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AN INERTIAL SELF ADAPTIVE ALGORITHM FOR SOLVING EQUILIBRIUM, FIXED POINT AND PSEUDOMONOTONE VARIATIONAL INEQUALITY PROBLEMS IN HILBERT SPACES.

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ABSTRACT. In this paper, we study an iterative approximation of a common solution to equilibrium, fixed point and variational inequality problems. We introduced an inertial Tseng method with a viscosity approach for approximating a solution to the problem in a Hilbert space. The two methods used in this research work enhance the convergence rate of the proposed algorithm. Under mild conditions, we show that the sequence generated converges strongly to a common solution of the fixed point and variational inequality problems associated with demicontractive and pseudomonotone operator which is also a solution to a generalized equilibrium problem. Our results extend and improve several existing results in literature.

1. INTRODUCTION

Throughout this paper, $\mathbb{R}$ denotes the set of real numbers, $\mathbb{N} = \{n \in \mathbb{Z} | n \geq 0\}$ denotes the set of natural numbers, H a real Hilbert space and C a nonempty, closed and convex subset of H. Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C be a nonempty, closed and convex subset of H, let $\Phi : C \times C \to \mathbb{R}$ and $\varphi : C \times C \to \mathbb{R}$ be two bifunctions. Then, the Generalized Equilibrium Problem (GEP) is defined as finding a point $x^* \in C$ such that

$$\Phi(x^*, x) + \varphi(x^*, x) \geq 0, \quad \forall x \in C. \tag{1.1}$$

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The set of solution of inequality (1.1) is denoted by GEP (Φ, φ). When $\varphi = 0$ the GEP reduces to classical Equilibrium Problem (EP) introduced by Blum and Oettli [7]. Over the years, the equilibrium problems have proven to be very effective for studying a wide class of problems in Network, transportation, image recovery, finance and Economics; see [2, 9, 10, 13, 14, 31, 32, 35, 37, 40, 44, 47] and the references therein. Also, the EP has served as a unifying framework for the study of Fixed Point Problems (FPP), Saddle Point Problems (SPP), variational inclusion problems, variational inequality problems, nonlinear complementary problems, the Nash equilibrium problems and so on. Many iterative algorithms for solving EPs and related OPs have been studied and proposed by several authors, see [1, 3, 11, 16, 24, 25, 28, 41, 54–56] .
Let $S : C \to C$ be a nonlinear mapping. A point $x^* \in C$ is called a fixed point of S if $Sx^* = x^*$. The set of all fixed points of S, denoted by $F(S)$ is given as

$$F(S) = \{x^* \in C : Sx^* = x^*\}. \tag{1.2}$$

A lot of problems in sciences and engineering can be formulated as finding solution of FPP of a nonlinear mapping. Recent studies of optimization problems (OPs) dealing with finding a common solution of the set of fixed points of a nonlinear mapping and the set of solutions to equilibrium problems have so far been carried out by many authors. Some authors who have considered the problem of finding a common solution between the sets of solution of equilibrium problems and fixed point problems of a nonlinear mapping in inequality (1.1) and equation (1.2) are in [26, 52, 55].
Let $A : C \to H$ be a nonlinear mapping. The VIP denoted by $VI(C,A)$ is to find $x^* \in C$ such that

$$\langle Ax^*, x - x^* \rangle \geq 0, \forall x \in C. \tag{1.3}$$

Ever since the independent introduction of variational inequality theory by Fichera and Stampacchia, it has become a vital tool in mathematical analysis and has applications in many fields of study such as optimization theory, physics, economics, engineering and many others (See [6, 20, 33, 34] and the references therein). Over the years, various effective solution methods have been investigated and developed to solve the problems of VIP (See [12, 46, 57] and the references therein).
It is our concern in this article to study a common solution to the equilibrium problem, variational inequality and fixed point problem. The potential application to mathematical models whose constraints can be expressed as fixed point problems and equilibrium problems is the motivation for studying such a common solution problem. For this reason,

several researchers have considered iterative approximations of solution to fixed point and optimization problems in different spaces of choice (see [26, 27, 29, 50] and the references therein). In 2008, Plubtieng et al. [43] introduced the following iterative scheme for finding the common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for an α-inverse stongly monotone mappings. They showed that the sequence converges strongly to a common element of the three sets under consideration. To be precised, they proposed the following iterative method:

Algorithm 1.1.

$$\begin{cases} F(u_n,y)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle \geq 0\ \forall y\in C. \\ y_n = P_C(u_n-\lambda_n Au_n). \\ x_{n+1}=\alpha_n u+\beta_n x_n+\gamma_n SP_C(y_n-\lambda_n Ay_n). \end{cases}$$

Recently, Gang et al. [22] proposed the following inertial Tseng's extragradient algorithm for approximating the common solution of pseudomonotone variational inequality problem and fixed point problem for nonexpansive mappings in real Hilbert spaces:

Algorithm 1.2.

$$\begin{cases} x_0,x_1\in H. \\ w_n = x_n+\theta_n(x_n-x_{n-1}). \\ y_n = P_C(w_n-\gamma Aw_n). \\ z_n = y_n-\gamma(Ay_n-Aw_n). \\ x_{n+1}=\alpha_n f(x_n)+(1-\alpha_n)[\beta_n Sz_n+(1-\beta_n)z_n]. \end{cases}$$

where f is a contraction, S is a nonexpansive mapping, A is pseudomonotone, L-Lipschitz and sequentially weakly continuous and $\gamma \in (0,\frac{1}{L})$. The authors proved a strong convergence result for the proposed algorithm under some suitable conditions.

Owolabi et al. [42] proposed the following iterative scheme for finding common solutions to equilibrium problem, variational inclusion problem and fixed point problem for an infinite family of strict pseudocontractive mappings.

Algorithm 1.3.

$$\hat{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\theta_n}{\|x_n-x_{n-1}\|}\right\}, & \textit{if } x_n \neq x_{n-1}, \\ \alpha, & \textit{otherwise.} \end{cases}$$

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ F(u_n, y) + \frac{1}{r_n}(y - u_n, u_n - w_n) \geq 0, \ \forall y \in H, \\ v_n = \delta_n w_n + (1-\delta_n)u_n, \\ z_n = (I + \lambda_n B)^{-1}(I - \lambda_n A)v_n, \\ x_{n+1} = \beta_n f(x_n) + \xi_n x_n + \mu_n W_n z_n. \end{cases}$$

Motivated by the above results, we consider the problem of approximating a common solution to an equilibrium problem, fixed point problem and variational inequality problem in a real Hilbert space. An inertial Tseng iterative method is introduced and combined with the viscosity technique. We showed that the ensuing sequences from this method converge under some mild conditions to a common solution to the fixed point problem and variational inequality problem associated with demicontractive and pseudomonotone operators which is also a solution to a generalized equilibrium problem.

The organizational structure of our paper is built as follows. In Section 2, we give relevant definitions and lemmas needed for use in the subsequent sections. In Section 3, we propose an algorithm and analyze its convergence in Section 4. We give numerical examples of our proposed algorithm in Section 5 and finally in Section 6, we give a concluding remark.

2. Preliminaries

Let H be a real Hilbert space and C be a nonempty, closed and convex subset of H. The weak convergence of x_n to x is denoted by $x_n \rightharpoonup x$ as $n \to \infty$, while the strong convergence of x_n to x is written as $x_n \to x$ as $n \to \infty$.

Lemma 2.1. *[15] For each $x, y \in H$ and $\delta \in \mathbb{R}$, there holds*

(1) $\|x+y\|^2 \leq \|x\|^2 + 2\langle y, x+y \rangle$
(2) $\|x+y\|^2 = \|x\|^2 + 2\langle x, y \rangle + \|y\|^2$.
(3) $\|\delta x + (1-\delta)y\|^2 = \delta\|x\|^2 + (1-\delta)\|y\|^2 - \delta(1-\delta)\|x-y\|^2$.

For all $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C.$$

P_C is called *metric projection* of H onto C. It is known that P_C is nonexpansive.

