Journal of the
Nigerian Mathematical Society

Vol.43, Issue 3, pp.273 - 286, 2024

QUALITATIVE CLASSIFICATION OF AN AUTONOMOUS DIFFERENTIAL EQUATION AND ITS APPLICATION TO IMAGE RECOGNITION SYSTEM

BAMIDELE OLUWADE[1]

ABSTRACT. In (face) image recognition system, a set of test images is compared to a set of training images for verification. A basic requirement for effective matching of these face images is that each image is structurally unique/distinct. This uniqueness is important in applications such as the issuance of national identification card and bank verification number. Standard algorithms and software exist on image recognition systems but there is still room for additional efforts. The present paper considers the problem of creating unique images for a hypothetical model of face image recognition system, such that the faces in the training data and the test data are simulated. The paper presents the classification of a first order autonomous ordinary differential equation having a nonlinear quintic polynomial part via the critical points of the equation. This is accomplished using a novel classification method, which may be called differential structure method, that was earlier developed by the author. The relevant qualitative properties are existence and uniqueness of solutions. It is shown that forty six (46) qualitative classes or unique images can be generated for matching in a simulated environment. A higher number of images can be generated by using a higher degree polynomial differential equation. Essentially, the results in the paper are also applicable to population modelling.

1. INTRODUCTION

In (face) image recognition system [1],[2], [3], a set of test images is compared to a set of training images for verification. A basic requirement for effective matching of these face images is that each image

Received by the editors April 28, 2024; Revised: August 08, 2024; Accepted: August 19, 2024

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp. it/jnms/

2020 *Mathematics Subject Classification.* 34C41, 68T10,34C28, 34M04.

Key words and phrases. Qualitative classification, Autonomous system, Image recognition system, Differential equation, Phase portrait.

is structurally unique/distinct. This uniqueness is important in applications such as the issuance of national identification card and bank verification number. Several algorithms and software exist on image recognition systems. These are essentially based on popular recognition algorithms such as the principal component analysis with eigenface, elastic bunch graph matching fisherface, neuronal motivated dynamic link matching, linear discriminate analysis and hidden Markov model e.g. see [4], [5], [6].The present paper presents a novel approach to the problem of creating unique images for a hypothetical model of face image recognition system, such that the faces in the training data and the test data are simulated.

Qualitative classification presents a reliable procedure for generating structurally distinct images. Qualitative classification of a set of differential equations or systems is important in computational problems because it enables the deduction of the qualitative properties of a member (A) of the set from the known properties of another member (B) of the set, both of which belong to the same qualitative class. This classification is based on the concept of qualitative equivalence of differential equations [7]. Generally, A and B are said to be qualitatively equivalent (i.e. belong to the same qualitative class) if there exists a continuous injection from the phase portrait of A onto the phase portrait of B such that the orientation of their phase portraits or trajectories is preserved.

That is, in the qualitative theory, differential equations are studied from the point of view of the geometry of their solution curves instead of solving them analytically. Two equations which belong to the same class have the same phase portraits i.e. same structural images. This is based on the fundamental theorem of equivalence relations in abstract algebra [8]. The relevant qualitative properties are existence and uniqueness of solutions. The qualitative equivalence of ordinary differential equations had earlier been shown to have applications in such practical areas of computing as the design of networks, the construction of fractals, the development of novel AI algorithms for resource-sharing protocols and the design of components of building/architectural structures.[9], [11], [46],[47].

In the present paper, the author presents the classification of a first order autonomous ordinary differential equation which has a polynomial nonlinear part of the form

$$x^{/} = f(x) = \sum_{i=0}^{n} a_i x^i \tag{1}$$

where $a_i \in \mathbb{R}$ and $a_n \neq 0$, $n = 5$. This is with respect to the critical points of the equation. The phase portrait of (1), which is a graphical or geometrical representation of the qualitative behavior of the equation, is completely determined by the nature of its critical points.

Qualitative classification is accomplished in this paper using a method, which may be called differential structure method, that was earlier developed by the author [11], [26]. The method involves a geometrical procedure for the ordering of the complex critical points (ccp) of (1) in space such that two ccp, or a ccp and a real critical point, may be computationally compared.

