{1}{2}}}{\partial g_{n+\frac{1}{2}}} + 72906752 \frac{\partial f_{n+1}}{\partial f_{n+1}} \frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}} \right) h - 146313216 \frac{\partial f_{n+\frac{1}{2}}}{\partial y_{n+\frac{1}{2}}},$$

for the denominator of v_1.

Since the echelon form (2.10) of J consists of four pivots, J is non-singular at the root. $\square$

2.4. **Algorithm.** Since we have shown that the Jacobians are non-singular, In the discussions below, we present Newton based algorithms for the solution of systems of initial value problems arising from the above method. We state clearly that the algorithm is self-starting and do not rely on predictors or correctors to start. The starting values are the initial values of the differential equations.

Algorithm 2.1. Input : h, `tol`, system of differential equations to be solved, their initial values and corresponding Jacobians.
For $k = 1, 2, 3, \cdots$, until convergence
Form

$$\mathbf{v}^{(k)} = \begin{bmatrix} y_{n+1}^{(k)} \\ \\ y_{n+\frac{1}{2}}^{(k)} \\ \\ y_{n+\frac{3}{2}}^{(k)} \\ \\ y_{n+2}^{(k)} \end{bmatrix},$$

Compute $J(\mathbf{v}^{(k)})$.
Find the L and U factors of $J(\mathbf{v}^{(k)})$, that is $J(\mathbf{v}^{(k)}) = LU$.
Solve the triangular system $L\mathbf{w}^{(k)} = -\mathbf{F}(\mathbf{v}^{(k)})$ for $\mathbf{w}^{(k)}$.
Solve the triangular system $U\Delta\mathbf{v}^{(k)} = \mathbf{w}^{(k)}$ for $\Delta\mathbf{v}^{(k)}$.
Increment $\mathbf{v}^{(k+1)} = \mathbf{v}^{(k)} + \Delta\mathbf{v}^{(k)}$.
Output: $y_{n+1}^{(k+1)}$.

Note that for a lack of space, we could not define $\mathbf{F}(\mathbf{v}^{(k)})$ used in the above algorithm and that is why we are using landscape, where

$$\mathbf{F}(\mathbf{v}^{(k)}) = \begin{bmatrix} y_{n+1} - y_n - \dfrac{h\left(759f_n + 1864f_{n+\frac{1}{2}} + 1134f_{n+1} + 24f_{n+\frac{3}{2}} - f_{n+2}\right) + h^2\left(45g_n - 120g_{n+\frac{1}{2}} - 135g_{n+1}\right)}{3780} \\ y_{n+\frac{1}{2}} - y_n - \dfrac{h\left(186367f_n + 235792f_{n+\frac{1}{2}} + 58212f_{n+1} + 3632f_{n+\frac{3}{2}} - 163f_{n+2}\right) + h^2\left(10590g_n - 57360g_{n+\frac{1}{2}} - 13500g_{n+1}\right)}{967680} \\ y_{n+\frac{3}{2}} - y_n - \dfrac{h\left(8333f_n + 23088f_{n+\frac{1}{2}} + 17388f_{n+1} + 5008f_{n+\frac{3}{2}} - 57f_{n+2}\right) + h^2\left(570g_n + 720g_{n+\frac{1}{2}} + 2700g_{n+1}\right)}{35840} \\ y_{n+2} - y_n - \dfrac{h\left(-128f_n + 1512f_{n+1} - 608f_{n+\frac{1}{2}} + 992f_{n+\frac{3}{2}} + 122f_{n+2}\right) + h^2\left(-30g_n - 480g_{n+\frac{1}{2}} - 540g_{n+1}\right)}{945} \end{bmatrix}.$$

2.5. **Numerical Experiments.** In this important section, we are concerned about the performance of the new second derivative block hybrid method on some stiff systems of ordinary differential equations. We compare our result with those of Yakubu et al [5], [10] and [11]. Results are presented by Table and Figures which validate the new method for the numerical solution of systems of stiff initial-valued ordinary differential equations. In each of the examples in this section, we used a constant step size of 0.1.