Lemma 2.2. *[23] Let C be a nonempty, closed and convex subset of a real Hilbert space H. For any $x \in H$ and $z \in C$, we have*

$$z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0 \; \forall y \in C.$$

Lemma 2.3. *[23] Let C be a closed and convex subset in a real Hilbert space H and let $x \in H$. Then we have the following:*

(1) $\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y \rangle \; \forall \; y \in H.$

(2) $\|x - P_C x\|^2 \leq \|x - y\|^2 - \|y - P_C x\|^2 \; \forall \; y \in C.$

Definition 2.4 ([48][58])**.** Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For any elements $x, y \in C$, a mapping $A : C \to H$ is said to be:
(1) c-strict pseudocontractive, if there exists $c \in [0,1)$ such that

$$\|Ax - Ay\|^2 \leq \|x - y\|^2 + c\|x - Ax - (y - Ay)\|^2,$$

(2) c-demicontractive, if for any $x \in C$ and $q \in F(A)$ with $c \in [0,1)$,

$$\|Ax - q\|^2 \leq \|x - q\|^2 + c\|x - Ax\|^2,$$

(3) pseudomonotone, if

$$\langle Ax, y - x \rangle \geq 0 \Rightarrow \langle Ay, y - x \rangle \geq 0,$$

(4) sequentially weakly continuous on H, if for each sequence $\{x_n\}$, we have

$$x_n \rightharpoonup x \Rightarrow Ax_n \rightharpoonup Ax.$$

Definition 2.5. [19] A bounded linear operator D on H is called strongly positive if there exists a constant $\hat{\gamma} > 0$ such that

$$\langle Dx, x \rangle \geq \hat{\gamma}\|x\|^2, \; \forall x \in H.$$

Lemma 2.6. *[18] Let $\Phi : C \times C \to \mathbb{R}$ and $\varphi : C \times C \to \mathbb{R}$ be two bifunctions satisfying the following assumptions:*
(C1) $\Phi(x,x) \geq 0 \; \forall x \in C$;
(C2) Φ *is monotone i.e* $\Phi(x,y) + \Phi(y,x) \leq 0 \; \forall x, y \in C$;
(C3) Φ *is upper hemicontinuous , i.e for each* $x, y, z \in C$ $\limsup_{t \to \infty} \Phi(tz + (1-t)x, y) \leq \Phi(x,y)$;
(C4) For each $x \in C$ fixed, the function $y \mapsto \Phi(x,y)$ is convex and lower semicontinuous;
(C5) $\varphi(x,x) \geq 0$ for all $x \in C$;

(C6) For each $y \in C$ fixed, the function $x \to \varphi(x,y)$ is upper semicontinuous;
(C7) For each $x \in C$ fixed, the function $y \to \varphi(x,y)$ is convex and lower semicontinuous,
and assume that for fixed $r > 0$ and $z \in C$, there exists a nonempty compact subset K of H and $x \in C \cap K$ such that:

$$\Phi(y,x) + \varphi(y,x) + \frac{1}{r}\langle y - x, x - z\rangle < 0, \forall y \in C \setminus K$$

.

Lemma 2.7. *[18] Let C be a nonempty, closed and convex subset of a real Hilbert space H. Assume that the bifunction $\Phi, \varphi : C \times C \to \mathbb{R}$ be bifunctions satisfying the assumptions C1-C7 in Lemma 2.6 and φ is monotone.*
For $r > 0$ and for all $x \in H$ define a mapping $T_r^{(\Phi,\varphi)} : H \to C$ as follows:

$$T_r^{(\Phi,\varphi)}(x) = \{z \in C : \Phi(z,y) + \varphi(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}.$$

Then, the following hold:
(i) $T_r^{(\Phi,\varphi)}$ is single-valued;
(ii) $T_r^{(\Phi,\varphi)}$ is firmly nonexpansive i.e

$$\|T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y\|^2 \leq \langle T_r^{(\Phi,\varphi)}x - T_r^{(\Phi,\varphi)}y, x - y\rangle \ \forall x, y \in H.$$

(iii) $F(T_r^{(\Phi,\varphi)}) = GEP(\Phi,\varphi)$.
(iv) $GEP(\Phi,\varphi)$ is compact and convex.

Lemma 2.8. *[45] Let $\{a_n\}$ be a sequence of non-negative real numbers, $\{\alpha_n\}$ be a sequence in $(0,1)$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{b_n\}$ be a sequence of real numbers. Assume that $a_{n+1} \leq (1-\alpha_n)a_n + \alpha_n b_n$ for all $n \geq 1$. If $\limsup_{k\to\infty} b_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfying*

$$\liminf_{k\to\infty}(a_{n_{k+1}} - a_{n_k}) \geq 0,$$

then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.9. *[39] Let D be a self-adjoint strongly positive bounded linear operator on a Hilbert space with coefficient $\bar{\gamma} > 0$ and $0 < \rho \leq \|D\|^{-1}$. Then $\|I - \rho D\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 2.10. *[5] Consider the problem with C being a nonempty, closed, convex subset of a real Hilbert space H and $A : C \to H$ being pseudomonotone and continuous. Then p is a solution of inequality* (1.3) *if and only if*

$$\langle Ax, x-p\rangle \geq 0,\ \forall x \in C.$$

3. MAIN RESULTS

In this section, we introduce a new inertial projection and viscosity approximation method with a self adaptive technique for solving generalized equilibrium problem, fixed point and variational inequality problems. The following conditions are assumed throughout the paper.
Assumption 3.1. Suppose:
Condition A.

- (A1) The feasible set C is a nonempty, closed and convex subset of a real Hilbert space H;
- (A2) The associated operator $A : H \to H$ is pseudomonotone, L-Lipschitz and sequentially weakly continuous on bounded subset of H;
- (A3) $\Phi, \varphi : C \times C \to \mathbb{R}$ are two bifunctions satisfying the assumptions in Lemma 2.6 and $GEP(\Phi, \varphi)$ is the solution set of generalized equilibrium problem of the two bifunctions.
- (A4) The mapping $S : H \longrightarrow H$ is c-demicontractive.
- (A5) The solution set $\Omega := GEP(\Phi, \varphi) \cap F(S) \cap VI(C, A)$ is nonempty;

Condition B.

- (B1) The function $f : H \longrightarrow H$ is ρ-contractive with $\rho \in [0, 1)$ and the mapping $D : H \to H$ is a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ such that $0 < \gamma < \frac{\bar{\gamma}}{\rho}$;
- (B2) The control sequence $\{\alpha_n\}$, satisfy
  $\{\alpha_n\} \subset (0, 1), \lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=0}^{\infty} \alpha_n = \infty$;
- (B3) The control sequence $\{\varepsilon_n\}$, such that $\{\varepsilon_n\} \subset [0, \infty)$ and $\varepsilon_n = \circ(\alpha_n)$,
- (B4) $\liminf\limits_{n\to\infty}(\beta_n - c) > 0$.

We now present our Algorithm.

Algorithm 3.1. *Inertial self adaptive method.*

Initialization: Given $\delta > 0, \gamma_1 > 0, \phi \in (0, 1)$. Let $x_0, x_1 \in H$ be two initial points.
Iterative steps: Calculate x_{n+1} as follows;

$$\begin{cases} s_n = x_n + \delta_n(x_n - x_{n-1}), \\ u_n = T_r^{(\Phi,\varphi)}(s_n), \\ v_n = P_C(u_n - \gamma_n A u_n), \\ z_n = v_n - \gamma_n(Av_n - Au_n), \\ x_{n+1} = \alpha_n \xi f(x_n) + (I - \alpha_n D)(\beta_n z_n + (1-\beta_n) S z_n), \end{cases} \tag{3.1}$$

where δ_n and γ_n are updated by (3.2) *and* (3.3) *respectively.*

$$\delta_n = \begin{cases} \min\left\{ \frac{\varepsilon_n}{\|x_n - x_{n-1}\|},\ \delta \right\}, & \text{if } x_n \neq x_{n-1}, \\ \delta, & \text{otherwise.} \end{cases} \tag{3.2}$$

$$\gamma_{n+1} = \begin{cases} \min\left\{ \frac{\phi\|u_n - v_n\|}{\|Au_n - Av_n\|},\ \gamma_n \right\}, & \text{if } Au_n - Av_n \neq 0 \\ \gamma_n, & \text{otherwise.} \end{cases} \tag{3.3}$$

Remark 3.2. By condition $B3$, one can verify from equation (3.3) that $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$. Indeed, we see that $\delta_n\|x_n - x_{n-1}\| \leq \varepsilon_n,\ \forall n$ combined with $\lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$ gives $\lim_{n\to\infty} \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| \leq \lim_{n\to\infty} \frac{\varepsilon_n}{\alpha_n} = 0$.

4. Convergence Analysis

We first establish some lemmas needed to prove the strong convergence theorem for the proposed algorithm.