Equation (1) is important because it arises in practice as a model of the rate of formation of substances, such as in chemical science and technology [7]. Suppose $R_1, R_2, \ldots, R_n$ are the n reactants and P the product in a chemical reaction. Then this reaction can be written as

$$R_1 + R_2 + \ldots + R_n \to P \tag{2}$$

Suppose further that $m_1, m_2, \ldots, m_n$ grams of $R_1, R_2, \ldots, R_n$ respectively, combine to give 1 gram of P. Now, the rate of formation of P at any instant of time t equals the product of the masses of $R_1, R_2, \ldots, R_n$ which remain uncombined at that instant. If $a_1, a_2, \ldots, a_n$ grams of $R_1, R_2, ..., R_n$ respectively are brought together at $t = 0$, then the mass $x(t)$ of P present at time $t > 0$ is governed by the equation

$$x^{/} = \prod_{i=1}^{n} (a_i \breve{} m_i x) \tag{3}$$

. Using $n = 5$, forty six qualitative classes of (1) are generated in the present paper. It follows that a total of forty six (46) unique human facial images can be generated for matching in a model of an image recognition system. More facial images can be generated using higher values of n.

By the Abel-Ruffini theorem and Galois theory of equations [10], there is no solution in radicals for the general nth degree polynomial when $n \geq 5$. However, by the Fundamental Theorem of Algebra, every polynomial equation of degree $n > 0$ necessarily has a solution! This does not suppose that solution exists in $\mathbb{R}$ $\forall$n ≥ 5. Nevertheless, the goal of the present paper is not to explicitly determine or deduce what the specific zeros of the polynomial are. The purpose of the paper is to consider all the possible phase portraits which arise with respect to the expected n zeros of the polynomial. This is in line with the principle of the qualitative theory of differential equations.

The rest of the paper is arranged as follows: In Section 2, a general review of literature on qualitative classification, image recognition systems and related areas is presented. Section 3 dwells on the qualitative classification of (1) when $n = 5$, thereby generating qualitative classes.

This arises from the phase portraits of (1) when the critical points of the equation are real and complex. Section 4 discusses the link between qualitative classification and the generation of images for image recognition systems. Section 5 concludes the paper.

2. LITERATURE REVIEW

In this section, a review of literature is carried out on the qualitative classification of differential equations, as well as on image recognition system and related areas.

A fair proportion of research activities in the field of differential equations focuses on the algebraic and topological classification of these equations. Generally, differential equations may be represented from an analytic, algebraic and geometric perspective. An overview of these approaches, based on the qualitative behavior of the solution spaces, is presented in [12]. Several works have been done in the literature on the qualitative classification of dynamical systems, incorporating the phase portraits and limit cycles of polynomial vector fields, via different methods. These include [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23].

One of the popular methods is the use of affine transformation with respect to vector fields associated with two-dimensional autonomous differential equations. These fields usually consist of real polynomials $P(x,y)$ of stated degrees k. For instance, for a quadratic system, $k = 2$ while $k = 3$ for a cubic system. Affine equivalence classes may then be obtained for the vector fields without the critical points at infinity. An affine function is a mapping whose graph is a linear function which doesn't necessarily pass through the origin [43]. Two curves are said to be affinely equivalent if there exists an affine mapping which transforms one of the curves onto another. An affine transformation preserves parallel lines, points and planes. Normally, affine transformation may include translation, rotation or scaling. In particular, affine classification of quadratic and cubic systems was carried out in [23] and [15]. It was shown in the former that quadratic systems with degenerate infinity in eight types have 13 phase portraits on the Poincare sphere while it was shown in the latter that quadratic vector fields without the critical points (cp) at infinity have exactly 5 affine equivalent classes.The calculation of the number of limit cycles of dynamical systems is the focus of [24],[25].