Example 2.1. The well-known stiff system of Kaps [9]

$$\mathbf{f}(x,\mathbf{y}) = \begin{bmatrix} y_1'(x) \\ \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -1002y_1(x) + 1000y_2^2(x) \\ \\ y_1(x) - y_2(x)^2 - y_2(x) \end{bmatrix},$$

which satisfies the $y_1(0) = 1$ and $y_2(0) = 1$.

The Jacobian of the above matrix is given by

$$\mathbf{f}_{\mathbf{y}}(x,\mathbf{y}) = \begin{bmatrix} -1002 & 2000\,y_2 \\ \\ 1 & -2\,y_2 - 1 \end{bmatrix}.$$

We need to calculate $\mathbf{g}(x,\mathbf{y})$ as follows

$$\begin{aligned} \mathbf{g}(x,\mathbf{y}) &= \mathbf{f}_x(x,\mathbf{y}) + \mathbf{f}_{\mathbf{y}}(x,\mathbf{y})\mathbf{f}(x,\mathbf{y}) \\ &= \mathbf{f}_{\mathbf{y}}(x,\mathbf{y})\mathbf{f}(x,\mathbf{y}) \\ &= \begin{bmatrix} -1002 & 2000\,y_2 \\ \\ 1 & -2\,y_2 - 1 \end{bmatrix} \begin{bmatrix} 1000y_2^2(x) - 1002y_1(x) \\ \\ -y_2(x)^2 - y_2(x) + y_1(x) \end{bmatrix} \\ &= \begin{bmatrix} -2000y_2^3 - 1004000y_2^2 + 2000y_1y_2 + 1004004y_1 \\ \\ 2y_2^3 + 1003y_2^2 + (1-2y_1)\,y_2 - 1003y_1 \end{bmatrix}. \end{aligned}$$

Its corresponding Jacobian is given by

$$\mathbf{g}_{\mathbf{y}}(x,\mathbf{y}) = \begin{bmatrix} 2000y_2 + 1004004 & 2000y_1 - 6000y_2^2 - 2008000y_2 \\ \\ -2y_2 - 1003 & -2y_1 + 6y_2^2 + 2006y_2 + 1 \end{bmatrix}.$$

We plugged these values of $\mathbf{f}(x,\mathbf{y}), \mathbf{f}_{\mathbf{y}}(x,\mathbf{y}), \mathbf{g}(x,\mathbf{y}), \mathbf{g}_{\mathbf{y}}(x,\mathbf{y})$ into Algorithm 2.1 and the results of our approximations are as shown in Fig. 2 and Table 2.

Table 2. Comparing the absolute errors of the New Block Hybrid Method with [5] on Example 2.1.

x	y_i	**Yakubu et al [5]**	**New Method**
5	y_1	$1.228938367083599 \times 10^{-03}$	$7.9078603859383074 \times 10^{-07}$
	y_2	$1.800318343625484 \times 10^{-06}$	$1.9170221584458025 \times 10^{-07}$
50	y_1	$3.325679258575631 \times 10^{-05}$	$6.3478672788605185 \times 10^{-46}$
	y_2	$5.804723043345561 \times 10^{-07}$	$4.0263603198079618 \times 10^{-26}$

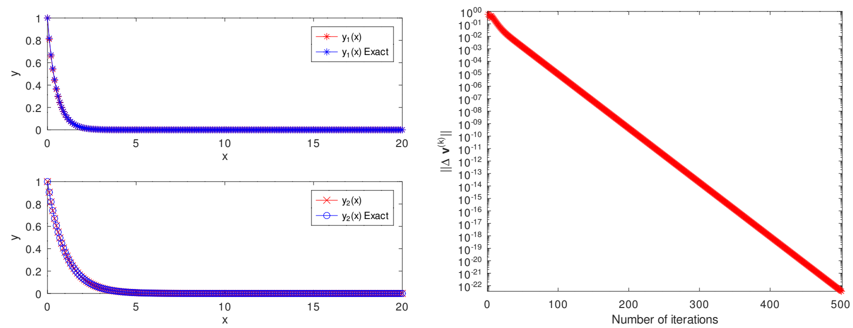

Fig. 2. A plot of the numerical and exact solutions on the left and the norm of $\Delta\mathbf{v}^{(k)}$ versus number of iterations on the right on Example 2.1.