Lemma 4.1. *[58] The sequence γ_n given by Equation* (3.3) *is nonincreasing and $\lim_{n\to\infty} \gamma_n = \gamma \geq \min\{\gamma_1, \frac{\phi}{L}\}$.*

Lemma 4.2. *Let $\{u_n\}$ and $\{v_n\}$ be two sequences generated by Algorithm 3.1 such that conditions (A1)-(A3) hold. If there exists a subsequence $\{u_{n_k}\}$ which is weakly convergent to $u \in H$ and $\lim_{k\to\infty}\|u_{n_k} - v_{n_k}\| = 0$, then $u \in VI(C,A)$.*

Proof. From Equation (3.1), $v_n = P_C(u_n - \gamma_n A u_n)$. By the characterization of the projection map we have

$$\langle u_{n_k} - \gamma_{n_k} A u_{n_k} - v_{n_k}, x - v_{n_k} \rangle \leq 0,\ \forall x \in C,$$

which implies that

$$\frac{1}{\gamma_{n_k}} \langle u_{n_k} - v_{n_k}, x - v_{n_k} \rangle \leq \langle A u_{n_k}, x - v_{n_k} \rangle,\ \forall x \in C.$$

From this we obtain

$$\frac{1}{\gamma_{n_k}}\langle u_{n_k}-v_{n_k},x-v_{n_k}\rangle+\langle Au_{n_k},v_{n_k}-u_{n_k}\rangle\leq\langle Au_{n_k},u_{n_k}\rangle,\ \forall x\in C. \quad (4.1)$$

Since the subsequence $\{u_{n_k}\}$ is weakly convergent to $u\in H$, then $\{u_{n_k}\}$ is a bounded subsequence. By the Lipschitz continuity of A and $\|u_{n_k}-v_{n_k}\|\to 0$, we have that $\{Au_{n_k}\}$ and $\{v_{n_k}\}$ are bounded as well. Since $\gamma_{n_k}\geq\min\{\gamma_1,\frac{\phi}{L}\}$, by applying inequality (4.1) we have

$$\liminf_{k\to\infty}\langle Au_{n_k},x-u_{n_k}\rangle\geq 0,\forall x\in C. \quad (4.2)$$

Observe that

$$\langle Av_{n_k},x-v_{n_k}\rangle=\langle Av_{n_k}-Au_{n_k},x-u_{n_k}\rangle+\langle Au_{n_k},x-u_{n_k}\rangle+\langle Av_{n_k},u_{n_k}-v_{n_k}\rangle. \quad (4.3)$$

Since $\|u_{n_k}-v_{n_k}\|\to 0$, then by the Lipschitz continuity of A we have $\lim_{k\to\infty}\|Au_{n_k}-Av_{n_k}\|=0$. This combined with inequality (4.2) and Equation (4.3) implies

$$\liminf_{k\to\infty}\langle Av_{n_k},x-v_{n_k}\rangle\geq 0.$$

Now, let $\{\Phi_k\}$ be a decreasing sequence of positive numbers such that $\Phi_k\to 0$ as $k\to\infty$. Let N_k represent the smallest positive integer for any k such that

$$\langle Av_{n_j},x-v_{n_j}\rangle+\Phi_k\geq 0\quad\forall j\geq N_k. \quad (4.4)$$

Clearly, the sequence $\{N_k\}$ is increasing since $\{\Phi_k\}$ is decreasing. From $\{v_{N_k}\}\subset C$, for any k, suppose $Av_{N_k}\neq 0$ (otherwise v_{N_k} is a solution of inequality (1.3)), let

$$\Psi_{N_k}=\frac{Av_{N_k}}{\|Av_{N_k}\|^2}.$$

Then, $\langle Av_{N_k},\Psi_{N_k}\rangle=1$ for each k. From inequality (4.4), we obtain

$$\langle Av_{N_k},x+\Phi_k\Psi_{N_k}-v_{N_k}\rangle\geq 0\quad\forall k.$$

By the pseudomonotonicity of A, we get

$$\langle A(x+\Phi_k\Psi_{N_k}),x+\Phi_k\Psi_{N_k}-v_{N_k}\rangle\geq 0,$$

which gives

$$\langle Ax,x-v_{N_k}\rangle\geq\langle Ax-A(x+\Phi_k\Psi_{N_k}),x+\Phi_k\Psi_{N_k}-v_{N_k}\rangle-\Phi_k\langle Ax,\Psi_{N_k}\rangle. \quad (4.5)$$

We now show that $\lim_{k\to\infty}\Phi_k\Psi_{N_k}=0$. We get that $v_{N_k}\rightharpoonup u$ since $u_{n_k}\rightharpoonup u$ and $\lim_{k\to\infty}\|u_{n_k}-v_{n_k}\|=0$. From $\{v_n\}\subset C$, we have $u\in C$. By the sequentially weak continuity of A on C, we have $\{Av_{n_k}\}\rightharpoonup Au$. We can assume that $Au\neq 0$ (otherwise, u is a solution of inequality (1.3)).

Since the norm mapping is sequentially weakly lower semicontinuous, we have:

$$0 < \|Au\| \leq \lim_{k\to\infty} \|Av_{n_k}\|.$$

By the fact that $\{v_{N_k}\} \subset \{v_{n_k}\}$ and $\Phi_k \to 0$ as $k \to \infty$, we have

$$0 \leq \limsup_{k\to\infty} \|\Phi_k \Psi_{N_k}\| = \limsup_{k\to\infty} \Big(\frac{\Phi_k}{\|Av_{n_k}\|} \Big) \leq \frac{\limsup\limits_{k\to\infty} \Phi_k}{\liminf\limits_{k\to\infty} \|Av_{n_k}\|} = 0,$$

which implies that $\limsup\limits_{k\to\infty} \Phi_k \Psi_{N_k} = 0$. From the facts that A is Lipschitz continuous, $\{v_{N_k}\}$ and $\{\Psi_{N_k}\}$ are bounded and $\lim\limits_{k\to\infty} \Phi_k \Psi_{N_k} = 0$, it follows from inequality (4.5) that

$$\liminf_{k\to\infty} \langle Ax, x - v_{N_k} \rangle \geq 0.$$

Hence, we obtain

$$\langle Ax, x-u \rangle = \lim_{k\to\infty} \langle Ax, x - v_{N_k} \rangle = \liminf_{k\to\infty} \langle Ax, x - v_{N_k} \rangle \geq 0, \ \forall x \in C.$$

By invoking Lemma 2.10, we obtain $u \in VI(C,A)$ as required. □

Lemma 4.3. *Let $\{u_{n_k}\}$ be a subsequence of $\{u_n\}$ defined by Algorithm 3.1 such that $u_{n_k} \rightharpoonup u \in C$. Suppose $\|u_{n_k} - s_{n_k}\| \to 0$ as $k \to \infty$. Then $u \in GEP(\Phi, \varphi)$.*

Proof. Since $u_{n_k} = T_r^{(\Phi,\varphi)}(s_{n_k})$, we have

$$\Phi(u_{n_k}, y) + \varphi(u_{n_k}, y) + \frac{1}{r} \langle y - u_{n_k}, u_{n_k} - s_{n_k} \rangle \geq 0, \ \forall y \in C.$$

It follows from the monotonicity of Φ and φ that

$$\frac{1}{r} \langle y - u_{n_k}, u_{n_k} - s_{n_k} \rangle \geq \Phi(y, u_{n_k}) + \varphi(y, u_{n_k}).$$

It follows from $\|u_{n_k} - s_{n_k}\| \to 0$ and $u_{n_k} \rightharpoonup u$, that

$$\Phi(y,u) + \varphi(y,u) \leq 0, \ \forall y \in C.$$

Let $y_t = ty + (1-t)u$ for any $t \in (0,1]$ and $y \in C$. Then, we have $y_t \in C$ and hence

$$\Phi(y_t, u) + \varphi(y_t, u) \leq 0.$$

Using the assumptions $C1$ and $C4$ in Lemma 2.6, we get

$$\begin{aligned} 0 &\leq \Phi(y_t, y_t) + \varphi(y_t, y_t) \\ &\leq t(\Phi(y_t, y) + \varphi(y_t, y)) + (1-t)(\Phi(y_t, u) + \varphi(y_t, u)) \\ &\leq \Phi(y_t, y) + \varphi(y_t, y). \end{aligned} \tag{4.6}$$