In a preceding paper [11],the author presented a combination of algebraic and geometrical methods for describing the phase portraits of (1) when $n = 4$, via the critical points of the equation. The representation of these phase portraits is carried out by providing a natural order for a subset of C (the set of complex numbers) consisting of all the critical points of (1) and using analogies in computer data/discrete structures. A basis for the analogy is that the number of critical points of the equation is always discrete and also that the number of phase portraits of the equation when $n = 1$ (which forms the fundamental building block) is discrete. In the paper, the set

$$\{x^{/} = f(x)\} \tag{4}$$

from (1) was classified into disjoint equivalence classes (qc) and a relationship established between the qualitative equivalence of (1) and fractals. It was proved that twenty eight (28) distinct qualitative classes can, up to isomorphism, be generated from (1) when $n = 4$. Ten (10) of these arise when some or all the critical points of (1) are complex-valued. It had earlier been shown that the same equation (1) has a total of sixteen (16) qualitative classes when $n = 3$ [26]; although there were gaps in part of the proof (not in the theorem) of the statement leading to this result, these were filled up in [11].

A detailed study of the concept of qualitative equivalence, especially when $n = 1$ (linear case) and $n = 2$ (quadratic case) can be found in [27].The equation has five qc in the former case and six in the latter. In [28], the author's focus was on the stability property of solutions of differential equations, instead of existence and uniqueness. He established a necessary condition for qualitative stability of first order linear autonomous ordinary differential system. The methodology developed in [29] for binary systems provides a procedural connection between qualitative equivalence and coded character sets since, for instance, the set of phase portraits of (1) when $n = 5$ can be considered as a (non-uniform quintic) code.

The study of phase portraits is important in physical systems and this is evident in many applied problems e.g. see [30] [31][32][33]. Qualitative classification naturally arises from the notion of qualitative equivalence. In [34], a necessary condition for qualitative equivalence of two first order autonomous ordinary differential equations of the transcendental type was established. This was accomplished by reducing one of the transcendental functions into the other as a polynomial via the power series expansion. Earlier in [35], the Galois group of the Chebyshev polynomials of the first kind of prime degree was studied. This is a class of polynomials which is widely applied in many areas including numerical computation such as Tau numerical method [36]. It was shown that this group is isomorphic to the cyclic group of order 2.

Severance [37] traced the historical development in biometric recognition system from about the early 1990s to about 2015. The focus of [1],[2] is on general principles and development on face recognition system. Some works on peculiar methodologies and applications of face recognition system include [4], [5],[6], [38], [39], [40]. In [3], a procedure was presented for characterizing and comparing images/characters based on their internal structure, in such a way that these are independent of image size and orientation. In this procedure, a library of reference images (training data) is first generated and stored. Each input image (test data) is then compared to the training data until a match is found. Storing of an image involved the creation of an image of the character followed by a reduction of the image to a skeleton image. This skeleton image is then represented in the form of a linked list as nodes, lines and curves. In their own work, [41] investigated the variational properties of face images using 10 samples of different face images of each of 48 different individuals. These images were processed under different light intensities. The resulting 480 face images were splitted into a training test and a test set (in a 75% to 25% ratio respectively), and then evaluated/analyzed using thirteen distinct modules available in MATLAB image processing toolbox. Based on the experiment, the authors recommended the principal component analysis with eigenfaces as a preferred algorithm.

3. QUALITATIVE CLASSIFICATION

This section presents the qualitative classification of (1) when $n = 5$ using the concept of qualitative equivalence of differential equations.

Let the critical points of (1) when $n = 5$ be $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5. The classification is based on the fact that the phase portrait of (1) on the line when there is a unique critical point (i.e. when $n = 1$) is a member of the set

$$S = \{Attractor(A), Repellor(R), PositiveShunt(P), NegativeShunt(N)\} \tag{5}$$

. The phase portrait of the equation for $n > 1$ is a combination of elements of S and is defined as a generation in [11],[26].When the critical points of (1) have at least a complex value, a geometric method was developed for comparing a real-valued critical point with a complex-valued critical point, and also for comparing two complex-valued critical points. This is to enable the points to be ordered in space. Ordering of a subset of C is possible in a cartesian coordinate space by considering the real part to lie along the $x-axis$ while the imaginary part lies along the $y-axis$. The above is accomplished as follows: Suppose $u_1 = (r_1, s_1) = r_1 + is_1$ and $u_2 = (r_2, s_2) = r_2 + is_2$ are two critical points in C, where $r_1, r_2, s_1, s_2 \in \mathbb{R}$. If $r_1 = r_2$, let $u_1 \leq u_2$ if $s_1 \leq s_2$, where "$\leq$" has the usual meaning of "*less than or equal to*". For example, $3-2i < 3+5i < 3+8i$. If $s_1 = 0$, let $u_1 \leq 0(\geq 0)$ if $r_1 \leq 0(\geq 0)$. This reduces to the usual ordering in R. If $r_1 \neq r_2$, let $u_1 \leq u_2$ if $r_1 \leq r_2$, $\forall s_1, s_2$. For example, $-3+70i < 8+20i < 10+5i$.

