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# THE RADIAL PART OF AN INVARIANT DIFFERENTIAL OPERATOR ON THE EUCLIDEAN MOTION GROUPS

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ABSTRACT. Let $G$ be the Euclidean motion group realised as the semi direct product of $\mathbb{R}^n$ and $SO(n)$, that is, $G = \mathbb{R}^n \rtimes SO(n)$. The pair $(\mathbb{R}^n \rtimes SO(n), SO(n))$ is called the Gelfand pair. In this work, among other things, spherical analysis on the pair is presented, including an explicit determination of spherical function for $G$.

## 1. INTRODUCTION

The notion of Gelfand pair was first noticed in the study of infinite dimensional irreducible unitary representations of semisimple Lie groups. Since then it has been applied also to analysis on symmetric spaces. Spherical analysis on a Gelfand pair (G,K) is the analogue of Fourier analysis on $\mathbb{R}^n$ or on the torus $\mathbb{T}$. Precisely, (G,K) is a Gelfand pair if the convolution algebra $L^1(K\backslash G/K)$ is commutative.

It has been recognised that many of the special functions introduced in analysis are closely related to the theory of linear representations of Lie groups. Prominent among such functions are the spherical function. The theory of spherical functions generalizes both the classical Laplace spherical harmonics and continuous characters of Lie groups. Spherical functions play an important part in the modern theory of infinite dimensional linear representations of Lie groups. In this work we discuss spherical functions on the Euclidean motion groups. Jean Dieudonne [4] has mentioned, without any proof, that the spherical function for $SE(2)$ is a Bessel function of order zero. It is our purpose in this work to develope a proof of this result and, as our result shows, it turns out

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that the spherical functions for the Gelfand pair $(G,K)$ is a spherical Bessel function.

This work is arranged as follows. Section 2 contains preliminaries concerning spherical functions on locally compact groups, one parameter subgroups and vector fields on $SE(2)$. The main results of this work concerning spherical functions on $G$ and the pair $(G,K)$ are given in section 3.

## 2. PRELIMINARIES

This section contains preliminaries about one-parameter subgroups, vector fields and spherical functions on $G$. We start with one parameter subgroups of $SE(2)$ in subsection 2.1 followed by vector fields in subsection 2.2 while spherical functions are discussed in subsection 2.3.

### 2.1. **One-parameter subgroup of SE(2).**

We begin this section by identifying the Lie algebra of $SE(n)$ denoted by $\mathfrak{se}(n)$. This Lie algebra is the sub-algebra of $gl(n+1,\mathbb{R})$ given as

$$\mathfrak{se}(n)=\left\{X=\begin{pmatrix} T & E \\ 0 & 0 \end{pmatrix}: E\in\mathbb{R}^n,\ T\in\mathfrak{so}(n)\right\}.$$

Here, $gl(n+1,\mathbb{R})$ is the Lie algebra of the general linear group $GL(n+1,\mathbb{R})$ consisting of real matrices $A$ of order $n+1$ with the Lie bracket $[A,B]=AB-BA$ for $A,B\in gl(n+1,\mathbb{R})$; $so(n)=\Big\{A\in gl(n+1,\mathbb{R})|A+A^t=0\Big\}$ is the Lie algebra of $SO(n)$; T is a skew - symmetric matrix; and E is a vector in $\mathbb{R}^n$ ([3],p.11). In particular, the Lie group $SE(2)$ admits a three dimensional Lie algebra $\mathfrak{se}(2)$ whose basis elements are

$$X_1=\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, X_2=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, X_3=\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

These basis elements obey the commutation relations $[X_1, X_2] = 0$, $[X_2, X_3] = X_1$ and $[X_3, X_1] = X_2$ as shown below

$$
\begin{aligned}
[X_1, X_2] &= X_1X_2 - X_2X_1 \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= 0.
\end{aligned}
$$

$$
\begin{aligned}
[X_2, X_3] &= X_2X_3 - X_3X_2 \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= X_1.
\end{aligned}
$$

$$
\begin{aligned}
[X_3, X_1] &= X_3X_1 - X_1X_3 \\
&= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}
$$

$$= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$
$$= X_2.$$

We are now ready to discuss the one-parameter subgroups of the Lie group $SE(2)$. This is important for defining the left or right invariant differential operators on $SE(2)$. Before going on, we define the one-parameter subgroup of a (real) Lie group $G$ in general.