Hence we have $\Phi(y_t,y)+\varphi(y_t,y)\geq 0$. Letting $t\to 0$ and using assumption $C3$ in Lemma 2.6, by the upper semicontinuity of φ, we have

$$\Phi(u,y)+\varphi(u,y)\geq 0,\ \forall y\in C.$$

Therefore $u\in GEP(\Phi,\varphi)$. □

Lemma 4.4. *Let sequences $\{u_n\}$, $\{v_n\}$ and $\{z_n\}$ be given as in Algorithm 3.1 and suppose Assumptions A and B hold. Then we have the following.*

$$\|z_n-p\|^2\leq\|u_n-p\|^2-(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2})\|u_n-v_n\|^2,\ \forall p\in\Omega.$$

Proof. Using the definition of z_n, we have

$$\begin{aligned}\|z_n-p\|^2&=\|v_n-\gamma_n(Av_n-Au_n)-p\|^2\\&=\|v_n-p\|^2+\gamma_n^2\|Av_n-Au_n\|^2-2\gamma_n\langle v_n-p,Av_n-Au_n\rangle\\&=\|u_n-p\|^2+\|v_n-u_n\|^2+2\langle v_n-u_n,u_n-p\rangle+\gamma_n^2\|Av_n-Au_n\|^2\\&-2\gamma_n\langle v_n-p,Av_n-Au_n\rangle\\&=\|u_n-p\|^2+\|v_n-u_n\|^2-2\langle v_n-u_n,v_n-u_n\rangle+2\langle v_n-u_n,v_n-p\rangle\\&+\gamma_n^2\|Av_n-Au_n\|^2-2\gamma_n\langle v_n-p,Av_n-Au_n\rangle\\&=\|u_n-p\|^2-\|v_n-u_n\|^2+2\langle v_n-u_n,v_n-p\rangle+\gamma_n^2\|Av_n-Au_n\|^2\\&-2\gamma_n\langle v_n-p,Av_n-Au_n\rangle.\end{aligned}\tag{4.7}$$

From $v_n=P_C(u_n-\gamma_nAu_n)$ and the property of metric projection, we get $\langle v_n-u_n+\gamma_nAu_n,v_n-p\rangle\leq 0$ which implies $\Rightarrow\langle v_n-u_n,v_n-p\rangle\leq -\gamma_n\langle Au_n,v_n-p\rangle$.
Using this and Equation (3.3) in Equation (4.7), we have:

$$\begin{aligned}\|z_n-p\|^2&\leq\|u_n-p\|^2-\|v_n-u_n\|^2-2\gamma_n\langle Au_n,v_n-p\rangle+\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\|u_n-v_n\|^2\\&-2\gamma_n\langle v_n-p,Av_n-Au_n\rangle\\&\leq\|u_n-p\|^2-\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2-2\gamma_n\langle v_n-p,Av_n\rangle.\end{aligned}\tag{4.8}$$

From $p\in VI(C,A)$, we have $\langle Ap,v_n-p\rangle\geq 0$. We obtain from the fact that A is pseudomonotone, that

$$\langle Av_n,v_n-p\rangle\geq 0.$$

It follows therefore, from inequality (4.8), that

$$\|z_n - p\|^2 \le \|u_n - p\|^2 - \Big(1 - \phi^2 \frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n - v_n\|^2.$$

□

Lemma 4.5. *Let $\{x_n\}$ be a sequence generated by Algorithm 3.1. Then, $\{x_n\}$ is bounded. Consequently, the sequences $\{z_n\}, \{v_n\}$, $\{u_n\}$ and $\{w_n\}$ are bounded.*

Proof. Let $p \in \Omega$, then from Equation 3.1 we have that

$$\begin{aligned}\|s_n - p\| &= \|x_n + \delta_n(x_n - x_{n-1}) - p\| \\ &\le \|x_n - p\| + \delta_n\|x_n - x_{n-1}\| \\ &= \|x_n - p\| + \alpha_n . \frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\|. \end{aligned} \tag{4.9}$$

From Remark 3.2, we have $\frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| \to 0$, thus there exists a constant $N_1 > 0$, such that

$$\frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| \le N_1,\ \forall n \ge 1. \tag{4.10}$$

Hence,

$$\|s_n - p\| \le \|x_n - p\| + \alpha_n N_1,\ \forall n \ge n_0. \tag{4.11}$$

Again from Algorithm 3.1, we have

$$\begin{aligned}\|u_n - p\| &= \|T_r^{\Phi,\varphi} s_n - T_r^{\Phi,\varphi} p\| \\ &\le \|s_n - p\|. \end{aligned} \tag{4.12}$$

Now, let $y_n = \beta_n z_n + (1-\beta_n)Sz_n$. Then,

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_n z_n + (1-\beta_n)Sz_n - p\|^2 \\ &= \|\beta_n(z_n - p) + (1-\beta_n)(Sz_n - p)\|^2 \\ &= \beta_n\|z_n - p\|^2 + (1-\beta_n)\|Sz_n - p\|^2 - \beta_n(1-\beta_n)\|z_n - Sz_n\|^2 \\ &\le \beta_n\|z_n - p\|^2 + (1-\beta_n)[\|z_n - p\|^2 + c\|z_n - Sz_n\|^2] - \beta_n(1-\beta_n)\|z_n - Sz_n\|^2 \\ &= \|z_n - p\|^2 + (1-\beta_n)c\|z_n - Sz_n\|^2 - \beta_n(1-\beta_n)\|z_n - Sz_n\|^2 \\ &= \|z_n - p\|^2 - (1-\beta_n)(\beta_n - c)\|z_n - Sz_n\|^2 \\ &\le \|z_n - p\|^2, \end{aligned} \tag{4.13}$$

which implies that $\|y_n - p\| \le \|z_n - p\|$.

Finally,

$$\begin{aligned}
\|x_{n+1}-p\| &= \|\alpha_n\xi f(x_n)+(I-\alpha_n D)y_n-p\| \\
&= \|\alpha_n\xi f(x_n)-\alpha_n Dp+(I-\alpha_n D)(y_n-p)\| \\
&\le \alpha_n\|\xi f(x_n)-Dp\|+(1-\alpha_n\hat{\gamma})\|z_n-p\| \\
&\le \alpha_n(\|\xi f(x_n)-\xi f(p)\|+\|\xi f(p)-Dp\|)+(1-\alpha_n\hat{\gamma})\|z_n-p\| \\
&\le \alpha_n\xi\rho\|x_n-p\|+\alpha_n\|\xi f(p)-Dp\|+(1-\alpha_n\hat{\gamma})(\|x_n-p\|+\alpha_n N_1) \\
&= (1-\alpha_n(\hat{\gamma}-\xi\rho))\|x_n-p\|+\alpha_n\|\xi f(p)-Dp\|+(1-\alpha_n\hat{\gamma})\alpha_n N_1 \\
&= (1-\alpha_n(\hat{\gamma}-\xi\rho))\|x_n-p\|+\alpha_n(\hat{\gamma}-\xi\rho)\Big\{\frac{\|\xi f(p)-Dp\|}{\hat{\gamma}-\xi\rho}+\frac{(1-\alpha_n\hat{\gamma})}{\hat{\gamma}-\xi\rho}N_1\Big\} \\
&\le (1-\alpha_n(\hat{\gamma}-\xi\rho))\|x_n-p\|+\alpha_n(\hat{\gamma}-\xi\rho)N_2, \qquad (4.14)
\end{aligned}$$

where $N_2 = sup_{n\in\mathbb{N}}\Big\{\frac{\|\xi f(p)-Dp\|}{\hat{\gamma}-\xi\rho}+\frac{(1-\alpha_n\hat{\gamma})}{\hat{\gamma}-\xi\rho}N_1\Big\}$. Therefore, $\{\|x_n-p\|\}$ is bounded and thus $\{x_n\}$ is bounded. Consequently, $\{s_n\},\{u_n\},\{z_n\},\{A(u_n)\}$ and $\{S(z_n)\}$ are bounded. □

Lemma 4.6. *Suppose $\{x_n\}$ is the sequence generated by Algorithm 3.1, then the following inequality hold for all $p\in\Omega$ and $n\in\mathbb{N}$*

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le& \left[1-\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\right]\|x_n-p\|^2 \\
&+\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\left(\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2\right. \\
&\left.+\frac{1}{(\hat{\gamma}-\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\right)-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2,
\end{aligned} \qquad (4.15)$$

and

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le& \left[1-\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\right]\|x_n-p\|^2 \\
&+\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\left(\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2\right. \\
&\left.+\frac{1}{(\hat{\gamma}-\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\right)-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\|u_n-s_n\|^2 \\
&-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2. \qquad (4.16)
\end{aligned}$$