Some other basic definitions of the method of differential structures in C [11],[26] are presented below, followed by the statements of the results.

Definition 3.1:(Fundamental Differential Structure) A fundamental differential structure (fds) refers to a line segment formed by joining two complex-valued critical points. An fds is perpendicular to the real axis of the complex plane.

Definition 3.2:(Inferior (Superior) Endpoint) An inferior (superior) endpoint of an fds (abbreviated as inf (sup) endpoint) is the point which has the smaller (greater) numeric value.

Definition 3.3:(Derived Differential Line) A derived differential line (ddl)is a straight line drawn from one end of an fds to meet a third critical point.

Definition 3.4:(Derived Differential Structure) A derived differential structure (dds) refers to the geometrical figure formed when a straight line is drawn from one end of an fds to meet another critical point.

Definition 3.5:(Inferior-left (Inferior-right) Structure) An inferior-left(inferior-right) structure (abbreviated as infleft(infright) structure) is a dds in which the ddl is produced from the inferior endpoint of an fds and drawn towards the left(right) side of the fds.

Definition 3.6:(Superior-left(Superior-right) Structure) A superior-left (superior-right) structure (abbreviated as supleft(supright) structure refers to a dds in which the ddl is produced from the superior endpoint of an fds and drawn towards the left(right) side of the fds.

Definition 3.7:(Trivial Differential Structure) A trivial differential structure (tds) is a straight line joining two real-valued critical points. A tds is perpendicular to an fds.

Definition 3.8:(Linked Structure) Two or more differential structures are said to form a linked structure (ls) if they have a common ddl.

Definition 3.9:(Linker) A linker refers to each constituent structure of an ls. An ls is named from left to right. The first linker encountered is said to be linker I, the second linker is called linker II, etc.

Theorem 3.1: (Number of qualitative classes for real critical points of equation (1)). The number of qualitative classes of (1) when all the critical points of f are real is twenty six.

Proof: The set of distinct generations arising from the subsets of (1) when all the critical points are real is $\{RARAR$, ARARA, NRAR, PARA, RPAR, ANRA, RARP, ARAN, RANR, ARPA, RAR, ARA, RAN, ARP, NNR, PPA, NRP, PAN, RPP, ANN, NR, PA, RP, AN, R, A$\}$. For example, the phase portrait *PARA* arises when $a_n > 0$, $\alpha_1 = \alpha_2 < 0$ and α_3, α_4, $\alpha_5 > 0$. The order of this set is seen to be 26 and hence the result.

Theorem 3.2: (Number of qualitative classes when there are exactly two complex-valued critical points). If (1) has exactly two complex-valued critical points, then the number of qualitative classes of (4) is eight.

Proof: Let α_4 and α_5 be the two complex-valued critical points of (1). Then the equation has four isomorphic differential structures which correspond to (i) $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ (ii) $\alpha_4 < \alpha_5 < \alpha_1 < \alpha_2 < \alpha_3$ (iii) $\alpha_1 < \alpha_4 < \alpha_5 < \alpha_2 < \alpha_3$ (iv) $\alpha_1 < \alpha_2 < \alpha_4 < \alpha_5 < \alpha_3$. In (i), two trivial differential structures (tds) joined to each other are in turn joined to an infleft. In (ii), a supright is joined to two tds (which are joined to each other). In the case of (iii), a 2-linker linked structure is joined to a tds where linker I is an infleft and linker II is a supright. In (iv), a tds is joined to a 2-linker linked structure in which linker I is an infleft and linker II a supright. The generation of each structure is RARAR when $a_n > 0$ and ARARA when $a_n < 0$. And the result follows.