**2.1.1 Definition[7].** Let $G$ be a linear Lie group. A function $\gamma : \mathbb{R} \to G$ is a one-parameter subgroup of $G$ if
(a) $\gamma$ is continuous,
(b) $\gamma(0) = I$, where $I$ is the identity element of $G$,
(c) $\gamma(t+s) = \gamma(t)\gamma(s)$, for all $s, t \in \mathbb{R}$.

There is an important theorem for calculating the one parameter subgroup of a linear Lie group which we state in theorem 2.1.2 below.

**2.1.2 Theorem** ([8], Theorem 1.1.1, p. 4) Let $\gamma : \mathbb{R} \to GL(n, \mathbb{R})$ be a one-parameter subgroup of $GL(n, \mathbb{R})$. Then $\gamma$ is a $C^\infty$ and $\gamma(t) = exp(tA)$, with $A = \gamma'(0)$. In fact, $\gamma$ is even and real analytic.

Putting $t = 1$, then $\gamma(1) = exp(A)$ and since the exponential map $exp(.)$ is always defined from the Lie algebra of a Lie group to the Lie group itself, it stands to reason that $A$ is an element of the Lie algebra $gl(n, \mathbb{R})$. We can now calculate the one-parameter subgroups of $SE(2)$ using the formula specified in theorem 2.1.2. For $X_1$, the corresponding one-parameter subgroup is

$$\begin{aligned} \gamma_1(t) &= exp(tX_1) \\ &= I + tX_1 + \frac{t^2X^2}{2!} + \frac{t^3X^3}{3!} + \ldots. \end{aligned}$$

Since

$$X_1^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_1^n, \; \forall \, n \; > \; 2,$$

we have

$$\gamma_1(t) = I + tX_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & t \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Similarly, the one-parameter subgroup of $SE(2)$ corresponding to $X_2$ is

$$\gamma_2(t) = exp(tX_2) = I + tX_2 + \frac{t^2X_2^2}{2!} + \frac{t^3X_2^3}{3!} + ....$$

But

$$X_2^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = X_2^n, \ \forall \, n \ > \ 2.$$

Therefore,

$$\gamma_2(t) = I + tX_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}.$$

Finally, $\gamma_3(t)$ is obtained as

$$\gamma_3(t) = exp(tX_3) = I + tX_3 + \frac{t^2X_3^2}{2!} + \frac{t^3X_3^3}{3!} + ... \tag{2.1}$$

Now

$$X_3^2 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^4 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$X_3^6 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $X_3^6 = X_3^2$, the exponential series (2.1) terminates at $X_3^5$. Therefore

$$\begin{aligned} \gamma_3(t) &= I + tX_3 + \frac{t^2X_3^2}{2!} + \frac{t^3X_3^3}{3!} + \frac{t^4X_3^4}{4!} + \frac{t^5X_3^5}{5!} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + t\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^3}{3!}\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &\quad + \frac{t^4}{4!}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{t^5}{5!}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots & -t + \frac{t^3}{3!} - \frac{t^5}{5!} + \ldots & 0 \\ t - \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots & 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} cost & -sint & 0 \\ sint & cost & 0 \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

## 2.2. **Vector Fields on SE(2).**

We begin this section by describing the tangent vector and the vector field of a given differentiable manifold $M$. Thereafter, an explicit form of vector fields for $SE(2)$ are presented.