Proof. Clearly, it follows from Equation (3.2) that

$$\begin{aligned}\|s_n - p\|^2 &= \|x_n + \delta_n(x_n - x_{n-1}) - p\|^2 \\ &= \|x_n - p + \delta_n(x_n - x_{n-1})\|^2 \\ &= \|x_n - p\|^2 + 2\delta_n\langle x_n - p, x_n - x_{n-1}\rangle + \delta_n^2\|x_n - x_{n-1}\|^2 \\ &\le \|x_n - p\|^2 + 2\delta_n\|x_n - p\|\|x_n - x_{n-1}\| + \delta_n^2\|x_n - x_{n-1}\|^2 \\ &= \|x_n - p\|^2 + \delta_n\|x_n - x_{n-1}\|(2\|x_n - p\| + \delta_n\|x_n - x_{n-1}\|) \\ &\le \|x_n - p\|^2 + \alpha_n \cdot \frac{\delta_n}{\alpha_n}N_3\|x_n - x_{n-1}\|,\end{aligned}$$

where $N_3 = \sup_{n\in\mathbb{N}}(2\|x_n - p\| + \delta_n\|x_n - x_{n-1}\|)$. From Equation (3.2), we have,

$$\begin{aligned}\|x_{n+1} - p\|^2 &= \|\alpha_n(\xi f(x_n) - Dp) + (I - \alpha_n D)(y_n - p)\|^2 \\ &\le (1 - \alpha_n\hat{\gamma})^2\|y_n - p\|^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle.\end{aligned} \tag{4.17}$$

Observe that

$$\begin{aligned}2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle &= 2\alpha_n\xi\langle f(x_n) - f(p), x_{n+1} - p\rangle \\ &\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\le \alpha_n\xi\rho(\|x_n - p\|^2 + x_{n+1} - p\|^2) \\ &\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &= \alpha_n\xi\rho\|x_n - p\|^2 + \alpha_n\xi\rho\|x_{n+1} - p\|^2 \\ &\quad + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle.\end{aligned} \tag{4.18}$$

Combining inequality (4.17) and inequality (4.18), we have

$$\begin{aligned}\|x_{n+1} - p\|^2 &\le (1 - \alpha_n\hat{\gamma})^2\|y_n - p\|^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1} - p\rangle \\ &= (1 - \alpha_n\hat{\gamma})^2\|y_n - p\|^2 + \alpha_n\xi\rho\|x_n - p\|^2 + \alpha_n\xi\rho\|x_{n+1} - p\|^2 \\ &+ 2\alpha_n\langle \xi f(p) - Dp, x_{n+1} - p\rangle \\ &\le (1 - \alpha_n\hat{\gamma})^2(\|z_n - p\|^2 - (1 - \beta_n)(\beta_n - c)\|z_n - Sz_n\|^2)\end{aligned}$$

$$
\begin{aligned}
&+\alpha_n\xi\rho\|x_n-p\|^2+\alpha_n\xi\rho\|x_{n+1}-p\|^2+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&\le(1-\alpha_n\hat{\gamma})^2\Big(\|u_n-p\|^2-(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2\Big)\\
&+\alpha_n\xi\rho\|x_{n+1}-p\|^2+\alpha_n\xi\rho\|x_n-p\|^2+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&\le(1-\alpha_n\hat{\gamma})^2\Big(\|s_n-p\|^2-(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2\Big)\\
&+\alpha_n\xi\rho\|x_{n+1}-p\|^2+\alpha_n\xi\rho\|x_n-p\|^2+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&\le(1-\alpha_n\hat{\gamma})^2\Big(\|x_n-p\|^2+\alpha_n\frac{\delta_n}{\alpha_n}N_3\|x_n-x_{n-1}\|-(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2\Big)\\
&+\alpha_n\xi\rho\|x_{n+1}-p\|^2+\alpha_n\xi\rho\|x_n-p\|^2+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&=[1-2\alpha_n\hat{\gamma}+(\alpha_n\hat{\gamma})^2+\alpha_n\xi\rho]\|x_n-p\|^2+2\alpha_n\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&+(1-\alpha_n\hat{\gamma})^2\Big(\alpha_n\frac{\gamma_n}{\alpha_n}N_3\|x_n-x_{n-1}\|^2-(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2\Big)\\
&+\alpha_n\xi\rho\|x_{n+1}-p\|^2.
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\|x_{n+1}-p\|^2&\le\frac{(1-2\alpha_n\hat{\gamma}+(\alpha_n\hat{\gamma})^2+\alpha_n\xi\rho)}{(1-\alpha_n\xi\rho)}\|x_n-p\|^2+\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(\alpha_n\frac{\delta_n}{\alpha_n}N_3\|x_n-x_{n-1}\|\\
&-(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2\Big)+\frac{2\alpha_n}{(1-\alpha_n\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle.\\
&=\frac{(1-2\alpha_n\hat{\gamma}+\alpha_n\xi\rho)}{(1-\alpha_n\xi\rho)}\|x_n-p\|^2+\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(\alpha_n\frac{\delta_n}{\alpha_n}N_3\|x_n-x_{n-1}\|\\
&-(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2\Big)+\frac{(\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\|x_n-p\|^2\\
&+\frac{2\alpha_n}{(1-\alpha_n\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\\
&=\left[1-\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\right]\|x_n-p\|^2\\
&+\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\left(\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2\right.\\
&\left.+\frac{1}{(\hat{\gamma}-\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-p\rangle\right)-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2,
\end{aligned}
$$

where $N_4 := \sup\{\|x_n - p\|^2 : n \in \mathbb{N}\}$. Similarly, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n\hat{\gamma})^2\|y_n-p\|^2 + 2\alpha_n\langle \xi f(x_n) - Dp, x_{n+1}-p\rangle \\
&= (1-\alpha_n\hat{\gamma})^2\|y_n-p\|^2 + \alpha_n\xi\rho\|x_n-p\|^2 + \alpha_n\xi\rho\|x_{n+1}-p\|^2 \\
&+ 2\alpha_n\langle \xi f(p) - Dp, x_{n+1}-p\rangle \\
&\le (1-\alpha_n\hat{\gamma})^2\|z_n-p\|^2 + \alpha_n\xi\rho\|x_n-p\|^2 + \alpha_n\xi\rho\|x_{n+1}-p\|^2 \\
&+ 2\alpha_n\langle \xi f(p) - Dp, x_{n+1}-p\rangle \\
&\le (1-\alpha_n\hat{\gamma})^2\Big(\|u_n-p\|^2 - \Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2\Big) \\
&+ \alpha_n\xi\rho\|x_{n+1}-p\|^2 + \alpha_n\xi\rho\|x_n-p\|^2 + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1}-p\rangle,
\end{aligned}$$

which implies

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le (1-\alpha_n\hat{\gamma})^2\Big(\|s_n-p\|^2 - u_n - s_n\|^2 - \Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2\Big) \\
&+ \alpha_n\xi\rho\|x_{n+1}-p\|^2 + \alpha_n\xi\rho\|x_n-p\|^2 + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1}-p\rangle \\
&\le (1-\alpha_n\hat{\gamma})^2\Big\{\|x_n-p\|^2 + \alpha_n\frac{\delta_n}{\alpha_n}N_3\|x_n-x_{n-1}\| - \|u_n-s_n\|^2 \\
&- \Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2\Big\} + \alpha_n\xi\rho\|x_{n+1}-p\|^2 \\
&+ \alpha_n\xi\rho\|x_n-p\|^2 + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1}-p\rangle \\
&= [1-2\alpha_n\hat{\gamma} + (\alpha_n\hat{\gamma})^2 + \alpha_n\xi\rho]\|x_n-p\|^2 \\
&+ (1-\alpha_n\hat{\gamma})^2\Big\{\alpha_n\frac{\gamma_n}{\alpha_n}N_3\|x_n-x_{n-1}\|^2 - \|u_n-s_n\|^2 \\
&- \Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2\Big\} + \alpha_n\xi\rho\|x_{n+1}-p\|^2 + 2\alpha_n\langle \xi f(p) - Dp, x_{n+1}-p\rangle.
\end{aligned}$$

