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SPLIT EQUALITY EQUILIBRIUM PROBLEM WITH MULTIVALUED STRICTLY PSEUDOCONTRACTIVE MAPPINGS IN HILBERT SPACES

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ABSTRACT. In this paper, we introduce a hybrid projected subgradient method for approximating a common solution of the set of split equality equilibrium problem and multisets split equality fixed point problem for finite families of multivalued strictly pseudocontractive mappings in real Hilbert spaces. We establish weak and strong convergence theorems. Our results generalize and improve several recent results.

1. INTRODUCTION

Due to its broad applications in many areas of applied mathematics (image processing, phase retrieval, computer tomography and radiation therapy treatment planning see, for example, [5, 6, 10, 14, 16, 20, 31, 35], the split feasibility problems continue to recieve great attentions.

Let K and Q be two nonempty, closed and convex subsets of two real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \to H_2$ be a bounded linear map. Then the split feasibility problem (SFP) is

$$\text{to find } x^* \in K \text{ such that } Ax^* \in Q. \tag{1}$$

Moudafi [24], introduced the following new split feasibility problems which is also called a general split equality problem:

$$\text{to find } x^* \in K \text{ and } y^* \in Q \text{ such that } Ax^* = By^*, \tag{2}$$

where $A : H_1 \to H_3$ and $B : H_2 \to H_3$ are two bounded linear maps. Observe that problem (2) reduces to problem (1) if $H_2 = H_3$ and $B = I$ (where I is the identity map on H_2) in (2).

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Let K be a nonempty closed convex subset of a real Hilbert space H and $f : K \times K \to \mathbb{R}$ be an equilibrium bifunction. The equilibrium problem, (EP) is

$$\text{to find } x^* \in K \text{ such that } f(x^*, y) \geq 0,\ \forall\, y \in K. \tag{3}$$

The set of solution of equilibrium problem is denoted by $EP(f,K)$ i.e

$$EP(f,K) = \{x^* \in K : f(x^*, y) \geq 0,\ \forall\, y \in K\}.$$

The equilibrium problem involves many important problems arising in physics, engineering, transportation, economics, structural analysis, etc see for example [1, 2, 9, 28].

Let $f : K \times K \to \mathbb{R}$, $g : Q \times Q \to \mathbb{R}$ be two bifunctions and $A : H_1 \to H_2$ be a bounded linear map, then the split equilibrium problem (SEP) is

$$\begin{aligned} &\text{to find } x^* \in K \text{ such that } f(x^*, x) \geq 0,\ \forall\, x \in K,\\ &\text{and } y^* = Ax^* \in Q \text{ solves } g(y^*, y) \geq 0,\ \forall\, y \in Q. \end{aligned} \tag{4}$$

Several techniques have been developed to solve (4), see [19]. The split equality equilibrium problem (SEEP) is

$$\begin{aligned} \text{to find } x^* \in K,\ y^* \in Q \text{ such that } & f(x^*, x) \geq 0,\ \forall\, x \in K,\\ & g(y^*, y) \geq 0,\ \forall\, y \in Q \text{ and } Ax^* = By^*. \end{aligned} \tag{5}$$

The set of solutions of (5) is denoted by $SEEP(f,g)$.

A mapping $T : K \to K$ is said to be nonexpansive, see for example Wang et al. [35] if $||Tx - Ty|| \leq ||x - y||,\ \forall\, x,y \in K$.

T is called λ-strictly pseudocontractive see [3], if there exists $\lambda \in (0,1)$ such that

$$||Tx - Ty||^2 \leq ||x - y||^2 + \lambda ||x - y - (Tx - Ty)||^2,\ \forall\, x,y \in K. \tag{6}$$

A subset K of H is said to be proximinal, see, for example, [27, 30, 32] if for each $x \in H$ there exists $y \in K$ such that

$$||x - y|| = \inf_{z \in K} ||x - z|| = d(x,K).$$

Remark 1: It is clear that every closed, convex nonempty subset of a real Hilbert space H is proximinal.

Let $\mathscr{C}B(K)$ and $\mathscr{P}(K)$ denote the families of nonempty, closed bounded subsets and nonempty proximinal subsets of K, respectively. The Hausdorff distance on $\mathscr{C}B(K)$ is defined by

$$\mathscr{D}(A,B) = \max\left\{\sup_{a \in A} d(a,B),\ \sup_{b \in B} d(b,A)\right\},\ \forall A,B \in \mathscr{C}B(K).$$

Let $D(T)$ be the domain of T. A multivalued mapping $T : D(T) \to \mathscr{C}B(K)$ is called nonexpansive if $\mathscr{D}(Tx,Ty) \leq ||x-y||,\ \forall\ x,y \in D(T)$. $T : D(T) \to \mathscr{C}B(K)$ is called λ-strictly pseudocontractive, see Chidume et al. [11] if there exists $\lambda \in (0,1)$ such that

$$(\mathscr{D}(Tx,Ty))^2 \leq ||x-y||^2 + \lambda||(x-u)-(y-v)||^2,\ \forall x,y \in D(T),\ u \in Tx,\ v \in Ty, \tag{7}$$

T is called generalized λ-strictly pseudocontractive, see, Chidume and Okpala [13] if there exists $\lambda \in (0,1)$ such that $\forall x,y \in D(T)$, we have

$$(\mathscr{D}(Tx,Ty))^2 \leq ||x-y||^2 + \lambda(\mathscr{D}(Vx,Vy))^2,\ V = (I-T) \tag{8}$$

and I is the identity operator.

Remark 2:

(i) It is obvious from (7) that the class of multivalued nonexpansive mappings is contained in the class of λ-strictly pseudocontractive multivalued mappings for any $\lambda \in (0,1)$,
(ii) From (8), the class of generalized λ-strictly pseudocontractive multivalued mappings contains the class of λ-strictly pseudocontractive multivalued mappings, see, Chidume and Okpala [13].

A point $x \in D(T)$ is called a fixed point of T if $x \in Tx$. In this paper, we denote by $F(T)$ the set of all fixed points of T i.e. $F(T) = \{x \in D(T) : x \in Tx\}$. A fixed point $x \in D(T)$ is called an endpoint of T if $Tx = \{x\}$. The mapping T is said to have endpoint property if $Tx = \{x\}$ for every fixed point of T.