**2.2.1 Definition**. Let $M$ be a $C^\infty$ - manifold and let $p \in M$. A tangent vector at a point $p$ is a mapping $L : C^\infty(p) \to \mathbb{R}$ such that for $\alpha, \beta \in \mathbb{R}$ (i) $L(\alpha f_1 + \beta f_2) = \alpha L(f_1) + \beta L(f_2)$; (ii) $L(f_1 f_2)(p) = L(f_1) f_2(p) + f_1(p) L(f_2)$.
Condition (i) expresses linearity of $L$ while condition (ii) is the Leibnitz rule. The set of all tangent vectors at $p \in M$ is denoted by $T_p(M)$ and is

called tangent space. A tangent bundle $T(M)$ over an n - dimensional manifold $M$ is the union of all tangent spaces $T_p(M)$ of $M$.

We are now ready to define the concept of a vector field on a given differentiable manifold.

**2.2.2 Definition.** A vector field on $M$ (also called the global section of the tangent bundle) is a map

$$X : M \to T(M)$$

defined by $M \ni p \mapsto X(p) = X_p \in T_p(M)$ $\forall p \in M$ and such that $\pi \circ X = Id_M$, where $Id_M$ is the identity map on $M$ and $\pi : T(M) \to M$. A vector field $X$ on a manifold $M$ is called a $C^\infty$ - vector field if for every $f \in C^\infty(M)$, the function $Xf$ is in $C^\infty(M)$.

The explicit form of vector fields on $SE(2)$ are presented in what follows. Let $(x,y,\theta)$ be a system of coordinates on $SE(2)$, with $x,y \in \mathbb{R}$ and $\theta \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Let $\mathscr{D}(SE(2))$ be the space of $C^\infty$ functions on $SE(2)$ with compact support. For $X_1, X_2 \in \mathbb{R}^2$, $X_3 \in SO(2)$ and for $f \in \mathscr{D}(SE(2))$, the corresponding left invariant vector fields for $SE(2)$ are given as follows.

$$(X_{1SE(2)}f)(x,y,\theta) = cos\theta\frac{\partial f}{\partial x} + sin\theta\frac{\partial f}{\partial y},$$

$$(X_{2SE(2)}f)(x,y,\theta) = -sin\theta\frac{\partial f}{\partial x} + cos\theta\frac{\partial f}{\partial y},$$

$$(X_{3SE(2)}f)(x,y,\theta) = \frac{\partial f}{\partial \theta}.$$

Also, for $g \in \mathscr{D}(\mathbb{R}^2)$ and $h \in \mathscr{D}(SO(2))$, the vector fields are given as

$$(X_{\mathbb{R}^2}g)(x,y) = \frac{\partial g}{\partial x},$$

$$(X_{\mathbb{R}^2}g)(x,y) = \frac{\partial g}{\partial y},$$

$$(X_{SO(2)}h)(\theta) = \frac{\partial h}{\partial \theta}.$$

These vector fields are now used to discuss invariant differential operators on $SE(2)$ in what follows below.

**2.2.3 Definition**. Let $G$ be a Lie group. Let $x,g \in G$, then the left and right actions of $G$ are respectively defined as

$$L_g(x) = gx$$

and

$$R_g(x) = xg.$$

A linear differential operator $P$ on $G$ is said to be left or right invariant if it commutes with the left or right action of $G$. That is to say, $P$ satisfies the following condition

$$P(fL_g) = (Pf)L_g$$

or

$$P(fR_g) = (Pf)R_g,$$

for all $g \in G$, $f \in C^\infty(G)$. The operator $P$ is said to be bi-invariant if it is left and right invariant. Let $\mathscr{U}(\mathfrak{g})$ be regarded as the algebra of left invariant differential operators on $G$, where $\mathfrak{g}$ is the Lie algebra of $G$. Let $\mathrm{Z}(\mathfrak{g})$ be the center of $\mathscr{U}(\mathfrak{g})$, then the elements of $\mathrm{Z}(\mathfrak{g})$ are the bi-invariant differential operators on $G$. A differential operator $P$ on $SE(2)$ defined as ([2], p.7)