Thus

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\le \frac{(1-2\alpha_n\hat{\gamma} + (\alpha_n\hat{\gamma})^2 + \alpha_n\xi\rho)}{(1-\alpha_n\xi\rho)}\|x_n-p\|^2 \\
&+ \frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(\alpha_n\frac{\delta_n}{\alpha_n}N_3\|x_n-x_{n-1}\| - \|u_n-s_n\|^2
\end{aligned}$$

$$
\begin{aligned}
&-\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2\Big)+\frac{2\alpha_n}{(1-\alpha_n\xi\rho)}\langle \xi f(p)-Dp, x_{n+1}-p\rangle.\\
&=\frac{(1-2\alpha_n\hat{\gamma}+\alpha_n\xi\rho)}{(1-\alpha_n\xi\rho)}\|x_n-p\|^2+\frac{(\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\|x_n-p\|^2\\
&+\frac{2\alpha_n}{(1-\alpha_n\xi\rho)}\langle \xi f(p)-Dp, x_{n+1}-p\rangle\\
&+\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(\alpha_n\frac{\delta_n}{\alpha_n}N_3\|x_n-x_{n-1}\|-\|u_n-s_n\|^2-\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2\Big)\\
&=\left[1-\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\right]\|x_n-p\|^2+\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\Bigg(\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4\\
&+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2\\
&+\frac{1}{(\hat{\gamma}-\xi\rho)}\langle \xi f(p)-Dp, x_{n+1}-p\rangle\Bigg)-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\|u_n-s_n\|^2\\
&-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2.
\end{aligned}
$$

Thus the required inequalities are obtained. □

Theorem 4.7. *Suppose that conditions A and B hold. Then the iterative sequence $\{x_n\}$ generated by Algorithm 3.1 converges to $u\in\Omega$ where $u=P_\Omega f(u)$.*

Proof. Let $u=P_\Omega f(u)$. From Lemma 4.6, we obtain

$$
\begin{aligned}
\|x_{n+1}-u\|^2 \le& \left[1-\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\right]\|x_n-u\|^2\\
&+\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\Big(\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2\\
&+\frac{1}{(\hat{\gamma}-\xi\rho)}\langle \xi f(p)-Dp, x_{n+1}-u\rangle\Big)-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2,
\end{aligned}
\tag{4.19}
$$

and

$$\begin{aligned}\|x_{n+1}-u\|^2 \leq & \Big[1-\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\Big]\|x_n-u\|^2 \\ & +\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}\Big(\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2 \\ & +\frac{1}{(\hat{\gamma}-\xi\rho)}\langle\xi f(p)-Dp,x_{n+1}-u\rangle\Big)-\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\|u_n-s_n\|^2 \\ & -\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}\Big(1-\phi^2\frac{\gamma_n^2}{\gamma_{n+1}^2}\Big)\|u_n-v_n\|^2, \qquad (4.20)\end{aligned}$$

From inequality (4.19), it follows that

$$\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_n\xi\rho)}(1-\beta_n)(\beta_n-c)\|z_n-Sz_n\|^2 \leq \|x_n-u\|^2-\|x_{n+1}-u\|^2+\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}M', \qquad (4.21)$$

where $M'=sup\{b_n:n\in\mathbb{N}\}$ and
$b_n:=\frac{\alpha_n\hat{\gamma}^2}{2(\hat{\gamma}-\xi\rho)}N_4+\frac{(1-\alpha_n\hat{\gamma})^2}{2\alpha_n(\hat{\gamma}-\xi\rho)}\alpha_n\|x_n-x_{n-1}\|^2+\frac{1}{(\hat{\gamma}-\xi\rho)}\langle\xi f(u)-Du,x_{n+1}-u\rangle$.
Next we show that $\{x_n\}$ converges to u. Set $a_n:=\|x_n-u\|^2$ and $\xi_n:=\frac{2\alpha_n(\hat{\gamma}-\xi\rho)}{(1-\alpha_n\xi\rho)}$. It is easy to see from inequality (4.19) that the inequality:

$$a_{n+1}\leq(1-\xi_n)a_n+\xi_n b_n.$$

holds. To conclude, we have to show that $\limsup_{k\to\infty} b_{n_k}\leq 0$ whenever a subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfies

$$\liminf_{k\to\infty}(a_{n_{k+1}}-a_{n_k})\geq 0. \qquad (4.22)$$

Indeed, let $\{a_{n_k}\}$ be a subsequence of $\{a_n\}$ satisfying inequality (4.22), we obtain from inequality (4.19) that

$$\begin{aligned}\limsup_{k\to\infty}\frac{(1-\alpha_n\hat{\gamma})^2}{(1-\alpha_{n_k}\xi\rho)}(1-\beta_{n_k})(\beta_{n_k}-c)\|z_{n_k}-Sz_{n_k}\|^2 \leq \limsup_{k\to\infty}(a_{n_k}-a_{n_{k+1}}) \\ +2M'\frac{(\hat{\gamma}-\xi\rho)}{(1-\alpha_{n_k}\xi\rho)}\lim_{k\to\infty}\alpha_{n_k}=-\liminf_{k\to\infty}(a_{n_{k+1}}-a_{n_k})\leq 0 \qquad (4.23)\end{aligned}$$

which implies that

$$\lim_{k\to\infty}\|z_{n_k}-Sz_{n_k}\|=0. \qquad (4.24)$$

Using inequality (4.20) and following the same process as in inequality (4.19), we have that

$$\lim_{k\to\infty} \|u_{n_k} - s_{n_k}\| = \lim_{k\to\infty} \|u_{n_k} - v_{n_k}\| = 0. \tag{4.25}$$

From Algorithm 3.1, we see that $\|s_{n_k} - x_{n_k}\| = \alpha_{n_k} \frac{\delta_{n_k}}{\alpha_{n_k}} \|x_{n_k} - x_{n_{k-1}}\|$. Thus

$$\lim_{k\to\infty} \|s_{n_k} - x_{n_k}\| = 0. \tag{4.26}$$

It follows from $\|u_{n_k} - x_{n_k}\| \le \|u_{n_k} - s_{n_k}\| + \|s_{n_k} - x_{n_k}\|$, Equation (4.25) and Equation (4.26), that

$$\lim_{k\to\infty} \|u_{n_k} - x_{n_k}\| = 0. \tag{4.27}$$

Observe also that

$$\begin{aligned}\|y_{n_k} - z_{n_k}\| &= \|\beta_{n_k}(z_{n_k} - z_{n_k}) + (1-\beta_{n_k})(Sz_{n_k} - z_{n_k})\| \\ &\le \beta_{n_k}\|(z_{n_k} - z_{n_k})\| + (1-\beta_{n_k})\|(Sz_{n_k} - z_{n_k})\|.\end{aligned}$$

Thus, by Equation (4.24), we obtain

$$\lim_{k\to\infty} \|y_{n_k} - z_{n_k}\| = 0. \tag{4.28}$$

By using Equation (4.25) and Equation(4.27), we get

$$\|v_{n_k} - x_{n_k}\| \le \|v_{n_k} - u_{n_k}\| + \|u_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{4.29}$$

From the definition of z_n, Equation (4.24) and Equation (4.29) we have

$$\begin{aligned}\|z_{n_k} - x_{n_k}\| &= \|v_{n_k} - x_{n_k} - \gamma_{n_k}(Av_{n_k} - Au_{n_k})\| \\ &\le \|v_{n_k} - x_{n_k}\| + \gamma_{n_k}\|Av_{n_k} - Au_{n_k}\| \\ &\le \|v_{n_k} - x_{n_k}\| + \frac{\phi\gamma_{n_k}}{\gamma_{n_{k+1}}}\|v_{n_k} - u_{n_k}\| \to 0, \text{ as } k \to \infty. \end{aligned} \tag{4.30}$$

By using triangular inequality, Equation (4.28) and Equation (4.30), we obtain

$$\|y_{n_k} - x_{n_k}\| \le \|y_{n_k} - z_{n_k}\| + \|z_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty. \tag{4.31}$$

Again, from Algorithm 3.1, we have by using assumption B_2 and inequality (4.31), that

$$\begin{aligned}\|x_{n_{k+1}} - x_{n_k}\| &= \|\alpha_{n_k}\xi f(x_{n_k}) + (I - \alpha_{n_k}D)y_{n_k} - x_{n_k}\| \\ &= \|\alpha_{n_k}\xi f(x_{n_k}) - \alpha_{n_k}Dx_{n_k} + (I - \alpha_{n_k}D)(y_{n_k} - x_{n_k})\| \\ &\le \alpha_{n_k}\|\xi f(x_{n_k}) - Dx_{n_k}\| + (1 - \alpha_{n_k}\hat{\gamma})\|y_{n_k} - x_{n_k}\| \to 0, \text{ as } k \to \infty.\end{aligned} \tag{4.32}$$