Moudafi and Al-Shemas [25] introduced the following simultaneous iterative scheme to solve the split equality problem (2)

$$\begin{cases} x_{k+1} = U(x_k - \gamma_k A^*(Ax_k - By_k)) \\ y_{k+1} = T(y_k + \gamma_k B^*(Ax_k - By_k)), \end{cases} \tag{9}$$

where H_1, H_2 and H_3 are real Hilbert spaces, $U : H_1 \to H_1$, $T : H_2 \to H_2$ are two firmly quasi-nonexpansive mappings, $A : H_1 \to H_3$, $B : H_2 \to H_3$ are bounded linear maps with their adjoints A^*, B^* respectively and $\gamma_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B} - \varepsilon)$, λ_A and λ_B stand for the spectral radii of A^*A and B^*B respectively. He proved weak and strong convergence of (9) to a solution of (2)

Recently, Ma et al. [22] introduced the following algorithm for solving split equality equilibrium problem in a real Hilbert space:

$$\begin{cases} f(u_n,u)+\frac{1}{r_n}\langle u-u_n,u_n-x_n\rangle \geq 0,\ \forall\ u\in C,\\ g(v_n,v)+\frac{1}{r_n}\langle v-v_n,v_n-y_n\rangle \geq 0,\ \forall\ v\in Q,\\ x_{n+1}=\alpha_n u_n+(1-\alpha_n)T(u_n-\rho_n A^*(Au_n-Bv_n)),\\ y_{n+1}=\alpha_n v_n+(1-\alpha_n)S(v_n+\rho_n B^*(Au_n-Bv_n)),\ \forall\ n\geq 1, \end{cases} \tag{10}$$

where λ_A and λ_B are spectral radii of A^*A and B^*B respectively, ρ_n is a positive real sequence such that $\rho_n \in (\varepsilon, \frac{2}{\lambda_A+\lambda_B}-\varepsilon)$ for some $\varepsilon>0$, $\{\alpha_n\}\subset(0,1)$ and $\{r_n\}\subset(0,\infty)$. Under some mild conditions on two bifunctions f and g, they proved weak and strong convergence of the scheme (10) to the solution of *SEEP* (5) which solve some fixed point problems of two nonexpansive mappings T and S. Moreover, several methods have been investigated to solve split equality problems; see for example [17, 33] and the references contained therein.

Let $(x_1,y_1)\in K\times Q$ and $\{x_n,y_n\}$ be iteratively defined by

$$\begin{cases} w_n\in\partial_{\varepsilon_n}f(x_n,.)x_n,\\ z_n\in\partial_{\varepsilon_n}g(y_n,.)y_n,\\ u_n=P_K(x_n-\gamma_n w_n), \qquad \gamma_n=\frac{\beta_n}{max\{\sigma_n,||w_n||\}},\\ v_n=P_Q(y_n-\delta_n z_n), \qquad \delta_n=\frac{\eta_n}{max\{\xi_n,||z_n||\}},\\ x_{n+1}=P_K\Big(\alpha_{n,0}(u_n-\beta A^*(Au_n-Bv_n))+\sum_{i=1}^N\alpha_{n,i}s_n^i\Big),\\ y_{n+1}=P_Q\Big(\alpha_{n,0}(v_n+\beta B^*(Au_n-Bv_n))+\sum_{j=1}^N\alpha_{n,j}t_n^j\Big), \end{cases} \tag{11}$$

where $\beta_n,\ \eta_n,\ \sigma_n$ and ξ_n are positive real numbers, $s_n^i\in T_i(u_n-\beta A^*(Au_n-Bv_n))$, $t_n^j\in U_j(v_n+\beta B^*(Au_n-Bv_n))$, $\alpha_{n,i}\in(0,1)$ satisfying $\sum_{i=0}^N\alpha_{n,i}=1$. $\beta\in(0,\frac{1}{\|A\|^2+\|B\|^2})$.

It is our purpose in this paper to prove weak and strong convergence of (11) to an element in the solution set of common split equality equilibrium problems and set of split fixed points problem of two finite families of multivalued strictly pseudocontractive mappings in real Hilbert spaces.

2. PRELIMINARIES

Let K be a nonempty, closed, convex subset of a real Hilbert space H. It is well known see [21, 37] that for every $x\in H$ there exists a nearest point $P_K(x)\in K$ with the following property;

$$||x-P_K(x)||\leq||x-y||,\ \forall y\in K,$$

where P_K is called the metric projection of H onto K. Furthermore for metric projection P_K, the following properties are well known see [34];

(1) $\langle x-P_K(x), y-P_K(x)\rangle \leq 0,\ \forall x\in H, y\in K,$
(2) $||P_K(x)-P_K(y)||^2 \leq \langle x-y, P_K(x)-P_K(y)\rangle,\ ,\ \forall x,y\in H.$

Through out this paper, $\rightharpoonup$ and $\to$ denote weak and strong convergence respectively.

Definition 1:[11] Let X be a Banach space. A multivalued mapping $T: D(T)\subset X\to 2^X$ is said to be hemicompact if for any sequence $\{x_n\}$ in K and $\lim\limits_{n\to\infty} d(x_n, Tx_n)=0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\to x\in D(T)$.

Definition 2: [15] Let X be a Banach space. Let $T: D(T)\subseteq X\to 2^X$ be a multivalued mapping. $I-T$ is said to be *demiclosed* at zero if for any sequence $\{x_n\}_{n\geq 1}\subseteq D(T)$ such that $x_n\rightharpoonup x$ and $\lim\limits_{n\to\infty} d(x_n, Tx_n)=0$, we have $x\in Tx$ i.e. $d(x,Tx)=0$.

Definition 3: [4, 18] Let $\phi: K\to\mathbb{R}\cup\{\infty\}$ be a proper convex function. For a given $\varepsilon>0$, the ε-subdifferential of ϕ at $x_0\in K$ is given by

$$\partial_\varepsilon\phi(x_0)=\{x\in K: \phi(y)-\phi(x_0)\geq\langle x, y-x_0\rangle-\varepsilon,\ \forall y\in K\}.$$

Remark 3: It is known that if the function ϕ is proper, lower semicontinuous and convex, then for each $x_0\in D(\phi)$ the ε-subdifferential $\partial_\varepsilon\phi(x_0)$ is a nonempty closed convex set.

Definition 4: A bifunction $f: K\times K\to\mathbb{R}$ is said to be:

(1) γ-strongly monotone on K if there exists $\gamma>0$ such that
$$f(x,y)+f(y,x)\leq -\gamma||x-y||^2,\ \forall\, x,y\in K,$$
(2) monotone on K if
$$f(x,y)+f(y,x)\leq 0,\ \forall\, x,y\in K,$$
(3) pseudomonotone on K with respect to $x\in K$ if
$$f(x,y)\geq 0\ \Rightarrow\ f(y,x)\leq 0\ ,\ \forall\, y\in K,$$
(4) pseudomonotone on K with respect to $B\subseteq K$, if it is pseudomonotone on K with respect to every $x\in B$ (See [29]).