$$P = \sum_{\alpha,\beta,\gamma\in\mathbb{N}} a_{\alpha\beta\gamma} X^{\alpha}_{1SE(2)} X^{\beta}_{2SE(n)} X^{\gamma}_{3SE(2)}$$

is an invariant differential operator and so $P \in \mathscr{U}(\mathfrak{g})$. The next proposition gives explicit forms of P when it is K-bi-invariant and $SE(2)$-bi-invariant.

**2.2.4 Proposition**([2], lemma 11). Let P be left invariant linear differential operator on $SE(2)$.
(i) If P is K- bi-invariant, then

$$P = \sum_{\alpha,\beta\in\mathbb{N}} a_{\alpha\beta} (X^2_{1SE(2)} + X^2_{2SE(2)})^{\alpha} X^{\beta}_{3SE(2)}.$$

(ii) If P is $SE(2)$ - bi - invariant then

$$P = \sum_{\alpha\in\mathbb{N}} a_{\alpha} \left( X^2_{1SE(2)} + X^2_{2SE(2)} \right)^{\alpha}.$$

**Proof**: (i) P is K - bi- invariant if and only if it satisfies $[P, X_3] = 0$. This condition, in symmetric algebra, is written as

$$\begin{aligned}[P, X_3] &= \frac{\partial p}{\partial X_1}[X_1, X_3] + \frac{\partial P}{\partial X_2}[X_2, X_3] \\ &= -\frac{\partial P}{\partial X_1} X_2 + \frac{\partial P}{\partial X_2} X_1 \\ &= 0.\end{aligned}$$

This implies that P is of the form

$$P = Q(X_1^2 + X_2^2, X_3^2)$$

where $Q$ is a polynomial of two variables, that is,

$$P = \sum_{\alpha,\beta \in \mathbb{N}} a_{\alpha\beta} (X_{1SE(2)}^2 + X_{2SE(2)}^2)^{\alpha} X_{3SE(2)}^{\beta}$$

(ii) P is SE(2)-bi-invariant if and only if it satisfies $[P,X_1] = [P,X_2] = [P,X_3] = 0$. In the symmetric algebra, we have

$$\begin{aligned}[P,X_1] &= \frac{\partial P}{\partial Y}[X_2,X_1] + \frac{\partial P}{\partial X_3}[X_3,X_1] \\ &= \frac{\partial P}{\partial X_3} X_2 \\ &= 0.\end{aligned}$$

Therefore, $\frac{\partial P}{\partial X_3} = 0$. Also,

$$[P,X_2] = 0 \Rightarrow \frac{\partial P}{\partial X_3} = 0$$

and since $P$ is K-bi-invariant, we have

$$P = \sum_{\alpha \in \mathbb{N}} a_{\alpha} \left( X_{1SE(2)}^2 + X_{2SE(2)}^2 \right)^{\alpha}. \qquad \square$$

An explicit invariant differential operator on $SE(2)$, which is of interest in what follows in section 3, is the Laplace-Beltrami operator given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2}.$$

### 2.3. **Spherical functions on locally compact groups.**

Let G be a locally compact group, let K be a compact subgroup of G and let $L^1(G)$ be a convolution algebra of integrable functions on G. A bi-invariant function of a locally compact group is defined in 2.3.1 below.

**2.3.1 Definition**. A function

$$f : G \to \mathbb{C}$$

is said to be bi-invariant under K if it is constant on double coset of K. That is, if $f(k_1 g k_2) = f(g)\ \forall k_1, k_2 \in K$ and $\forall g \in G$ ([6] p.1).