Theorem 3.3: (Number of qualitative classes when there exists only one real critical point). Suppose (1) has only one real critical point, then (4) has twelve qualitative classes.

Proof: Let α_1 be the single real-valued critical point of (1).Then there are six isomorphic differential structures arising from the equation. These correspond to (i)$\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4 < \alpha_5$ (ii)$\alpha_2 < \alpha_3 < \alpha_4 < \alpha_5 < \alpha_1$ (iii)$\alpha_2 < \alpha_3 < \alpha_1 < \alpha_4 < \alpha_5$ (iv)$\alpha_1 < \alpha_2 = \alpha_4 < \alpha_3 = \alpha_5$ (v)$\alpha_2 = \alpha_4 < \alpha_3 = \alpha_5 < \alpha_1$ (vi)$\alpha_2 = \alpha_4 < \alpha_3 = \alpha_5 < \alpha_1$. (i) is a linked structure in which linker I is an infleft, linker II a upright while linker III is an infleft. (ii) is a linked structure in which linker I is a supright, linker II an infleft while linker III is a supright. In the case of (iii), the structure has two linkers such that linker I is a supright and linker II an infleft.The structures formed in (iv),(v) and (vi) are respectively an infleft, an infright and a supright. The generation of each of (i) - (iii) is RARAR when $a_n > 0$ and ARARA when $a_n < 0$. For each of (iv)

- (vi), there is a distinct generation when $a_n > 0$ and another distinct generation when $a_n < 0$. And the result follows.

Theorem 3.4: (Total number of qualitative classes when $n = 5$). The total number of qualitative classes of (1) is forty six.

Proof: This follows by adding the number of qualitative classes in Theorem 3.1, Theorem 3.2 and Theorem 3.3.

4. DISCUSSION

Based on the results in Section 3 above, 46 unique face images can be created for recognition. The procedure discussed in the section works only perfectly when the population (PL) is exactly equal to the total number of qualitative classes for an arbitrary n. This is depicted in Table 4.1.

Table 4.1. Degree n of polynomial corresponding to an exact number of face images

Population (PL)	Degree (n) of Polynomial in Equation (1)
5	1
6	2
16	3
28	4
46	5

Thus, for instance, when $1 \leq PL \leq 4$ and $PL = 8$, the procedure will not provide an exact value of n, though an approximate scheme may be used. In these cases, $n = 1$ and $n = 2$, respectively, may be used for the population. Also, when $PL = 40$, the value of $n = 5$ may be used. In general, for a given PL that does not have an exact corresponding n in Table 4.1, the nearest PL is first determined. Then the value of n which corresponds to this nearest PL is assumed.

By extension, consider the current world human population which is estimated to be $PL = 8,094,640,535$[42].Then there exists a value n of (1) which generates all the PL human faces or generates an approximate number of faces. The determination of this exact or approximate value of n is non-trivial and involves understanding the pattern of the qualitative classes for several values of n. And once the algorithm is understood, a computer software can be developed which would deduce the value of n for any possible human population in a community, town, country, continent or the entire world. The software can be a standalone one or can be integrated into existing image recognition systems.

In Section 3, the total number of qualitative classes of (1) when $n = 5$ was first calculated via the critical points (cp) of the equation. By using the concept of differential structures developed in [11], a total of 46 qualitative classes were generated. When the equation has exactly two complex-valued cp ($n = 5$), two(out of the four resulting) differential structures are isomorphic to two (out of the three) structures formed when there are exactly two complex-valued cp in the case $n = 4$. When (1) ($n = 5$) has exactly four cp, one (out of the six) structures is isomorphic to one (out of the two) structures formed when there are exactly four cp in the case $n = 4$. Two other structures (when $n = 5$) are isomorphic to two of the structures formed when exactly two cp are complex for $n = 3$.

In general, it can be said that for any particular value of n in (1), there is a 1-1 correspondence between the set of unique faces and the set of phase portraits.

From the perspective of machine learning and artificial intelligence (AI), the procedure of qualitative equivalence of differential equations is a classification algorithm. That is, it is in the same category as popular algorithms such as K-nearest neighbor (KNN), Decision Tree, Naive Bayes algorithm, Artificial Neural Network (ANN)/Deep Learning, Support Vector Machine (SVM), Perceptron and Logistic Regression [37][44],[45],[46].