To study equilibrium problem (3), we assume that the bifunction f satisfies the following conditions:

(C1) $f(x,x)=0$ for every $x\in K$ and $f(x,.)$ is convex and lower semicontinuous on K,
(C2) $f(.,y)$ is weakly upper semicontinuous for every $y\in K$,
(C3) f is pseudomonotone on K with respect to $EP(f,K)$ and satisfies the strict paramonotonicity property i.e $f(y,x)=0$ for $x\in EP(f,K)$ and $y\in K$ implies $y\in EP(f,K)$,
(C4) If $\{x_n\}\subseteq K$ is bounded and $\varepsilon_n\to 0$ as $n\to\infty$, then the sequence $\{w_n\}$, $w_n\in\partial_{\varepsilon_n}f(x_n,.)x_n$ is bounded.

The following Lemmas will be needed in the proof of our main results:

Lemma 1: [12] Let H be a real Hilbert space. Let $\{x_i\}_{i=1}^{\infty}\subset H$ be a sequence and $\alpha_i\in(0,1)$, $i=1,2,3,\dots,n$ such that $\sum_{i=1}^{n}\alpha_i=1$. Then the following identity holds:

$$||\sum_{i=1}^{n}\alpha_i x_i||^2=\sum_{i=1}^{n}\alpha_i||x_i||^2-\sum_{i,j=1,\ i\neq j}^{n}\alpha_i\alpha_j||x_i-x_j||^2.$$

Lemma 2: [23] Let H be a real Hilbert space. For every $x,y\in H$, we have the following identity:

$$||x-y||^2=||x||^2-||y||^2-2\langle x-y,y\rangle.$$

Lemma 3: [13] Let K be a nonempty subset of a real Hilbert space H and $T:K\to CB(K)$ be generalized λ-strictly pseudocontractive multivalued mapping. Then T is Lipschizian with Lipschitz constant $L=\frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}}$.

Lemma 4: [36] Let $\{a_n\}$ and $\{b_n\}$ be two sequences of nonnegative real numbers such that

$a_{n+1}\leq a_n+b_n,\ n\geq 0$. If $\sum_{n=0}^{\infty}b_n<\infty$. Then $\lim\limits_{n\to\infty}a_n$ exists.

Lemma 5: Opial, [26] Let H be a real Hilbert space and $\{\mu_n\}$ be a sequence in H such that there exists a nonempty set $W\subset H$ satisfying the following conditions:

(i) For every $\mu\in W$, $\lim\limits_{n\to\infty}\|\mu_n-\mu\|$ exists,
(ii) Any weak-cluster point of the sequence $\{\mu_n\}$ belongs to W.

Then there exists $w^*\in W$ such that $\{\mu_n\}$ converges weakly to w^*.

3. MAIN RESULTS

Theorem 1: Let H_1, H_2, H_3 be real Hilbert spaces and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear maps with their adjoints A^*, B^* respectively. Let $K \subset H_1$ and $Q \subset H_2$ be nonempty, closed, convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $f : K \times K \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be bifunctions satisfying $(C1)-(C4)$. Let $T_i : H_1 \to CB(H_1)$, $i = 1,2,3,\ldots,N$ and $U_j : H_2 \to CB(H_2)$, $j = 1,2,3,\ldots,N$ be two finite families of multivalued strictly pseudocontractive mappings such that $\Omega = \Big\{(p,q) \in K \times Q : p \in \Big(\bigcap_{i=1}^{N} F(T_i)\Big) \cap EP(f,K),\ q \in \Big(\bigcap_{j=1}^{N} F(U_j)\Big) \cap EP(g,Q),\ Ap = Bq\Big\} \neq \emptyset$. Let $(p,q) \in \Omega$ such that for each $j = 1,2,3,\ldots,N$, T_j and U_j have endpoint property . Let $\{\sigma_n\}$, $\{\xi_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_n\}$, $\{\eta_n\}$ and $\{\varepsilon_n\}$ be sequences of positive real numbers satisfying:

(i) $\sigma_n,\ \xi_n \in (c,d),,\ \forall n \in \mathbb{N}$ for some numbers c,d with $0 < c < d$,

(ii) $\alpha_{n,i} \in (0,1)$, $\alpha_{n,0} \in (\lambda,1)$, $\liminf_{n\to\infty} \alpha_{n,0} > \lambda$ with $\lambda^* = \max_{1\le i\le N}\{\lambda_i\}$, $\lambda^{**} = \max_{1\le j\le N}\{\lambda_j\}$, $\lambda = \max\{\lambda^*,\ \lambda^{**}\}$, $\liminf_{n\to\infty} \alpha_{n,i} > 0$ and $\liminf_{n\to\infty} \alpha_{n,j} > 0$, for $i,j \in \{1,2,3,\ldots,N\}$,

(iii) $\sum_{n=1}^{\infty} \beta_n = \infty$, $\sum_{n=1}^{\infty} \beta_n^2 < \infty$ and $\sum_{n=1}^{\infty} \beta_n \varepsilon_n < \infty$ and

(iv) $\sum_{n=1}^{\infty} \eta_n = \infty$, $\sum_{n=1}^{\infty} \eta_n^2 < \infty$ and $\sum_{n=1}^{\infty} \eta_n \varepsilon_n < \infty$.

If $T_i,\ i = 1,2,3,\ldots N$ and $U_j,\ j = 1,2,3,\ldots,N$ are demiclosed at zero, then the sequence $\{(x_n,y_n)\}$ generated by (11) converges weakly to the solution of problem (5).

Proof: Since $(p,q) \in \Omega$, it follows from Lemma 1 2 and (11) that

$$
\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_K\Big(\alpha_{n,0}(u_n-\beta A^*(Au_n-Bv_n))+\sum_{i=1}^{N}\alpha_{n,i}s_n^i\Big)-p\|^2\\
&\le \|\alpha_{n,0}(u_n-\beta A^*(Au_n-Bv_n))+\sum_{i=1}^{N}\alpha_{n,i}s_n^i-p\|^2\\
&= \|\alpha_{n,0}(u_n-\beta A^*(Au_n-Bv_n)-p)+\sum_{i=1}^{N}\alpha_{n,i}(s_n^i-p)\|^2\\
&= \alpha_{n,0}\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2+\sum_{i=1}^{N}\alpha_{n,i}\|s_n^i-p\|^2\\
&\quad -\sum_{i,k=1,i<k}\alpha_{n,i}\alpha_{n,k}\|s_n^i-s_n^k\|^2-\sum_{i=1}^{N}\alpha_{n,0}\alpha_{n,i}\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2
\end{aligned}
$$

$$\begin{aligned}
&\leq \alpha_{n,0}\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2+\sum_{i=1}^{N}\alpha_{n,i}d^2(s_n^i,T_ip)\\
&- \sum_{i=1}^{N}\alpha_{n,0}\alpha_{n,i}\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2\\
&\leq \alpha_{n,0}\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2\\
&\quad+\sum_{i=1}^{N}\alpha_{n,i}\mathscr{D}^2(T_i(u_n-\beta A^*(Au_n-Bv_n)),T_ip)\\
&\quad-\sum_{i=1}^{N}\alpha_{n,0}\alpha_{n,i}\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2.
\end{aligned}$$