We next show that $\limsup_{k\to\infty} b_{n_k} \le 0$. Clearly, it suffices to show that $\limsup_{k\to\infty}\langle \xi f(u) - Du, x_{n_{k+1}} - u\rangle \le 0$. Let $\{x_{n_{k_j}}\}$ be a subsequence of $\{x_{n_k}\}$ such that

$$\lim_{j\to\infty}\langle \xi(u) - Du, x_{n_{k_j}} - u\rangle = \limsup_{k\to\infty}\langle \xi f(u) - u, x_{n_k} - u\rangle. \tag{4.33}$$

Since $\{x_{n_k}\}$ is bounded, there exists a subsequence $\{x_{n_{k_j}}\}$ of $\{x_{n_k}\}$ such that $x_{n_{k_j}} \rightharpoonup u^* \in H$. Thus, we have by inequality (4.30), that $z_{n_{k_j}} \rightharpoonup u^*$. Since I-S is demiclosed at zero, it follows from (4.24), that $u^* \in F(S)$. Also, from Equation (4.25), and Lemma 4.2, we obtain that $u^* \in VI(C,A)$. Again from Equation (4.25), inequality (4.27) and Lemma 4.3, we get that $u^* \in GEP(\Phi,\varphi)$. We therefore conclude that $u^* \in \Omega$. From $u = P_\Omega f(u)$, it follows from Equation (4.33), that

$$\begin{aligned}\limsup_{k\to\infty}\langle \xi f(u) - D(u), x_{n_k} - u\rangle &= \lim_{j\to\infty}\langle \xi f(u) - D(u), x_{n_{k_j}} - u\rangle \\ &= \langle \xi f(u) - Du, u^* - u\rangle \le 0.\end{aligned}$$

We obtain from this and inequality (4.32), that

$$\begin{aligned}\limsup_{k\to\infty}\langle \xi f(u) - D(u), x_{n_{k+1}} - u\rangle &= \limsup_{k\to\infty}\langle \xi f(u) - D(u), x_{n_{k+1}} - x_{n_k}\rangle \\ &\quad + \limsup_{k\to\infty}\langle \xi f(u) - D(u), x_{n_k} - u\rangle \\ &= \langle \xi f(u) - Du, u^* - u\rangle \le 0. \end{aligned} \tag{4.34}$$

Applying Lemma 2.8 to inequality (4.19) together with the fact that $\lim_{n\to\infty}\frac{\delta_n}{\alpha_n}\|x_n - x_{n-1}\| = 0$ and condition $B2$, we deduce that $\lim_{k\to\infty}\|x_n - u\| = 0$ as desired. □

Remark 4.8. This result has extended the result obtained in [42] and other results in this direction.

As direct consequences of Theorem 4.7, we obtain the following corollaries.
If $\varphi = 0$, then we obtain the following as a consequence of Theorem 4.7.

Corollary 4.9. *Suppose conditions A and B hold. Let C be a nonempty, closed and convex subset of H. Let $S : H \to H$ be a c-demicontractive mapping and Φ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying conditions $(C1)-(C4)$ in Lemma 2.6 such that $\Omega = EP(\Phi)\cap F(S)\cap VI(C,A) \neq \emptyset$. Choose $\delta, \gamma_1 > 0$, $\phi \in (0,1)$ and δ_n be given as in* (3.2). *Let $\{x_n\}$ be the* *sequence generated as follows:*

$$\begin{cases} s_n = x_n + \delta_n(x_n - x_{n-1}), \\ \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - s_n\rangle \geq 0 \ \forall y \in H, \\ y_n = P_C(u_n - \gamma_n A u_n), \\ z_n = y_n - \gamma_n(Ay_n - Au_n), \\ x_{n+1} = \alpha_n \varphi f(x_n) + (I - \alpha_n D)(\beta_n z_n + (1-\beta_n)Sz_n, \end{cases} \tag{4.35}$$

and

$$\gamma_{n+1} = \begin{cases} \min\left\{ \frac{\phi\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \gamma_n \right\}, & \text{if } Ax_n - Ay_n \neq 0, \\ \gamma_n, & \text{otherwise.} \end{cases}$$

Then $\{x_n\}$ converges strongly to $u \in \Omega$ where $u = P_\Omega f(u)$.

If $S = I$ and $\delta_n = 0$, then we obtain the following as a consequence of Theorem 4.7.

Corollary 4.10. *Suppose conditions A and B hold. Let C be a nonempty, closed and convex subset of H. Let $S : H \to H$ be a c-demicontractive mapping and let Φ be a bifunction from $C \times C \to \mathbb{R}$ satisfying $(C1) - (C4)$ such that $\Omega = EP(\Phi) \cap VI(C,A) \neq \emptyset$. Choose $\gamma_1 > 0$ and $\phi \in (0,1)$. Let $\{x_n\}$ be the sequence generated as follows:*

$$\begin{cases} \Phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0 \ \forall y \in H, \\ y_n = P_C(u_n - \gamma_n A u_n), \\ z_n = y_n - \gamma_n(Ay_n - Au_n), \\ x_{n+1} = \alpha_n \varphi f(x_n) + (I - \alpha_n D)z_n, \end{cases} \tag{4.36}$$

and

$$\gamma_{n+1} = \begin{cases} \min\left\{ \frac{\phi\|x_n - y_n\|}{\|Ax_n - Ay_n\|}, \gamma_n \right\}, & \text{if } Ax_n - Ay_n \neq 0, \\ \gamma_n, & \text{otherwise.} \end{cases}$$

Then $\{x_n\}$ converges strongly to $u \in \Omega$ where $u = P_\Omega f(u)$.

5. Numerical Example

In this section, we present a numerical example to illustrate the behaviour of the sequence generated by Algorithm 3.1. All the programs are implemented in MATLAB 2023b on a Intel(R)Core(TM) i5-8250S CPU @1.60 GHz computer with RAM 8.00 GB.

Example 5.1. Let $H = \ell_2(\mathbb{R})$ be the linear space whose elements are all 2-summable sequence $\{x_i\}_{i=1}^{\infty}$ of scalars in $\mathbb{R}$ that is $\ell_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \cdots), x_i \in \mathbb{R}$ and $\sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ with the inner product $\langle\cdot,\cdot\rangle : \ell_2 \times \ell_2 \to \mathbb{R}$ defined by $\langle x,y\rangle := \sum_{i=1}^{\infty} x_i y_i$ and the norm $\|.\| : \ell_2 \to \mathbb{R}$ by $\|x\|_{\ell_2} := (\sum_{i=1}^{\infty} |x_i|^2)^{\frac{1}{2}}$, where $x = \{x_i\}_{i=1}^{\infty}, y = \{y_i\}_{i=1}^{\infty}$. Let $C := \{x \in \ell_2(\mathbb{R}) : \|x\|_{\ell_2} \leq 1\}$ and $A : C \to \ell_2$ be defined by $A(x) = \Big(\|x\|_{\ell_2} + \frac{1}{\|x\|_{\ell_2}+1}\Big)x,\ \forall x \in C$. Let $\Phi : C \times C \to \mathbb{R}$ be defined by $\Phi(x,y) = x^2 + xy - 2y^2$ and $\varphi(x,y) = -2x^2 + xy + y^2$ for all $x, y \in C$.
By some simple calculations, we arrive at

$$T_r^{\phi,\varphi} s_n = \frac{s_n}{12r+1}.$$

Let $S : \ell_2(\mathbb{R}) \to \ell_2(\mathbb{R})$ be defined by $Sx = \frac{1}{5}x$ with $\rho = \frac{1}{5}$. Let $D : H \to H$ be defined by $Dx = \frac{x}{3}$ for all $x \in H$ with $\bar{\gamma} = \frac{1}{3}$, then we take $\gamma = 1$ which satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$.
We choose $f(x) = \frac{x}{2}, \xi = \frac{1}{3}, \delta = 0.8, \delta_n = \frac{1}{(n+1)^2}, \gamma_1 = 3.1, \phi = 0.7, \alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{2n}{5n+4}$. We demonstrate the numerical behavior of the sequences generated by Algorithm 4.7 using different starting points of x_0 and x_1. The process is terminated by using the stopping criterion $\|x_{n+1} - x_n\| \leq \varepsilon$, where $\varepsilon = 10^{-4}$.