5. CONCLUSION

The paper presents a link between the qualitative classification of a set of differential equations and face image recognition system. Essentially, the concept of qualitative equivalence was used to generate 46 structurally distinct human face images which are available for matching in a simulated experiment. More number of images can be generated by increasing the value of n in (1). Conjecturally, there is a direct proportionality between the value of n and the number of faces that can be generated.

In particular, further work can be done to find the exact (or approximate) value of n which will give the qualitative classes corresponding to the population $p = 8,094,640,535$, which is the estimated population of the world as at 3rd March, 2024 [42]. Similarly, this can be done for other estimated populations whether in a community, town, nation, continent or the entire world at a particular time.

Since the degree of a polynomial, by definition, is necessarily a non-negative integer, the question of finding an exact value of n for every population using the method described in this paper appears to be impossible! This calls for new approaches to the subject matter using other methods. For instance, qualitative study of fractional differential equations may be explored for associated problems, towards ensuring that every PL has a unique $n \in \Re$ or C. Also, studies on the qualitative classification of second and higher order ordinary differential equations may be pursued. Furthermore, qualitative classification of first and higher order partial differential equations, differential-difference equations and functional differential equations may be explored using the technique of differential structure described in this paper. The technique of differential structure may be applied to other dynamical systems by depicting the phase portraits of these systems in terms of the terminologies of differential structure. This has the potential of opening up a new approach to understanding the behaviours of the systems. In general, the limitations of the methods of qualitative classification may be improved by exploring the blending of novel mathematical methods into their procedure. This will metamorphose into a qualitative-analytic classification method, similar to the one used in [34].

This paper has thus applied the result of the qualitative classification of (1) to the generation of unique images in a model of face image recognition system. Essentially, a relationship is established between the population of a community and the total qualitative classes of (1).

Further work may also be done by using existing optimization techniques to find the most optimally valid value of n for a given population. In the present paper, one starts with n and generates a corresponding population PL. Using reverse engineering, it will be interesting if one can start with a given population and then use the value of PL to obtain an optimal n. Further work may also be done to investigate how a step change in the value of n (i.e. when n is increased to $n+1\ \forall n$) affects the increment in population corresponding to n and $n+1$ respectively. It is envisaged that related works in the future will enable the prediction of population growth in a community, which may complement statistical results/techniques on birth-death process and related phenomena.

6. ACKNOWLEDGEMENTS

This research is supported by a generous research grant from a group of foreign-based Nigerians. The author is appreciative of this kind gesture. Furthermore, the author thanks the referees for valuable comments and suggestions that have improved this paper.

REFERENCES

[1] Asit Kumar Datta, Madhura Datta, Pradipta Kumar Banerjee, *Face detection and recognition: theory and practice*, CRC, 2015.

[2] Stan Z. Li, Anil K. Jain, *Handbook of face recognition*, Springer Science & Business Media, 2005.

[3] Mark A. Walch, John A Pawlicki, *Image recognition system*, United States Patent 5267332, 30th November, 1993.

[4] Tom Simonite, *Facebook creates software that matches faces almost as well as you do*, MIT Technology Review, http://www.technologyreview.com, 2014 (accessed 03 March 2024).

[5] Kevin Bryson, *Evaluating anti-facial recognition tools*, The University of Chicago (Physical Sciences), https://www.physicalsciences.uchicago.edu, 2023 (accessed 03 March 2024).

[6] Benjamin Riggan, Nathaniel Short, Shuowen Hu, *Thermal to visible synthesis of face images using multiple regions*, 2018 IEEE Winter Conference on Applications of Computer Vision (WACV), 30-38, 2018.

[7] D.K.Arrowsmith and C. M. Place, *Ordinary differential equations:A qualitative approach with applications*, Chapman and Hall, London, 1982.

[8] (Bami)dele Oluwade, *A note on the algebra of qualitative equivalence of ordinary differential equations*, African Journal of Computing & ICT **3**(12) 39-44, 2010.

[9] (Bami)dele Oluwade, *A model for constructing subgraphs of hypercubes*, IEEE AFRICON 2007, EE Publishers, South Africa, 1-7, 2007.