Since $T_i,\ i=1,2,3,\ldots N$ are λ_i-multivalued strictly pseudocontractive, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &\leq \alpha_{n,0}\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2\\
&\quad+\sum_{i=1}^{N}\alpha_{n,i}\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2\\
&\quad+\sum_{i=1}^{N}\alpha_{n,i}\lambda_i\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2\\
&\quad-\sum_{i=1}^{N}\alpha_{n,0}\alpha_{n,i}\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2.
\end{aligned}$$

Since $\lambda^*=\max\limits_{1\leq i\leq N}\{\lambda_i\}$, we have

$$\begin{aligned}
&\|x_{n+1}-p\|^2\leq\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2\\
&\quad-\sum_{i=1}^{N}\alpha_{n,i}(\alpha_{n,0}-\lambda^*)\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2. \qquad (12)
\end{aligned}$$

But

$$\begin{aligned}
\|u_n-\beta A^*(Au_n-Bv_n)-p\|^2 &= \|u_n-p\|^2+\beta^2\|A^*(Au_n-Bv_n)\|^2\\
&\quad-2\beta\langle u_n-p,A^*(Au_n-Bv_n)\rangle\\
&= \|u_n-p\|^2+\beta^2\|A^*(Au_n-Bv_n)\|^2\\
&\quad-2\beta\langle Au_n-Ap,Au_n-Bv_n\rangle. \qquad (13)
\end{aligned}$$

Using Lemma 2 2, we have

$$\begin{aligned}
\|u_n-p\|^2 &= \|x_n-p-(x_n-u_n)\|^2\\
&= \|x_n-p\|^2-\|x_n-u_n\|^2-2\langle u_n-p,x_n-u_n\rangle\\
&\leq \|x_n-p\|^2+2\langle p-u_n,x_n-u_n\rangle.
\end{aligned}$$

Since $u_n = P_K(x_n - \gamma_n w_n)$, then for $p \in K$ we have,

$$\langle x_n - u_n, p - u_n\rangle \le \gamma_n \langle w_n, p - u_n\rangle,$$

so that

$$\begin{aligned} \|u_n - p\|^2 &\le \|x_n - p\|^2 + 2\gamma_n\langle w_n, p - u_n\rangle \\ &= \|x_n - p\|^2 + 2\gamma_n\langle w_n, p - x_n\rangle + 2\gamma_n\langle w_n, x_n - u_n\rangle \\ &\le \|x_n - p\|^2 + 2\gamma_n\langle w_n, p - x_n\rangle + 2\gamma_n\|w_n\|\|x_n - u_n\|. \end{aligned} \tag{14}$$

But

$$\begin{aligned} ||x_n - u_n||^2 = \langle x_n - u_n, x_n - u_n\rangle &\le \gamma_n\langle w_n, x_n - u_n\rangle \\ &\le \gamma_n ||w_n|| ||x_n - u_n|| \\ &\le \gamma_n \max\{\sigma_n, ||w_n||\} ||x_n - u_n|| \\ &= \beta_n ||x_n - u_n||. \end{aligned}$$

Hence,

$$||x_n - u_n|| \le \beta_n. \text{ Similarly } ||\mathrm{y_n} - \mathrm{v_n}|| \le \eta_{\mathrm{n}}. \tag{15}$$

Therefore, from (14) we obtain

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + 2\gamma_n\langle w_n, p - x_n\rangle + 2\beta_n^2.$$

Putting the above inequality in (13), we have

$$\begin{aligned} \|u_n - \beta A^*(Au_n - Bv_n) - p\|^2 \le\ & \|x_n - p\|^2 + 2\gamma_n\langle w_n, p - x_n\rangle + 2\beta_n^2 \\ & + \beta^2\|A^*(Au_n - Bv_n)\|^2 \\ & - 2\beta\langle Au_n - Ap, Au_n - Bv_n\rangle. \end{aligned}$$

Since $w_n \in \partial_{\varepsilon_n} f(x_n, .)x_n$ and $f(x_n, x_n) = 0$, $\forall n \in \mathbb{N}$, we get

$$\begin{aligned} \langle w_n, p - x_n\rangle &\le f(x_n, p) - f(x_n, x_n) + \varepsilon_n \\ &= f(x_n, p) + \varepsilon_n, \end{aligned}$$

so that,

$$\begin{aligned} \|u_n - \beta A^*(Au_n - Bv_n) - p\|^2 \le\ & \|x_n - p\|^2 + 2\gamma_n f(x_n, p) + 2\gamma_n\varepsilon_n \\ & + 2\beta_n^2 + \beta^2\|A^*(Au_n - Bv_n)\|^2 \\ & - 2\beta\langle Au_n - Ap, Au_n - Bv_n\rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \|x_{n+1} - p\|^2 \le\ & \|x_n - p\|^2 + 2\gamma_n f(x_n, p) + 2\gamma_n\varepsilon_n + 2\beta_n^2 \\ & + \beta^2\|A^*(Au_n - Bv_n)\|^2 - 2\beta\langle Au_n - Ap, Au_n - Bv_n\rangle \\ & - \sum_{i=1}^{N} \alpha_{n,i}(\alpha_{n,0} - \lambda^*)\|u_n - \beta A^*(Au_n - Bv_n) - s_n^i\|^2. \end{aligned} \tag{16}$$

Similar computations by taking into account of the assumptions on δ_n, we have

$$\begin{aligned}\|y_{n+1}-q\|^2 &\le \|y_n-q\|^2+2\delta_n g(y_n,q)+2\delta_n\varepsilon_n+2\eta_n^2\\ &+\beta^2\|B^*(Au_n-Bv_n)\|^2+2\beta\langle Bv_n-Bq, Au_n-Bv_n\rangle\\ &-\sum_{j=1}^{N}\alpha_{n,j}(\alpha_{n,0}-\lambda^{**})\|v_n+\beta B^*(Au_n-Bv_n)-t_n^j\|^2. \qquad (17)\end{aligned}$$