(Case 1): $x_0 = (0.6, 0.5, \cdots, 0, \cdots)$ and $x_1 = (1.1, 0.1, \cdots, 0, \cdots)$;
(Case 2): $x_0 = (0.1, 0.5, \cdots, 1, \cdots)$ and $x_1 = (1.1, 1.2, \cdots, 0, \cdots)$;
(Case 3): $x_0 = (1.1, 1.5, \cdots, 0, \cdots)$ and $x_1 = (0.7, 0.9, \cdots, 0, \cdots)$;
(Case 4): $x_0 = (1.3, 0.0, \cdots, 0, \cdots)$ and $x_1 = (1.9, 0.0, \cdots, 0, \cdots)$.

The report of this example is given in Figure 1.

Example 5.2. Let $H = \mathbb{R}^2$ be the two dimensional Euclidean space of the real number with an inner product $\langle\cdot,\cdot\rangle : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\langle x,y\rangle = x \cdot y = x_1 \cdot y_1 + x_2 \cdot y_2$ where $x = (x_1, x_2) \in \mathbb{R}^2$ and a usual norm $\mathbb{R}^2 \to \mathbb{R}$ be defined by $\|x\| = \sqrt{x_1^2 + x_2^2}$ where $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $A : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Ax = (2x_1 - x_2, x_1 + 2x_2)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$. Let the mapping $\Phi, \varphi : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ be defined by $\Phi(x,y) = -x^2 + y^2\ \forall x, y \in \mathbb{R}^2$, and $\varphi(x,y) = -2x^2 + xy + y^2\ \forall x, y \in \mathbb{R}^2$.
As before, it is easy to see that

$$T_r^{\phi,\varphi} s_n = \frac{s_n}{5r+1}.$$

Let the mapping $S : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Sx = \frac{3x}{4}$. Let $D : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $Dx = \frac{x}{6}, \forall x \in \mathbb{R}^2$ with $\bar{\gamma} = \frac{1}{6}$, then we take $\gamma = 1$ which

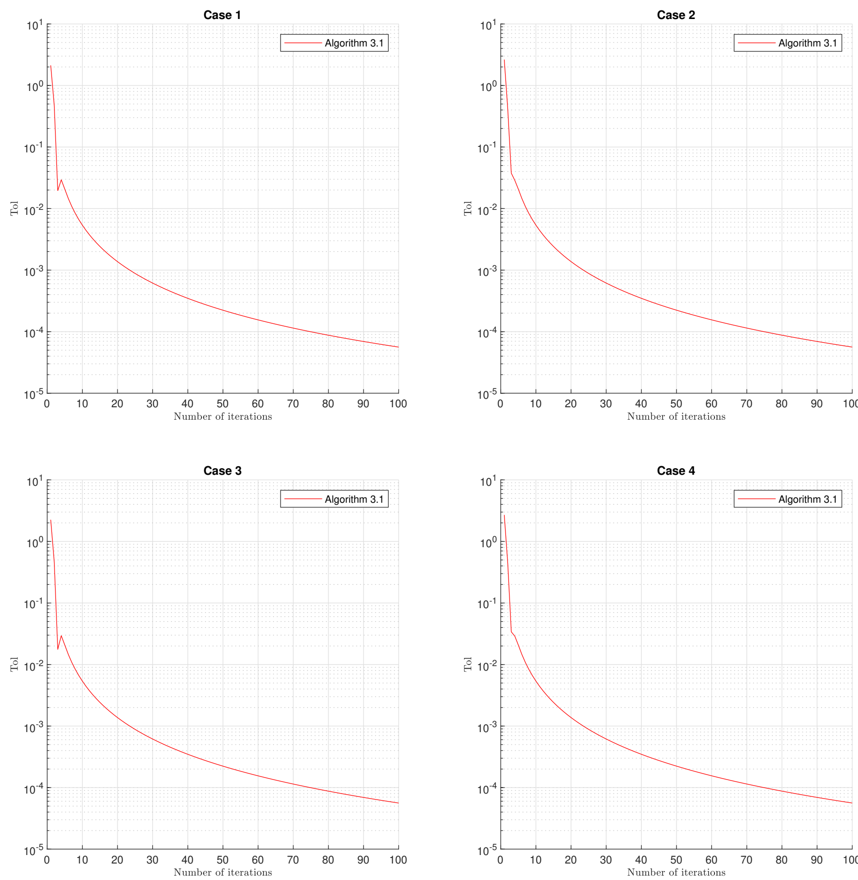

FIGURE 1. Example 5.1, Top left: Case 1; Top right: Case 2; Bottom left : Case 3; Bottom right: Case 4.

satisfies $0 < \gamma < \frac{\bar{\gamma}}{\rho}$. Let the mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by $f(x) = \frac{x}{5}, \forall x \in \mathbb{R}^2$ then with $\rho = \frac{1}{5}$ and let $r = 1$. The numerical behaviour of the sequences generated by Algorithm 3.1, using different starting points is shown as follows. We choose $\xi = \frac{1}{3}, \delta = 0.9, \delta_n = \frac{1}{(n+1)^2}, \phi = 0.8, \alpha_n = \frac{1}{n+5}, \gamma_1 = 3.3, \varepsilon_n = \frac{1}{(n+5)^3}$ and $\beta_n = \frac{2n}{5n+4}$.

(Case A): $x_0 = (0.6, 0.5)$ and $x_1 = (1.1, 0.1)$;
(Case B): $x_0 = (0.1, 0.5)$ and $x_1 = (1.1, 1.2)$;
(Case C): $x_0 = (1.1, 1.5)$ and $x_1 = (0.7, 0.9)$;
(Case D): $x_0 = (1.1, 0)$ and $x_1 = (1.9, 0.7)$.

The report of this example is given in Figure 2.

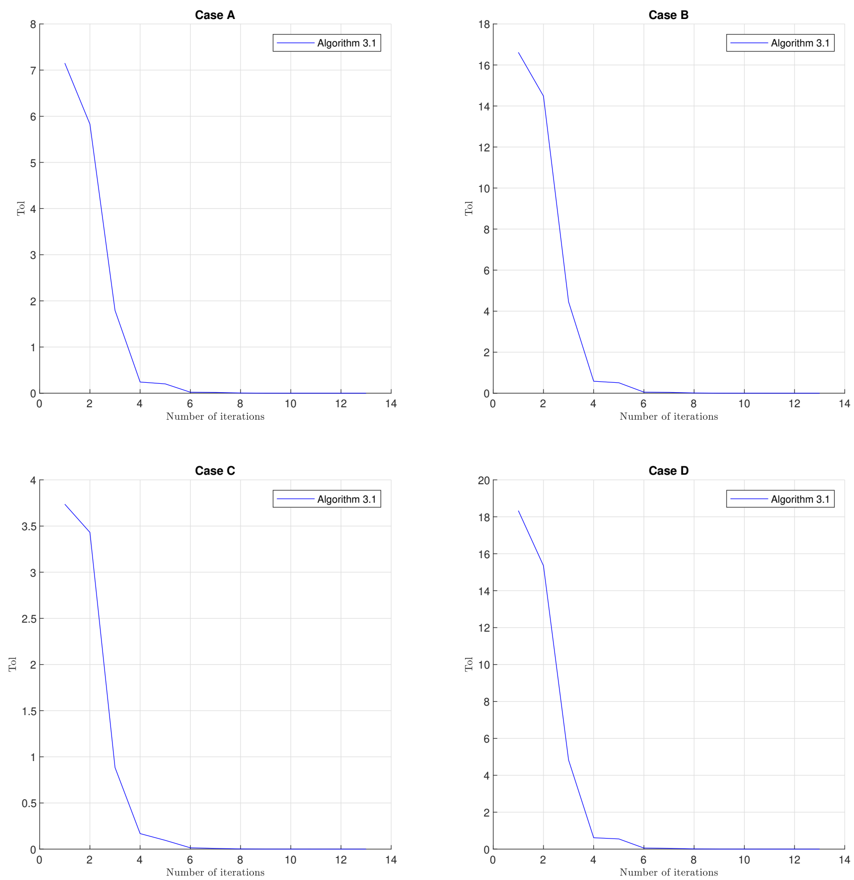

FIGURE 2. Example 5.2, Top left: Case 1; Top right: Case 2; Bottom left : Case 3; Bottom right: Case 4.

6. CONCLUSION

In this paper, we have considered a problem of finding a common solution to an equilibrium, variational inequality and fixed point problems. We introduced an iterative method which combines the inertial, Tseng and viscosity techniques. The method is self-adaptive and thus independent of the Lipschitz constant of the cost operator. Using these methods we have established a strong convergence result for approximating a common solution to variational inequality problem and fixed point problem associated with pseudomonotone and demicontractive operators and also a solution to a generalized equilibrium problem. By the way of numerical illustrations, we displayed the convergence of our proposed method.

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