[10] Ian Stewart,*Galois theory (5th edition)*,Chapman and Hall/CRC, 2022.

[11] (Bami)dele Oluwade, *Modelling fractal patterns via the qualitative equivalence of a nonlinear ODE*, Nonlinear Analysis **63**(5–7) e2409–e2414, 2005.

[12] F. Neuman, *Smooth and discrete systems-algebraic, analytic, and geometrical representations*, Advances in Difference Equations **2** 111–120, 2004.

[13] A. Cima and J. Llibre, *Algebraic and topological classification of the homogeneous cubic vector fields in the plane*, Journal of Mathematical Analysis and Applications **147** 420– 448, 1990.

[14] L. Dongnei, T. Xvan and T. Guangyue, *The qualitative classification of a special type of higher order ordinary differential system*, Journal of Mathematical Analysis and Applications **13**(3) 232–247, 1997.

[15] C. Guang-Qing and L. Zhao-Jun, *Affine classification for the quadratic vector fields without the critical points at infinity*, Journal of Mathematical Analysis and Applications **172** 62–72, 1993.

[16] X.A.Yang and F.F. Zhang, *Algebraic classification of polynomial systems in the plane*, Annals of Differential Equations **6**(4) 463–480, 1990.

[17] Stavros Anastassiou, Spyros Pnevmatikos, Tassos Bountis, *Classification of dynamical systems based on a decomposition of their vector fields*, Journal of Differential Equations **253** (Issue 7) 2252-2262, 2012.

[18] Cristina Bujac, Nicole Vulpe, *Classification of cubic differential systems with invariant straight lines of total multiplicity eight and two distinct infinite singularities*, Electronic Journal of the Qualitative Theory of Differential Equations **74** 1-38, 2015.

[19] Hebai Chen, Rui Zhang, *Dynamics of polynomial Rayleigh-Duffing system near infinity and it's global phase portraits with a center*, Advances in Mathematics **433** (Paper No. 109326) 37pp., 2023.

[20] Meryem Belattar, Rachid Cheurfa, Ahmed Bendjeddou, *Cubic planar differential systems with non-algebraic limit cycles enclosing a focus*, International Journal of Dynamical Systems and Differential Equations, **3** 197-208, 2023.

[21] Qibao Jiang, Jaume Llibre, *Qualitative classification of singular points*, Qualitative Theory of Dynamical Systems **6** (Issue 1) 87-167, 2005.

[22] J.Chavarrige and I. A. Garcia, *Some criteria for the existence of limit cycles for quadratic vector fields*, Journal of Mathematical Analysis and Applications **282** 296–304, 2003.

[23] A. Gasull and R. Prohens, *Quadratic and cubic systems with degenerate infinity*, Journal of Mathematical Analysis and Applications **198** 25–34, 1996.

[24] Amel Boulfoul, Quarda Saifia, *On the number of limit cycles in a class of planar differential systems*, Nonlinear Studies **30** (3) 855-870, 2023.

[25] N. G. Lloyd, *A note on the number of limit cycles in certain two- dimensional systems*, Journal of the London Mathematical Society **2**(20) 277–286, 1979.

[26] (Bami)dele Oluwade, *On the qualitative classes of $x^/ = ax^3 + bx^2 + cx + d$*, Proceedings of the National Mathematical Centre, Abuja, Nigeria **1**(1)75–82, 2000.

[27] Bamidele Oluwade, *Qualitative equivalence of autonomous ordinary differential equations*, M.Sc. thesis, Obafemi Awolowo University, Ile-Ife, Nigeria, 1999.

[28] (Bami)dele Oluwade, *An algebraic condition for qualitative stability of first order linear autonomous ordinary differential systems*, Kragujevac Journal of Mathematics **23** 53-58, 2001.

[29] Bamidele Oluwade,*Design and analysis of computer-coded character sets*, Ph.D. thesis, University of Ibadan, Nigeria, 2004.

[30] Baoli Hao, Ming Zhong and Kevin O'keeffe, *Attractive and repulsive interactions in the one-dimensional swarmalator model*, Physical Review E **108**, no 6, Paper No 064214, 9pp, 2023.