Adding (16) and (17) and taking $\lambda=\max\{\lambda^*,\ \lambda^{**}\}$, we have

$$\begin{aligned}\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 &\le \|x_n-p\|^2+\|y_n-q\|^2\\ &+2\gamma_n f(x_n,p)+2\delta_n g(y_n,q)+2\gamma_n\varepsilon_n+2\delta_n\varepsilon_n+2\beta_n^2\\ &+2\eta_n^2-2\beta\Big(1-\beta(\|A\|^2+\|B\|^2)\Big)\|(Au_n-Bv_n)\|^2\\ &-\sum_{i=1}^{N}\alpha_{n,i}(\alpha_{n,0}-\lambda)\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2\\ &-\sum_{j=1}^{N}\alpha_{n,j}(\alpha_{n,0}-\lambda)\|v_n+\beta B^*(Au_n-Bv_n)-t_n^j\|^2. \qquad (18)\end{aligned}$$

Since f,g are pseudomonotones, $p\in EP(f,K)$ and $q\in EP(g,Q)$, we have that $f(x_n,p)\le 0$ and $g(y_n,q)\le 0$, so that

$$\begin{aligned}\|x_{n+1}-p\|^2+\|y_{n+1}-q\|^2 &\le \|x_n-p\|^2+\|y_n-q\|^2\\ &+2\gamma_n\varepsilon_n+2\delta_n\varepsilon_n+2\beta_n^2+2\eta_n^2\\ &-2\beta\Big(1-\beta(\|A\|^2+\|B\|^2)\Big)\|(Au_n-Bv_n)\|^2\\ &-\sum_{i=1}^{N}\alpha_{n,i}(\alpha_{n,0}-\lambda)\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|^2\\ &-\sum_{j=1}^{N}\alpha_{n,j}(\alpha_{n,0}-\lambda)\|v_n+\beta B^*(Au_n-Bv_n)-t_n^j\|^2. \qquad (19)\end{aligned}$$

Let $\Omega_n(p,q)=\|x_n-p\|^2+\|y_n-q\|^2$. By the assumptions $\beta\in\Big(0,\frac{1}{\|A\|^2+\|B\|^2}\Big)$ and $\alpha_{n,0}\in(\lambda,1)$, it follows from (19) that

$$\Omega_{n+1}(p,q)\le\Omega_n(p,q)+2\gamma_n\varepsilon_n+2\delta_n\varepsilon_n+2\beta_n^2+2\eta_n^2.$$

Using (iii), (iv) and applying Lemma 4 2, we have $\lim_{n\to\infty}\Omega_n(p,q)$ exists, where $b_n=2\gamma_n\varepsilon_n+2\delta_n\varepsilon_n+2\beta_n^2+2\eta_n^2$ and $\sum_{n=1}^{\infty}b_n<\infty$. Taking $\mu_n=(x_n,y_n)$, $\mu=(p,q)$ and $W=\Omega$, then the first condition of Lemma 5 2 is satisfied.

Observe from (19), we obtain

$$2\beta\Big(1-\beta(\|A\|^2+\|B\|^2)\Big)\|(Au_n-Bv_n)\|^2 \le \Omega_n(p,q)$$
$$-\Omega_{n+1}(p,q)+2\gamma_n\varepsilon_n+2\delta_n\varepsilon_n+2\beta_n^2+2\eta_n^2.$$

Thus, we have

$$\sum_{n=1}^{\infty}2\beta\Big(1-\beta(\|A\|^2+\|B\|^2)\Big)\|(Au_n-Bv_n)\|^2 \le \Omega_1(p,q)$$
$$+\sum_{n=1}^{\infty}2\gamma_n\varepsilon_n+\sum_{n=1}^{\infty}2\delta_n\varepsilon_n\sum_{n=1}^{\infty}2\beta_n^2+\sum_{n=1}^{\infty}2\eta_n^2<\infty.$$

It follows that

$$\lim_{n\to\infty}\|(Au_n-Bv_n)\|=0. \tag{20}$$

Also from (19), we have that

$$\lim_{n\to\infty}\|u_n-\beta A^*(Au_n-Bv_n)-s_n^i\|=0\ \forall\ i,\in\{1,2,3,\dots,N\}. \tag{21}$$

$$\lim_{n\to\infty}\|v_n+\beta B^*(Au_n-Bv_n)-t_n^j\|=0\ \forall\ j\in\{1,2,3,\dots,N\}. \tag{22}$$

Furthermore, since $\lim_{n\to\infty}\Omega_n(p,q)$ exists, it follows that $\{x_n\}$ and $\{y_n\}$ are bounded. Let x', y' be clusters points of $\{x_n\}$ and $\{y_n\}$ respectively. Then there exists a subsequence of $\{(x_n,y_n)\}$ with out loss of generality still denoted by $\{(x_n,y_n)\}$ such that $x_n\rightharpoonup x'$ and $y_n\rightharpoonup y'$. Since $\|x_n-u_n\|\le\beta_n$ and $\sum_{n=1}^{\infty}\beta_n^2<\infty$, we have

$$\lim_{n\to\infty}\|x_n-u_n\|=0. \text{ Similarly } \lim_{\mathrm{n}\to\infty}\|\mathrm{y_n}-\mathrm{v_n}\|=0. \tag{23}$$

Now

$$\begin{aligned} d(x_n,T_ix_n) &\le \|x_n-(u_n-\beta A^*(Au_n-Bv_n))\| \\ &\quad+\|u_n-\beta A^*(Au_n-Bv_n-s_n^i\| \\ &\quad+\mathscr{D}(T_i(u_n-\beta A^*(Au_n-Bv_n),T_ix_n). \end{aligned}$$

By Lemma 3 2 and Remark 2 1 T_i is Lipschitzian for each $i = 1,2,3,\ldots N$. Therefore,

$$\begin{aligned} d(x_n, T_i x_n) &\leq \|x_n - u_n + \beta A^*(Au_n - Bv_n)\| \\ &\quad + \|u_n - \beta A^*(Au_n - Bv_n) - s_n^i\| \\ &\quad + \frac{1+\sqrt{\lambda_i}}{1-\sqrt{\lambda_i}} \Big\| u_n - \beta A^*(Au_n - Bv_n) - x_n \Big\| \\ &\leq \|x_n - u_n\| + \beta \|A^*\| \|Au_n - Bv_n\| \\ &\quad + \|u_n - \beta A^*(Au_n - Bv_n) - s_n^i\| \\ &\quad + \frac{1+\sqrt{\lambda}}{1-\sqrt{\lambda}} \Big[\|u_n - x_n\| + \beta \|A^*\| \|Au_n - Bv_n\| \Big] \\ &= \frac{2}{1-\sqrt{\lambda}} \Big[\|x_n - u_n\| + \beta \|A^*\| \|Au_n - Bv_n\| \Big] \\ &\quad + \|u_n - \beta A^*(Au_n - Bv_n) - s_n^i\|. \end{aligned}$$

From (20), (21) and (23), we obtain

$$\lim_{n\to\infty} d(x_n, T_i x_n) = 0, \ \forall\ i = 1,2,3,\ldots N. \tag{24}$$

Similarly, we have that

$$\lim_{n\to\infty} d(y_n, U_j y_n) = 0, \ \forall\ j = 1,2,3,\ldots N. \tag{25}$$

Since $x_n \rightharpoonup x'$, T_i is demiclosed at zero for each $i = 1,2,3,\ldots N$ and using (24), we have $x' \in T_i x'$ for each $i = 1,2,3,\ldots N$, showing that $x' \in \bigcap_{i=1}^{N} F(T_i)$. Similarly $y' \in \bigcap_{j=1}^{N} F(U_j)$.