We plotted the performance of the new second derivative block hybrid method with the exact solution on Example 2.1 which is shown on the left side of Fig. 2 (the new method depicts a great promise), while to the right of the same figure is the variation of the norm of $\Delta\mathbf{v}^{(k)}$ versus number of iterations. For $x = 50$, we see that aside the new method performing better than that of Yakubu et al., [5], Table 2 column 4 shows that the new method also outperforms those of [10] and [11] albeit their respective absolute errors were $y_1 = 7.14\times10^{-21}, y_2 = 3.34\times10^{-19}$ and $y_1 = 7.38\times10^{-24}, y_2 = 4.83\times10^{-25}$.

Example 2.2. The non-linear stiff problem

$$\mathbf{f}(x,\mathbf{y}) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} -0.013y_1 - 1000y_1y_3 \\ -2500y_2y_3 \\ -0.013y_1 - 1000y_1y_3 - 2500y_2y_3, \end{bmatrix},$$

satisfying the initial conditions

$$[y_1(0), y_2(0), y_3(0)]^T = [1,1,0]^T,$$

is from Gear [12] and has no known exact solution so we compare our solution with `octave ode15s` [13].

The Jacobian of the above system is

$$\mathbf{f}_{\mathbf{y}}(x,\mathbf{y}) = \begin{bmatrix} -0.013 - 1000y_3 & 0 & -1000y_1 \\ 0 & -2500y_3 & -2500y_2 \\ -0.013 - 1000y_3 & -2500y_3 & -1000y_1 - 2500y_2 \end{bmatrix},$$

and

$$\mathbf{g}(x,\mathbf{y}) = \mathbf{f}_{\mathbf{y}}(x,\mathbf{y})\mathbf{f}(x,\mathbf{y})$$
$$= \begin{bmatrix} -0.013-1000y_3 & 0 & -1000y_1 \\ 0 & -2500y_3 & -2500y_2 \\ -0.013-1000y_3 & -2500y_3 & -1000y_1-2500y_2 \end{bmatrix} \begin{bmatrix} -0.013y_1-1000y_1y_3 \\ -2500y_2y_3 \\ -0.013y_1-1000y_1y_3-2500y_2y_3 \end{bmatrix}$$

$$= \begin{bmatrix} 1000000y_1y_3^2+(2500000y_1y_2+1000000y_1^2+26y_1)y_3+13y_1^2+1.69\times10^{-4}y_1 \\ 6250000y_2y_3^2+(6250000y_2^2+2500000y_1y_2)y_3+32.5y_1y_2 \\ \alpha \end{bmatrix},$$

where

$$\begin{aligned} \alpha \;=\; & (6250000y_2+1000000y_1)y_3^2+(6250000y_2^2+5000000y_1y_2 \\ & +1000000y_1^2+26y_1)y_3+32.5y_1y_2+13y_1^2+1.69\times10^{-4}y_1. \end{aligned}$$

Similarly, the Jacobian of $\mathbf{g}(x,\mathbf{y})$ is

$$\mathbf{g}_{\mathbf{y}}(x,\mathbf{y}) = \begin{bmatrix} \tau & \kappa & \omega \end{bmatrix},$$

where

$$\tau = \begin{bmatrix} 10^{-6}(10^{12}y_3^2+(2.5\times10^{12}y_2+2\times10^{12}y_1+2.6\times10^{7})y_3+2.6\times10^{7}y_1+169) \\ 0.5(5000000y_2y_3+65y_2) \\ 10^{-6}(10^{12}y_3^2+(5\times10^{12}y_2+2\times10^{12}y_1+2.6\times10^{7})y_3+3.25\times10^{7}y_2+2.6\times10^{7}y_1+169) \end{bmatrix},$$

obtained by differentiating $\mathbf{g}(x,\mathbf{y})$ partially with respect to y_1. After differentiating $\mathbf{g}(x,\mathbf{y})$ partially with respect to y_2, we arrived at