Let $C_c(G)^K$ (resp. $L^1(G)^K$) be the set of continuous compactly supported (resp. $L^1$) functions that are bi-invariant under K. Then $C_c(G)^K$

(resp. $L^1(G)^K$) is a subalgebra of $C_c(G)$ (resp. $L^1(G)$), and the pair $(G,K)$ is a Gelfand pair if certain conditions are satisfied. These conditions are provided in the following definition.

**2.3.2 Definition** ([6], Definition 2). The pair (G,K) is called a Gelfand pair if $L^1(G)^K$ is a commutative algebra. In another formulation, the pair (G,K) is a called a Gelfand pair if the Banach $^*$ - algebra $L^1(K\backslash G/K)$ of a K-bi-invariant integrable functions on G is commutative.

An alternative definition of a Gelfand pair presented below may be found in ([5]) as proposition 6.1.3.

**2.3.3 Proposition**. Let G be a locally compact group and K a compact subgroup of G. Assume there exists a continuous involutive automorphism $\phi$ of G such that

$$\phi(x) \in Kx^{-1}K$$

for all $x \in G$. Then $(G,K)$ is a Gelfand pair.

Given a function $\varphi \in C(G)$ ( not necessarily compactly supported), a linear functional

$$\chi_\varphi : C_c(G) \to \mathbb{C}$$

is defined as

$$\chi_\varphi(f) = \int_G f(x)\varphi(x^{-1})dx.$$

We are now ready to define spherical function for a locally compact group $G$, and it is given below.

**2.3.4 Definition [1]**. A spherical function

$$\varphi : G \to \mathbb{C}$$

for the Gelfand pair (G,K) is a K-bi-invariant $C^\infty$ - function on K with $\varphi(e) = 1$ and satisfies one of the following three equivalent conditions

(1) $\int_K \varphi(xky)dx = \varphi(x)\varphi(y)$,
(2) $f \mapsto \int_G f(g)\overline{\varphi(g)}dg$ is a homomorphism of $C_c(K\backslash G/K)$ into $\mathbb{C}$,
(3) $\varphi$ is an eigenfunction of each $D \in \mathscr{D}(G/K)$, where $\mathscr{D}(G/K)$ is the algebra of K-invariant differential operators on $G/K$.

Also, a function $\varphi \in C(G)$, $\varphi \neq 0$, is said to be spherical if it is bi-invariant under K and $\chi_\varphi$ is a character of $C_c(G)^K$. That is, $\forall f, g \in C_c(G)^K$,

$$\chi_\varphi(f * g) = \chi_\varphi(f) \cdot \chi_\varphi(g).$$

## 3. Main Results

In this section, we determine an explicit form of the spherical function for the Euclidean Motion group $G = \mathbb{R}^n \rtimes SO(n)$. Before going on, we show that the pair $(\mathbb{R}^n \rtimes SO(n), SO(n))$ is a Gelfand pair, where $G = \mathbb{R}^n \rtimes SO(n)$ is the general Euclidean motion group. Elements of G are written as the pair $g = (a,k)$ with $a \in \mathbb{R}^n$, $k \in SO(2) =: K$. The pair is seen as the product of the rotation and the translation over $a$. This product is generally considered as an action of $G$ on $\mathbb{R}^n$ defined as

$$g \cdot x = k \cdot x + a,$$

where $a, x \in \mathbb{R}^n$ and $k \in SO(n)$. Now, given $k, k' \in K$ and $a, a' \in \mathbb{R}^n$, one can define a product in G as $(k,a)(k',a') = (k' \cdot a + a', kk')$. This shows that $(k,a) = (k,0)(1,a)$ following the product defined above. Let us define a map

$$\phi : G \to G$$

as

$$(k,a) \mapsto (k,-a) \text{ } or \text{ } \phi(k,a) = (k,-a)$$

Then $\phi$ is a continuous involutive automorphism of $G$ and $\phi(k,a) = [(k,0)(1,a)] = (k,0)(1,-a) = (k,0)(k,a)^{-1}(k,0)$. Therefore, $\phi(g) \in Kg^{-1}K$, $\forall g \in G$. Hence, $(G,K)$ is a Gelfand pair. This claim is supported by Proposition 2.1.3 above. Let us give an explicit derivation of spherical function for $G = \mathbb{R}^n \rtimes SO(n)$ as follows.