Next we show $x' \in EP(f,K)$ and $y' \in EP(g,Q)$.

Now from (18), we have

$$\begin{aligned} \Omega_{n+1}(p,q) &\leq \Omega_n(p,q) + 2\gamma_n f(x_n,p) + 2\gamma_n \varepsilon_n \\ &\quad + 2\delta_n \varepsilon_n + 2\beta_n^2 + 2\eta_n^2, \end{aligned}$$

so that

$$\begin{aligned} 2\gamma_n[-f(x_n,p)] &\leq \Omega_n(p,q) - \Omega_{n+1}(p,q) + 2\gamma_n \varepsilon_n + 2\delta_n \varepsilon_n \\ &\quad + 2\beta_n^2 + 2\eta_n^2. \end{aligned}$$

Since $\sum_{n=1}^{\infty} \beta_n \varepsilon_n < \infty$ and $\sum_{n=1}^{\infty} \beta_n = \infty$, we have $\lim_{n\to\infty} \varepsilon_n = 0$. Since $\{w_n\}$ is bounded and $\lim_{n\to\infty} \varepsilon_n = 0$, it follows from (C_4) that $\{w_n\}$ is bounded. Hence $\|w_n\| \leq M$, $\forall\ n \in \mathbb{N}$, for some $M > 0$. Let σ be such that $0 < \sigma_n < \sigma$, $\forall\ n \in \mathbb{N}$. Let

$$L = \max\{\mathrm{M}, \sigma\}. \tag{26}$$

Therefore using (26), we have

$$\frac{2}{L}\sum_{n=1}^{\infty}\beta_n[-f(x_n,p)] \leq 2\sum_{n=1}^{\infty}\gamma_n[-f(x_n,p)] \leq \Omega_1(p,q) + 2\sum_{n=1}^{\infty}\gamma_n\varepsilon_n + 2\sum_{n=1}^{\infty}\delta_n\varepsilon_n + 2\sum_{n=1}^{\infty}\beta_n^2 + 2\sum_{n=1}^{\infty}\eta_n^2.$$

By pseudomonotone property of f, we have $-f(x_n,p)\geq 0$. Since $\sum_{n=1}^{\infty}\beta_n = 0$, it follows that $\limsup_{n\to\infty} f(x_n,p)=0$. Similarly $\limsup_{n\to\infty} g(y_n,q)=0$.

Since $f(.,p)$ is weakly upper semicontinous, we have $f(x',p)\geq \limsup_{n\to\infty} f(x_n,p) = 0$. This shows that $f(x',p)\geq 0,\ p\in K$, and by pseudomonotone property of f we get $f(x',p)\leq 0$. Hence $f(x',p)=0$. Using (C_3) it follows that $x'\in EP(f,K)$.

Similar argument yields $y'\in EP(g,Q)$.

Since $\lim_{n\to\infty}\|x_n-u_n\|=0$, $\lim_{n\to\infty}\|y_n-v_n\|=0$, $x_n\rightharpoonup x'$ and $y_n\rightharpoonup y'$ as $n\to\infty$, we have that $u_n\rightharpoonup x'$ and $v_n\rightharpoonup y'$ as $n\to\infty$. As $A:H_1\to H_3$ and $B:H_2\to H_3$ are bounded linear maps, we get $Au_n\rightharpoonup Ax'$ and $Bv_n\rightharpoonup By'$ as $n\to\infty$.
These convergences imply that

$$Au_n - Bv_n \rightharpoonup Ax'-By' \text{ as n} \to\infty.$$

Hence as norm is weakly lower semicontinous, we have

$$\|Ax'-By'\| \leq \liminf_{n\to\infty}\|Au_n-Bv_n\| = \lim_{n\to\infty}\|Au_n-Bv_n\| = 0.$$

This implies $Ax'=By'$. That is $(x',y')\in\Omega$. Since every Hilbert space satisfies Opial condition, there is no more than one cluster point of the sequence $\{(x_n,y_n)\}$ and so the weak convergence of $\{(x_n,y_n)\}$ follows by applying Lemma 5 2 with $W=\Omega$. This completes the proof.

We now prove the following strong convergence theorem:

Theorem 2. If in addition to theorem (3) at least one of $T_{i's}$, for some $i\in\{1,2,3,\ldots,N\}$ and at least one of $U_{j's}$, for some $j\in\{1,2,3,\ldots,N\}$ is hemi-compact, then the iterative sequence $\{(x_n,y_n)\}$ generated by (11) converges strongly the solution of (5).

Proof: Assume T_s and U_h are hemicompacts for some $s,\ h\in\{1,2,3,\ldots,N\}$. Since $\{x_n\}$ and $\{y_n\}$ are bounded and $\lim_{n\to\infty} d(x_n,T_sx_n)=0$ and $\lim_{n\to\infty} d(y_n,U_hy_n) = 0$, then there exists subsequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ of $\{x_n\}$ and $\{y_n\}$ respectively such that $x_{n_k}\to\hat{x}$ and $y_{n_k}\to\hat{y}$ as $k\to\infty$. Since $x_{n_k}\rightharpoonup x'$

and $y_{n_k} \rightharpoonup y'$ as $k \to \infty$, we have $x' = \hat{x}$ and $y' = \hat{y}$. It follows from demiclosedness of T_i, $i = 1,2,3,\dots,N$ and U_j, $i = 1,2,3,\dots,N$ that $x' \in \bigcap_{i=1}^N F(T_i)$ and $y' \in \bigcap_{j=1}^N F(U_j)$. Using the same arguments as in the proof of theorem (3) we have $x' \in EP(f,K)$ and $y' \in EP(g,Q)$.