$$\kappa = \begin{bmatrix} 2500000y_1y_3 \\ 0.5(1.25\times10^{7}y_3^2+(2.5\times10^{7}y_2+5000000y_1)y_3+65y_1) \\ 10^{-6}(6.25\times10^{12}y_3^2+(1.25\times10^{13}y_2+5\times10^{12}y_1)y_3+3.25\times10^{7}y_1) \end{bmatrix},$$

and

$$\omega = \begin{bmatrix} 10^{-6}(2\times10^{12}y_1y_3+2.5\times10^{12}y_1y_2+1.0\times10^{12}y_1^2+2.6\times10^{7}y_1) \\ 0.5(2.5\times10^{7}y_2y_3+1.25\times10^{7}y_2^2+5000000y_1y_2) \\ 10^{-6}(2(6.25\times10^{12}y_2+10^{12}y_1)y_3+6.25\times10^{12}y_2^2+5\times10^{12}y_1y_2+10^{12}y_1^2+2.6\times10^{7}y_1) \end{bmatrix},$$

obtained by differentiating $\mathbf{g}(x,\mathbf{y})$ partially with respect to y_3. We plugged these values of $\mathbf{f}(x,\mathbf{y}),\mathbf{f_y}(x,\mathbf{y}),\mathbf{g}(x,\mathbf{y}),\mathbf{g_y}(x,\mathbf{y})$ into Algorithm 2.1 and the results are as shown in Fig. 3.

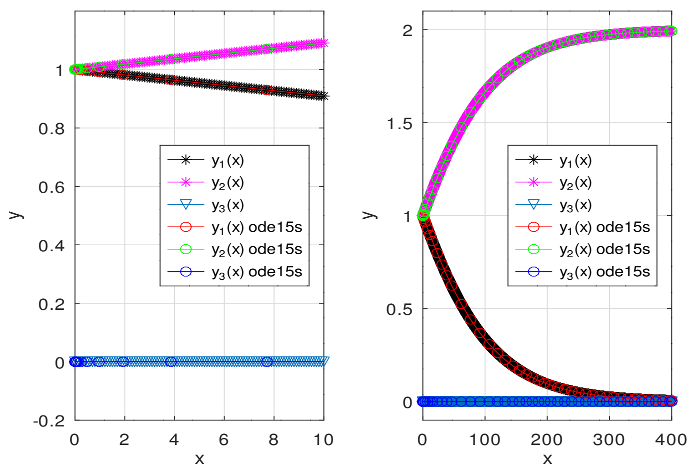

Fig. 3. The figure above shows the result of comparing the new method with `octave ode15s`.

Fig. 3 showed that the new second derivative block hybrid method performs at par with the well known stiff `ode15s` solver. Notice from the figure on the left that we plotted the results in the range $[0,10]$ and $[0,400]$ for x. This is to properly see the trajectory of the solution.

Example 2.3. The well-known Fatunla problem [14]

$$\mathbf{f}(x,\mathbf{y}) = \begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \\ y_5'(x) \\ y_6'(x) \end{bmatrix} = \begin{bmatrix} -10 & 100 & 0 & 0 & 0 & 0 \\ -100 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix},$$

which satisfies the initial conditions

$$[y_1(0),y_2(0),y_3(0),y_4(0),y_5(0),y_6(0)] = [1,1,1,1,1,1].$$

Observe that

$$\mathbf{g}(x,\mathbf{y}) = \mathbf{f}_{\mathbf{y}}(x,\mathbf{y})\mathbf{f}(x,\mathbf{y})$$
$$= \begin{bmatrix} -10 & 100 & 0 & 0 & 0 & 0 \\ -100 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \end{bmatrix} \begin{bmatrix} -10y_1 + 100y_2 \\ -100y_1 - 10y_2 \\ -4y_3 \\ -y_4 \\ -0.5y_5 \\ -0.1y_6 \end{bmatrix}$$

$$= \begin{bmatrix} -9900y_1 - 2000y_2 \\ 2000y_1 - 9900y_2 \\ 16y_3 \\ y_4 \\ \frac{y_5}{4} \\ \frac{y_6}{100} \end{bmatrix},$$

and the Jacobian

$$\mathbf{g}_{\mathbf{y}}(x,\mathbf{y}) = \begin{bmatrix} -9900 & -2000 & 0 & 0 & 0 & 0 \\ 2000 & -9900 & 0 & 0 & 0 & 0 \\ 0 & 00 & 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{100} \end{bmatrix}.$$

Table 3. Comparing the absolute errors of the new method with those of [5] on Example 2.3.

