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ULTIMATE BOUNDEDNESS OF SOLUTIONS FOR A CERTAIN NONLINEAR THIRD ORDER DIFFERENTIAL EQUATION

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ABSTRACT. Sufficient conditions are established for the ultimate boundedness of solutions for the third order nonlinear differential equation.

1. INTRODUCTION

We consider the third order nonlinear differential equation

$$x''' + \phi(x,x',x'') + q(x') + h(x) = e(t,x,x',x'') \qquad (1)$$

or its equivalent

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= -\phi(x,y,z) - q(y) - h(x) + e(t,x,y,z), \end{aligned} \qquad (2)$$

where ϕ, q, $h \in C(\mathbb{R},\mathbb{R})$, $e \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \mathbb{R}^+ = [0,\infty)$ and $\mathbb{R} = (-\infty,\infty)$. It is assumed that the functions ϕ, q, h and e depend only on the arguments displayed explicitly and the $'$ denote differentiation with respect to t. We shall take it that $h'(x) = \dfrac{dh(x)}{dx}$ exists and continuous, also the uniqueness of (1) will be assumed. Ultimate boundedness of solutions is very important in the theory and applications of differential equations, and an effective method for studying the ultimate boundedness of nonlinear differential equations is still the Lyapunov's direct method (see [1] - [13]).

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Besides, Ademola et al. [1] discussed on the boundedness and ultimate boundedness of solutions of nonlinear differential equation

$$\dddot{x} \quad + \quad f(\ddot{x}) + g(\dot{x}) + h(x) = p(t, x, \dot{x}, \ddot{x})$$

while the equivalent is given as

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= z \\ \dot{z} &= p(t,x,y,z) - f(z) - g(y) - h(x), \end{aligned} \tag{3}$$

where ϕ, q, $h \in C(\mathbb{R},\mathbb{R})$, $e \in C(\mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), \mathbb{R}^+ = [0,\infty)$ and $\mathbb{R} = (-\infty,\infty)$. Sufficient conditions were given with appropriate Lyapunov function to establish their results.

The motivation for this study come from the work by Ademola et al. [1] and Omeike [9]. Our aim is to improve on the ultimate boundedness of solutions result obtained in [1] and to correct some terms in the result achieved. In the next section, we establish a criterion for the ultimate boundedness of solutions of Eq. (2).

2. MAIN RESULT

Our main result is the following theorem.

Theorem 2.1. Suppose that b, k, m, η_0 are positive constants, $e(t,x,y,z) \equiv e(t)$ and that

(i) $h(0) = 0$, $\dfrac{h(x)}{x} \geq \eta_0$, for all $x \neq 0$;

(ii) $h'(x) \leq m$ for all x;

(iii) $\dfrac{q(y)}{y} \geq k$, for all $y \neq 0$;

(iv) $\dfrac{\phi(x,y,z)}{z} \geq b$ for all $z \neq 0$;

(v) $\displaystyle\int_0^t |e(s)|ds \leq e_0 < \infty$, where e_0 is a positive constant.

Then for any finite constants x_0, y_0, z_0, there exist a constant $\Phi = \Phi(x_0, y_0, z_0)$, such that any solution $(x(t), y(t), z(t))$ of the Eq. (2) ultimately satisfies

$$|x(t)| \leq \Phi, \ |y(t)| \leq \Phi, \ |z(t)| \leq \Phi$$

for all $t \geq 0$.

Remarks 2.2. If $\phi(x,x',x'') = ax'', q(x') = bx', h(x) = cx$ and $e(t,x,x',x'') = 0$. Eq. (1) reduces to a linear constant coefficient differential equation and with the conditions in Theorem 2.1, it satifies the Routh-Hurwitz criterion $a > 0$, $ab > c$ $c > 0$.

Remarks 2.3. Theorem 2.1 generalizes the results of Ademola et al [1], if we set $\phi(x,x',x'') = f(x'')$.

3. PRELIMINARIES

In order to prove Theorem 2.1, we need to show that any solution $(x(t),y(t),z(t))$ of (2) satifies

$$|x(t)| \le \Phi,\ |y(t)| \le \Phi,\ |z(t)| \le \Phi \tag{4}$$

for all sufficiently large t, where Φ is a suitable constant.