The Laplace-Beltrami operator on $SE(2)$(see ([9], p.5)) is given by

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial \psi^2} \tag{1}$$

Let the operator act on $\varphi = \varphi(r, \theta, \psi)$, then

$$\nabla^2 \varphi = \frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial \psi^2}. \tag{2}$$

We have the following elleptic partial differential equation

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} + \frac{\partial^2 \varphi}{\partial \psi^2} = 0. \tag{3}$$

Let us assume that the above equation has a solution of the form

$$\varphi(r, \theta, \psi) = R(r) F(\theta) \Psi(\psi). \tag{4}$$

Now,

$$\begin{cases} \dfrac{\partial\varphi}{\partial r} = F(\theta)\Psi(\psi)\dfrac{\partial R}{\partial r} \; and \; \dfrac{\partial^2\varphi}{\partial r^2} & = F(\theta)\Psi(\psi)\dfrac{\partial^2 R}{\partial r^2} \\ \dfrac{\partial^2\varphi}{\partial\theta^2} & = R(r)\Psi(\psi)\dfrac{\partial^2 F}{\partial\theta^2} \\ \dfrac{\partial^2\varphi}{\partial\psi^2} & = R(r)F(\theta)\dfrac{\partial^2\Psi}{\partial\psi^2}. \end{cases} \tag{5}$$

Substituting (5) into (3), we have

$$F(\theta)\Psi(\psi)\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{r}F(\theta)\Psi(\psi)\frac{\partial R(r)}{\partial r} + \frac{1}{r^2}R(r)\Psi(\psi)\frac{\partial^2 F}{\partial\theta^2} + R(r)F(\theta)\frac{\partial^2\Psi}{\partial\psi^2} = 0. \tag{6}$$

Divide (6) by $R(r)F(\theta)\Psi(\psi)$, then we get

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R(r)}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F(\theta)}{\partial\theta^2} + \frac{1}{\Psi(\psi)}\frac{\partial^2\Psi(\psi)}{\partial\psi^2} = 0,$$

so that

$$\frac{1}{R(r)}\frac{\partial^2 R(r)}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R(r)}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F(\theta)}{\partial\theta^2} = -\frac{1}{\Psi(\psi)}\frac{\partial^2\Psi(\psi)}{\partial\psi^2}. \tag{7}$$

The left hand side of (7) depends only on $(r,\theta)$ while the right hand side depends only on $\psi$. We can equate each side to a constant, say $-m^2$. Thus we get

$$-\frac{1}{\Psi(\psi)}\frac{\partial^2\Psi}{\partial\Psi^2} = -m^2 \;\Rightarrow\; \frac{1}{\Psi(\psi)}\frac{\partial^2\Psi}{\partial\Psi^2} = m^2 \Rightarrow \frac{\partial^2\Psi}{\partial\Psi^2} = m^2\Psi(\psi), \tag{8}$$

and

$$\frac{1}{R(r)}\frac{\partial^2 R}{\partial r^2} + \frac{1}{R(r)r}\frac{\partial R}{\partial r} + \frac{1}{F(\theta)r^2}\frac{\partial^2 F}{\partial\theta^2} = -m^2. \tag{9}$$

Multiplying (9) by $r^2$, we get:

$$\frac{r^2}{R(r)}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R(r)}\frac{\partial R}{\partial r} + \frac{1}{F(\theta)}\frac{\partial^2 F}{\partial\theta^2} = -m^2 r^2$$

so that

$$\frac{r^2}{R(r)}\frac{\partial^2 R}{\partial r^2} + \frac{r}{R(r)}\frac{\partial R}{\partial r} + m^2 r^2 = -\frac{1}{F(\theta)}\frac{\partial^2 F}{\partial\theta^2}. \tag{10}$$