Furthermore, since $x_{n_k} - u_{n_k} \to 0$ and $y_{n_k} - v_{n_k} \to 0$ as $k \to \infty$, we obtain $u_{n_k} \to x'$ and $v_{n_k} \to y'$ as $k \to \infty$. Thus,

$$\|Ax' - By'\| = \lim_{k\to\infty} \|Au_{n_k} - Bv_{n_k}\| = 0,$$

and so $Ax' = By'$. This implies $(x',y') \in \Omega$. On the other hand since $\Omega_n(p,q) = \|x_n - p\|^2 + \|y_n - q\|^2$ for any $(p,q) \in \Omega$, we know that $\lim_{k\to\infty} \Omega_{n_k}(x',y') = 0$. From Theorem (3) we have $\lim_{n\to\infty} \Omega_n(x',y')$ exists, therefore $\lim_{n\to\infty} \Omega_n(x',y') = 0$. Hence $\lim_{n\to\infty} \|x_n - x'\| = 0$ and $\lim_{n\to\infty} \|y_n - y'\| = 0$. The proof is complete.
For $N = 1$, we obtain the following corollary:

Corollary 1: Let H_1, H_2, H_3 be real Hilbert spaces and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear maps with their adjoints A^*, B^* respectively. Let $K \subset H_1$ and $Q \subset H_2$ be nonempty, closed, convex subsets of real Hilbert space H_1 and H_2 respectively. Let $f : K \times K \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be bifunctions satisfying $(C1)-(C4)$. Let $T : H_1 \to \mathscr{C}B(H_1)$ and $U : H_2 \to \mathscr{C}B(H_2)$, be two multivalued strictly pseudocontractive mappings such that $\Omega = \Big\{(p,q) \in K \times Q : p \in F(T) \cap EP(f,K),\ q \in F(U) \cap EP(g,Q),\ Ap = Bq\Big\} \neq \emptyset$. Let T and U satisfy endpoint property and $\{\sigma_n\}$, $\{\xi_n\}$, $\{\alpha_n\}$, $\{\beta_n\}$, $\{\eta_n\}$, $\{\varepsilon_n\}$ be sequences of positive real numbers satisfying:

(i) $\sigma_n,\ \xi_n \in (c,d)$, $\forall n \in \mathbb{N}$ for some numbers c,d with $0 < c < d$,
(ii) $\alpha_{n,0} \in (\lambda,1)$, $\liminf_{n\to\infty} \alpha_{n,0} > \lambda$, $\lambda = \max\{\lambda^*,\ \lambda^{**}\}$,
(iii) $\sum_{n=1}^\infty \beta_n = \infty$, $\sum_{n=1}^\infty \beta_n^2 < \infty$ and $\sum_{n=1}^\infty \beta_n\varepsilon_n < \infty$ and
(iv) $\sum_{n=1}^\infty \eta_n = \infty$, $\sum_{n=1}^\infty \eta_n^2 < \infty$ and $\sum_{n=1}^\infty \eta_n\varepsilon_n < \infty$.

If $I - T$ and $I - U$ are demiclosed at zero, then the sequence $\{(x_n,y_n)\}$ generated by (11) converges weakly to the solution of problem (5).
We have the following corollary for finite family of multivalued nonexpansive mappings in Hilbert space H

Corollary 2: Let H_1, H_2, H_3 be real Hilbert spaces and $A : H_1 \to H_3$, $B : H_2 \to H_3$ be bounded linear maps with their adjoints A^*, B^* respectively. Let $K \subset H_1$ and $Q \subset H_2$ be nonempty, closed, convex subsets of real Hilbert spaces H_1 and H_2 respectively. Let $f : K \times K \to \mathbb{R}$ and $g : Q \times Q \to \mathbb{R}$ be bifunctions satisfying $(C1)-(C4)$. Let $T_i : H_1 \to$

$\mathscr{C}B(H_1)$, $i=1,2,3,\ldots,N$ and $U_j : H_2 \to \mathscr{C}B(H_2)$, $j=1,2,3,\ldots,N$ be two finite families of multivalued nonexpansive mappings such that $\Omega = \Big\{(p,q)\in K\times Q : p\in \Big(\bigcap_{i=1}^{N} F(T_i)\Big)\cap EP(f,K),\ q\in \Big(\bigcap_{j=1}^{N} F(U_j)\Big)\cap EP(g,Q),\ Ap = Bq\Big\} \neq \emptyset$. Let T_i for $i=1,2,3,\ldots,N$ and U_j for $j=1,2,3,\ldots,N$ satisfy endpoint property and $\{\sigma_n\}$, $\{\xi_n\}$, $\{\alpha_{n,i}\}$, $\{\beta_n\}$, $\{\eta_n\}$, $\{\varepsilon_n\}$ be sequences of positive real numbers satisfying:

(i) $\sigma_n,\ \xi_n \in (c,d)$, $\forall n\in\mathbb{N}$ for some numbers c,d with $0<c<d$,
(ii) $\alpha_{n,i}\in(a,1-a)$ where $a\in(0,1)$, $\liminf\limits_{n\to\infty}\alpha_{n,i}>0$ and $\liminf\limits_{n\to\infty}\alpha_{n,j}>0$, for $i,j\in\{1,2,3,\ldots,N\}$,
(iii) $\sum_{n=1}^{\infty}\beta_n=\infty$, $\sum_{n=1}^{\infty}\beta_n^2<\infty$ and $\sum_{n=1}^{\infty}\beta_n\varepsilon_n<\infty$ and
(iv) $\sum_{n=1}^{\infty}\eta_n=\infty$, $\sum_{n=1}^{\infty}\eta_n^2<\infty$ and $\sum_{n=1}^{\infty}\eta_n\varepsilon_n<\infty$.

If at least one of $T_{i's}$, for some $i\in\{1,2,3,\ldots,N\}$ and at least one of $U_{j's}$, for some $j\in\{1,2,3,\ldots,N\}$ is hemi-compact, then the sequence $\{(x_n,y_n)\}$ generated by (11) converges strongly to the solution of problem (5).

Remark 4 Our theorems and corollaries generalise and improve the results in Ma et al. [22] in the following senses:

(a) For the mappings,we consider multivalued strictly pseudocontractive mappings which contains multivalued nonexpansive mappings as a special case, which itself generalises single-valued nonexpansive mappings. Furthermore, they considered two single-valued nonexpansive mappings, while in this paper we consider two finite families of multivalued strictly pseudocontractive mappings.
(b) In Ma et al. [22], proximal point method for split equality monotone equilibrium problem is considered, while in this paper a hybrid projected subgradient method for split equality pseudomonotone equilibrium problem is considered.