Our proof of (4) rests on two properties (stated in the lemma below) of the function $V = V(t,x,y,z)$ defined by

$$\begin{aligned} 2V &= 2b\int_0^x h(\xi)d\xi + 2\int_0^y q(\tau)d\tau + 2yh(x) + \beta kx^2 \\ &+ (\beta + b^2)y^2 + z^2 + 2\beta(bxy + xz + yz), \end{aligned} \tag{5}$$

where β is a positive fixed constant satifying

$$0 < \beta < k - \frac{m}{b}. \tag{6}$$

The function (6) can be rewritten as

$$2V = V_1 + V_2,$$

where

$$V_1 = 2b\int_0^x h(\xi)d\xi + 2\int_0^y q(\tau)d\tau + 2yh(x)$$

and

$$V_2 = \beta kx^2 + (\beta + b^2)y^2 + z^2 + 2\beta(bxy + xz + yz).$$

In view of the hypothesis (iii) in Theorem 2.1, $q(y) \ge ky$ for all $y \ne 0$, thus

$$2\int_0^y q(\tau)d\tau + 2yh(x) \ge (ky + h(x))^2k^{-1} - k^{-1}h^2(x) \ge -k^{-1}h^2(x). \tag{7}$$

Since $(ky + h(x))^2 \ge 0$ for all x, y. Besides, hypothesis (i) and (ii) of Theorem 2.1 imply that

$$\begin{aligned} 2b\int_0^x h(\xi)d\xi &= 2k^{-1}\int_0^x (bk - h'(\xi))h(\xi)d\xi + k^{-1}h^2(x) \\ &\ge (bk - m)k^{-1}\eta_0 x^2 + k^{-1}h^2(x). \end{aligned} \tag{8}$$

On combining the inequalities (7) and (8), we have

$$V_1 \ge (bk - m)k^{-1}\eta_0 x^2 \tag{9}$$

for all x. Furthermore, V_2 can be written as

$$V_2 = XM_0X^T,$$

where $X = (x\,y\,z)$, $M_0 = \begin{pmatrix} \beta k & \beta b & \beta \\ \beta k & \beta + b^2 & b \\ \beta & b & 1 \end{pmatrix}$ and the $detM_0 = \beta^2(k - \beta) > 0$, since $k - \beta > 0$ (which follows from (6)). Thus,

$$V_2 \geq \beta^2(x^2 + y^2 + z^2), \tag{10}$$

for all $(x,y,z) \in \mathbb{R}^2$ with $\beta > 0$. By addition of inequalities (9) and (10), these satisfy the condition stated in the lemma below.

Lemma 3.1. Subject to the conditions of Theorem 2.1, $V(0,0,0) = 0$ and there is a positive constant Φ_1 depending only on b, k, m and β such that

$$V(x,y,z) \geq \Phi_1(x^2 + y^2 + z^2), \tag{11}$$

for all x,y,z. Furthermore, there are finite constants $\Phi_2 > 0$, $\Phi_3 > 0$ dependent only on b, k, m and β such that for any solution $(x(t), y(t), z(t))$ of (2), we have

$$V' \equiv \frac{d}{dt}V(x(t), y(t), z(t)) \leq -\Phi_2 \tag{12}$$

provided that $x^2 + y^2 + z^2 \geq \Phi_3$.

Proof of Lemma 3.1. We have been able to establish inequality (11) that it is positive definite from equation (5). Now, differentiating (5) along any solution (x(t), y(t), z(t)) of the system (2), it follows that

$$\begin{aligned} V' &= -\beta h(x)x - bq(y)y + y^2h'(x) + \beta by^2 - \beta q(y)x + \beta kyx \\ &+ \beta yz - \beta\phi(x,y,z)x - b\phi(x,y,z)y - \phi(x,y,z)z + \beta bxz \\ &+ b^2yz + bz^2 + \beta yz + e(t,x,y,z)z + \beta e(t,x,y,z)x \\ &+ be(t,x,y,z). \end{aligned}$$

In view of hypothesis (i) to (iv) and applying the inequality of the form $|2uv| \leq u^2 + v^2$ where necessary, we have

$$\begin{aligned} V' &\leq -\frac{1}{2}\beta\eta_0x^2 - \frac{7}{8}(bk - \beta b - \beta - m)y^2 - \Big(\frac{\phi(x,y,z)}{z} - b\Big)z^2 \\ &+ \frac{1}{2}\beta z^2 - B_j + (\beta x + by + z)e(t) \ (j = 1,2,3), \end{aligned} \tag{13}$$

where

$$
\begin{aligned}
B_1 &= \beta\Big[\frac{1}{4}\eta_0 x^2 + (q(y)-ky)x + \frac{1}{16\beta}(bk-\beta b-\beta-m)y^2\Big] \quad (14)\\
B_2 &= \beta\Big[\frac{1}{4}\eta_0 x^2 + (\phi(x,y,z)-bz)x\Big] \quad (15)\\
B_3 &= b\Big[\frac{1}{16b}(bk-\beta b-\beta-m)y^2 + (\phi(x,y,z)-bz)y\\
&+ \frac{\beta}{4b}z^2\Big]. \quad (16)
\end{aligned}
$$