Again equate both sides of (10) to $n^2$, where $n$ is a constant:

$$\frac{r^2}{R(r)}\frac{d^2 R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + m^2 r^2 = n^2 \tag{11}$$

$$-\frac{1}{F(\theta)}\frac{d^2F(\theta)}{d\theta} = n^2 \tag{12}$$

We can now solve the ordinary differential equation:

$$\frac{r^2}{R(r)}\frac{d^2R(r)}{dr^2} + \frac{r}{R(r)}\frac{dR(r)}{dr} + m^2r^2 - n^2 = 0. \tag{13}$$

Next, we transform this equation into Bessel equation. To do this, we let $mr = x$ so that $\frac{dx}{dr} = m$. Then

$$\begin{aligned}\frac{dR(r)}{dr} &= \frac{dR(r)}{dx}\frac{dx}{dr}\\ &= m\frac{dR(r)}{dx}.\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{d^2R(r)}{dr^2} &= \frac{d}{dr}\left(\frac{dR}{dr}\right)\\ &= \frac{d}{dr}\left(m\frac{dR(r)}{dx}\right)\\ &= m\frac{d^2R(r)}{dx^2}\frac{dx}{dr}\\ &= m\frac{d^2R(x)}{dx^2}m\\ &= m^2\frac{d^2R(x)}{dx^2}.\end{aligned}$$

Equation (13) becomes

$$m^2\frac{r^2}{R(r)}\frac{d^2R(r)}{dx^2} + \frac{mr}{R(r)}\frac{dR(r)}{dx} + (m^2r^2 - n^2) = 0. \tag{14}$$

Multiply (14) by $R(r)$ to get

$$m^2r^2\frac{d^2R(r)}{dx^2} + mr\frac{dR(r)}{dx} + (m^2r^2 - n^2)R(r) = 0.$$

Set $mr = x$; $m^2r^2 = x^2$, therefore,

$$x^2\frac{d^2R(r)}{dx^2} + x\frac{dR(r)}{dx} + (x^2 - n^2)R(r) = 0.$$

This may be re-written as

$$\frac{d^2R(r)}{dx^2} + \frac{1}{x}\frac{dR(r)}{dx} + \left(1 - \frac{n^2}{x^2}\right)R(r) = 0. \tag{15}$$

The differential equation (15) is a Bessel differential equation and it has a solution of the form

$$J_\lambda(mr) = \Gamma\left(\frac{n}{2}\right)\sum_{k=0}^{\infty}\frac{\lambda^k}{k!\Gamma(k+\frac{n}{2})}\left(\frac{mr}{2}\right)^{2k} \tag{16}$$

$$= \Gamma\left(\frac{n}{2}\right)\left(\frac{\sqrt{\lambda}r}{2}\right)^{\frac{2-n}{2}} I_{\frac{n-2}{2}}\left(\sqrt{\lambda}r\right), \tag{17}$$

where $I_\nu$ is the Bessel function of index $\nu$. Different values of $\lambda$ will give different solutions. In our own case, we are considering $SE(2)$, that is, $n = 2$. Therefore, (17) can be further simplified to be

$$J_\lambda(mr) = \Gamma(1)\left(\frac{\sqrt{\lambda}r}{2}\right)^{0} I_{\frac{2-2}{2}}\left(\sqrt{\lambda}r\right) \tag{18}$$

$$= I_0\left(\sqrt{\lambda}r\right). \qquad \square \tag{19}$$

Expression (19) is the desired spherical function for $SE(2)$, generally referred to as the Bessel function of order zero. Thus we have proved, in particular, the following

**3.1 Theorem.** The spherical function on $SE(2)$ is the Bessel function $I_0$ of order zero.

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