4. NUMERICAL EXAMPLE

In this section, we demonstrate numerical example of Theorem 3 in real line $\mathbb{R}$.
Let $H_1=H_2=H_3=\mathbb{R}$ with $\|.\|=|.|$, $K=Q=[0,1]$ and set $N=4$. Let $T_ix:\mathbb{R}\to 2^{\mathbb{R}}$ be defined by $T_ix=[(-\frac{1}{2}-i)x,-ix]$ for $i=1,2,3,4$. Then T_i is λ_i-strictly pseudocontractive multivalued mapping with $\lambda_1=\frac{5}{16}$, $\lambda_2=\frac{21}{36}$, $\lambda_3=\frac{45}{64}$, $\lambda_4=\frac{77}{100}$. Also let $U_jx:\mathbb{R}\to 2^{\mathbb{R}}$ be defined by $U_jx=$

$[(-\frac{1}{3}-j)x,-x]$ for $j=1,2,3,4$. Then U_j is λ_j-strictly pseudocontractive multivalued mapping with $\lambda_1=\frac{7}{36},\ \lambda_2=\frac{40}{81},\ \lambda_3=\frac{91}{144},\ \lambda_4=\frac{160}{225}$. It is clear that $T_i(0)=U_j(0)=\{0\}$ for $i,\ j\in\{1,2,3,4\}$.

Observe that $\lambda^*=\max\{\frac{5}{16},\ \frac{21}{36},\ \frac{45}{64},\ \frac{77}{100}\}=\frac{77}{100}$ and $\lambda^{**}=\max\{\frac{7}{36},\ \frac{40}{81},\ \frac{91}{144},\ \frac{160}{225}\}=\frac{160}{225}=\frac{32}{45}$. Therefore $\lambda=\max\{\lambda^*,\ \lambda^{**}\}=\max\{\frac{77}{100},\ \frac{32}{45}\}=\frac{77}{100}$. Let $\alpha_{n,0}=\frac{4}{5}>\frac{77}{100}$ and $\alpha_{n,i}=\alpha_{n,j}=\frac{1}{20}$ for $i,j\in\{1,2,3,4\}$.

Now consider the nonsmooth equilibrium problem with bifunctions f,g defined by $f(x,y)=2xy(y-x)+xy|y-x|\ \forall x,y\in K$ and $g(x,y)=5xy(y-x)+xy|y-x|\ \forall x,y\in Q$. Then f,g are pseudomonotones. $f(x,.)$, $g(x,.)$ are convex, $0\in EP(f,K)$, $0\in EP(g,Q)$. Furthermore, For $\varepsilon_n=0$, $\partial f(x,.)x=[x^2,3x^2]$ and $\partial g(x,.)x=[4x^2,6x^2]$.

Define $A,\ B:\mathbb{R}\to\mathbb{R}$ by $Ax=5x$ and $Bx=3x$. Then $A,\ B$ are bounded linear maps with adjoints $A^*=A,\ B^*=B$, $|A|=5,\ |B|=3$ and $A(0)=B(0)$. Thus $(0,0)\in\Omega$.

Taking $\varepsilon_n=0,\ \beta_n=\frac{1}{n},\ \eta_n=\frac{2}{n},\ \sigma_n=1,\ \xi_n=1,\ \beta=\frac{1}{40}\in(0,\frac{1}{|A|^2+|B|^2})$, $h_n=u_n-\beta A^*(Au_n-Bv_n)$ and $m_n=v_n+\beta B^*(Au_n-Bv_n)$. Then $h_n=u_n-\frac{1}{8}(5u_n-3v_n)$, $m_n=v_n+\frac{3}{40}(5u_n-3v_n)$ and the iterative scheme (11) becomes

$$\begin{cases} (x_1,y_1)\in[0,1]\times[0,1], \\ w_n\in[x_n^2,3x_n^2], \\ z_n\in[4y_n^2,6y_n^2], \\ u_n=P_K(x_n-\gamma_n w_n), \qquad \gamma_n=\frac{\beta_n}{max\{1,|w_n|\}}, \\ v_n=P_Q(y_n-\delta_n z_n), \qquad \delta_n=\frac{\eta_n}{max\{1,|z_n|\}}, \\ s_n^i\in[(-\frac{1}{2}-i)h_n,-ih_n],\ i=1,2,3,4 \\ t_n^j\in[(-\frac{1}{3}-j)m_n,-m_n],\ j=1,2,3,4 \\ x_{n+1}=P_K\left(\frac{4}{5}h_n+\frac{1}{20}\sum_{i=1}^4 s_n^i\right), \\ y_{n+1}=P_Q\left(\frac{4}{5}m_n+\frac{1}{20}\sum_{j=1}^4 t_n^j\right). \end{cases} \tag{27}$$

Using (27) with initial points $x_1=0.18,\ y_1=0.24$, the numerical results using MATLAB is given in Figure 1 and Figure 2.

5. ACKNOWLEDGEMENTS

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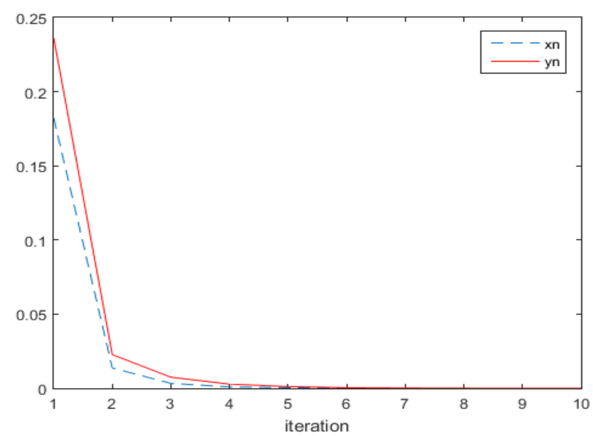

Fig. 1. Convergence process of $\{(x_n,\ y_n)\}$

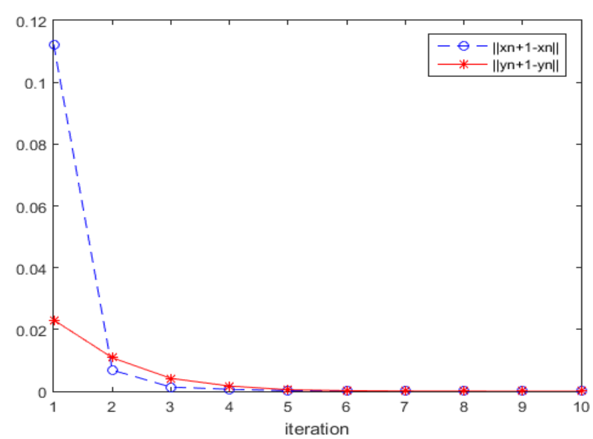

Fig. 2. Errors versus number of iterations

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