[31] Jaume Llibre and Claudia Valls, *The phase portrait of all polynomial Lienard isochronous centers*, Chaos Solitons Fractals **180** Paper No 114500, 4pp, 2024.

[32] Mo Jia, Desheng Li, Jinlong Bai and Yuxiang Zhang, *Attractor bifurcation for positive solutions of evolution equations*,Topology Applications **339** Paper No 108587, 15pp, 2023.

[33] Jaume Llibre and Y. Paulina Mancilla-Martinez, *Global attractor in the positive quadrant of the Lotka-Volterra system in $\mathbb{R}^2$*, International Journal of Bifurcation, Chaos and Applications in Science and Engineering, **33**, No 12, Paper No 2350147, 12pp, 2023.

[34] (Bami)dele Oluwade and Anthony Uyi Afuwape, *On the qualitative equivalence of first order autonomous ODEs of the transcendental type* (Appreciating Mathematics in Contemporary World) Proceedings of the International Conference in Honour of Prof. E. O. Oshobi and Dr. J. O. Amao, Department of Mathematics, Obafemi Awolowo University, Ile-Ife, Nigeria, 15th December, 2004, 87-93, 2005.

[35] (Bami)dele Oluwade, *The Galois group of the Chebyshev polynomials of the first kind of prime degree*, African Journal of Computing & ICT **3** Nos 1 & 2, 39-44, 2010.

[36] Bamidele Oluwade and O. A. Taiwo, *Asymptotic time complexity of an algorithm for generating the coefficients of the Chebyshev polynomials for the Tau numerical method*, Journal of the Nigerian Association of Mathematical Physics **12**, 2008, DOI: $10.4314/jonamp.v12i1.45534$.

[37] Charles Severance, *Anil Jain: 25 years of biometric recognition*, IEEE Computer, August 8-9, 2015.

[38] Mohammad Haghighat, Mohamed Abdel-Mottaleb, *Low resolution face recognition in surveillance systems using discriminant correlation analysis*, 2017 12th IEEE International Conference on Automatic Face & Gesture Recognition 912-917, 2017.

[39] A. J. O'Toole, P. J. Phillips, F. Jiang, J. Ayyad, N. Penard, H. Abdi, *Face recognition algorithms surpass humans matching faces over changes in illumination*, IEEE Transactions on Pattern Analysis and Machine Intelligence **29** (9) 1642-1646, 2007.

[40] Y. Yacoob, L. Davis, *Smiling faces are better for face recognition*, Proceedings of the Fifth IEEE International Conference on Automatic Face and Gesture Recognition, Washington DC, USA, 59-64, 20-21 May, 2002.

[41] Vincent Andrew Akpan, Reginald A. O. Osakwe, *Face image processing, analysis and recognition algorithms for enhanced optimal face recognition systems design: A comparative study*, African Journal of Computing & ICT **2** (2) 21-40, 2009.
[42] https://www.worldometers.info (accessed 03 March 2024).
[43] https://www.mathworks.com/discovery/affine-transformation (accessed 17 August 2024).
[44] Stuart Russell, Peter Norvig, *Artificial intelligence: A modern approach (4th edition)*, Prentice-Hall, New York, 2020.
[45] Bedole Omolu, Bamidele Oluwade, *On the application of the nearest neighbour classifier algorithm to character pattern recognition*, African Journal of Computing & ICT **10**(1&2) 1-17, 2017.
[46] Bamidele Oluwade, *A fair resource-sharing AI algorithm for humanitarian camps*, AI for Humanitarianism: Fostering Social Change through Emerging Technologies (Ed. Abel Ajibesin and Narasimha Rao Vajjhala), Chapman & Hall/CRC Press Artificial Intelligence and Robotics Series, Taylor & Francis, 2024, to appear.
[47] Bamidele Oluwade *Modelling of components of building structures using discrete structures of computer science* , University of Ibadan Journal of Science and Logics in ICT Research **1** 25-33, 2017.

[1]DEWADE SYSTEMS LIMITED (Software, Hardware & Renewable (Solar) Energy Systems Research & Development), POB 20253, U.I.P.O., Ibadan, Oyo 200005, Nigeria.
Email address :bamideleoluwade@gmail.com