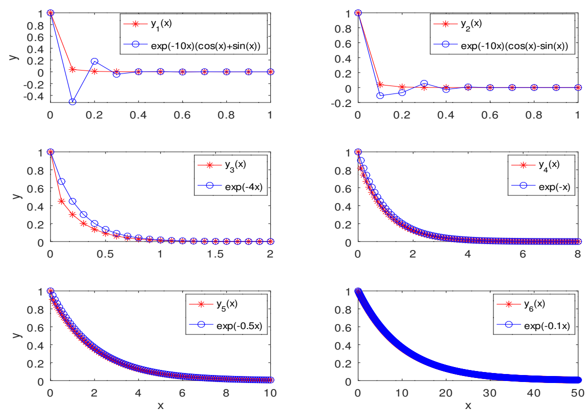

Fig. 4. Exact solution versus the new method on the Fatunla problem on Example 2.3.

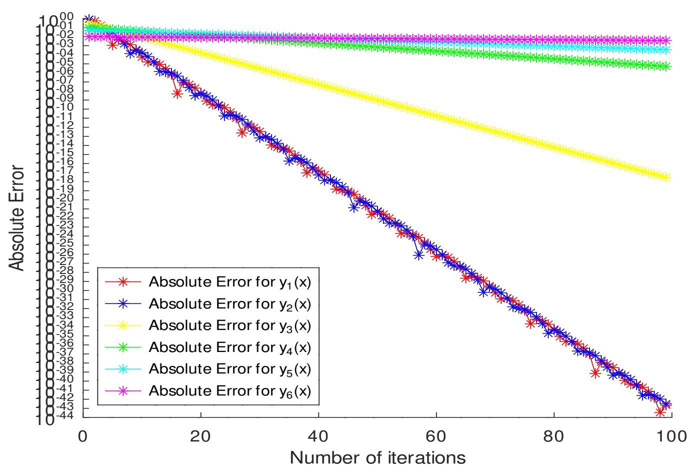

Fig. 5. Plot of the absolute error against the number of iterations on Example 2.3.

x	y_i	**Yakubu et al [5]**	**New Method**
5	y_1	$2.220446049250313\times10^{-16}$	$4.7592391018792229\times10^{-22}$
	y_2	$1.318389841742373\times10^{-16}$	$5.6855380861750618\times10^{-23}$
	y_3	0	$9.9704102211086080\times10^{-10}$
	y_4	0	$7.0850105622049034\times10^{-04}$
50	y_1	$3.330669073875470\times10^{-16}$	0
	y_2	$7.771561172376096\times10^{-16}$	0
	y_3	$4.440892098500626\times10^{-16}$	0
	y_4	0	$1.8319437858209123\times10^{-023}$

Results of numerical simulations on this example is as shown in Fig. 4, 5 and Table 3. Figure 4 is a plot of the exact solution versus the new 7th–order block hybrid method, it shows that as x increases, the exact coincides with the approximation solution. Since the step size is 0.1, it means we need 500 iterates to reach $x = 50$. However, because Fig. 5 is on a semilogy scale and in order to visualize the y-labels, we restricted the range of x values from 0 to 100. Furthermore, for $x = 5$, Table 3 showed that the new method outperforms those of [5] in two out of four y values. Nevertheless, for $x = 50$, we see that the accuracy of the new method increased to three out of four. This confirms the reliability of the new method for the numerical solution of stiff systems of ordinary differential equations.

3. CONCLUDING REMARKS

We have shown that the new second derivative block hybrid method of order seven with fewer function evaluations outperforms a fourteenth -order second derivative block hybrid method of Yakubu et al., [5] which uses more function evaluations and almost twice the size of the linear system presented in this paper. The accuracy of our method must have been due to the large region of stability. We recommend the new algorithm for the numerical approximation of stiff initial valued systems of ordinary differential equations.

4. ACKNOWLEDGEMENTS

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