So, using Eqs. (14) to (16) and taking into consideration the following inequalities

$$
\begin{aligned}
(q(y)-by)^2 &< \frac{\eta_0(bk-\beta b-\beta-m)}{16\beta}y^2\\
(\phi(x,y,z)-bz)^2 &< \frac{\eta_0}{4}z^2\\
(\phi(x,y,z)-bz)^2 &< \frac{\beta(bk-\beta b-\beta-m)}{16b^2}z^2
\end{aligned}
$$

such that we now obtain

$$
\begin{aligned}
B_1 &\geq \frac{\beta}{16}\Big(4\sqrt{\eta_0}|x| - \sqrt{\frac{bk-\beta b-\beta-m}{\beta}}|y|\Big)^2 \geq 0 \quad (17)\\
B_2 &\geq \frac{\beta}{4}\Big(\eta_0|x| + 2\Big(\frac{\phi(x,y,z)}{z}-b\Big)|z|\Big)^2\eta_0^{-1}\\
&- \eta_0^{-1}4^2\Big(\Big(\frac{\phi(x,y,z)}{z}-b\Big)^2 z^2\Big) \geq 0 \quad (18)\\
B_3 &\geq \frac{b}{16}\Big(\sqrt{\frac{bk-\beta b-\beta-m}{b}}|y| - 2\sqrt{\frac{\beta}{b}}|z|\Big)^2 \geq 0. \quad (19)
\end{aligned}
$$

Applying the estimates (17) to (18), we have

$$
\begin{aligned}
V' &\leq -\frac{1}{2}\beta\eta_0 x^2 - \frac{7}{8}(bk-\beta b-\beta-m)y^2 - \Big(\Big(\frac{\phi(x,y,z)}{z}-b\Big)-\frac{\beta}{2}\Big)z^2\\
&+ \max(\beta,b,1)(|x|+|y|+|z|)e(t)
\end{aligned}
$$

Then

$$
V' \leq -\Phi_4(x^2+y^2+z^2) + \Phi_5(|x|+|y|+|z|),
$$

where $\Phi_4 = \min\Big\{\frac{1}{2}\beta\eta_0;\ \frac{7}{8}(bk-\beta b-\beta-m);\ \Big(\frac{\phi(x,y,z)}{z}-b\Big)-\frac{\beta}{2}\Big\}$

and $\Phi_5 = \max\{\beta;\ b;\ 1\}$.
Furthermore,

$$V' \;\leq\; -\Phi_4(x^2+y^2+z^2)+\Phi_6(x^2+y^2+z^2)^{\frac{1}{2}}, \tag{20}$$

where $\Phi_6 = 3^{\frac{1}{2}}\Phi_5$.

If we take $(x^2+y^2+z^2)^{\frac{1}{2}} \geq \Phi_7 = 2\Phi_6\Phi_4^{-1}$, inequality (20) implies that

$$V' \;\leq\; -\frac{1}{2}\Phi_4(x^2+y^2+z^2). \tag{21}$$

Therefore,

$$V' \;\leq\; -\Phi_8, \tag{22}$$

provided that $x^2+y^2+z^2 \geq 2\Phi_8\Phi_4^{-1}$ and this completes the proof of Lemma 3.1 (with $\Phi_2 \equiv \Phi_8$).

4. PROOF OF THEOREM 2.1

Let $(x(t),y(t),z(t))$ be any solution of (2). Then there is a $t_0 \geq 0$ such that

$$x^2(t_0)+y^2(t_0)+z^2(t_0) < \Phi_3, \tag{23}$$

where Φ_3 is the constant in Lemma 3.1; for otherwise, that is if

$$x^2(t)+y^2(t)+z^2(t) \geq \Phi_3,\ t \geq 0. \tag{24}$$

Then by (12), we have

$$V' \leq -\Phi_2 < 0,\ t \geq 0. \tag{25}$$

and this in turn implies that $V \longrightarrow -\infty$ as $t \longrightarrow \infty$, which contradicts (11). So, to prove (5), it will suffice to show that if

$$x^2(t)+y^2(t)+z^2(t) < \Phi_9,\ for\ t = T, \tag{26}$$

where $\Phi_9 \geq \Phi_3$ is a finite constant, then there is a constant $\Phi_{10} > 0$, depending on b, k, m, η_0, β and Φ_9, such that

$$x^2(t)+y^2(t)+z^2(t) \leq \Phi_{10},\ for\ t \geq T. \tag{27}$$

Our proof of (27) is based essentially on an argument in the proof of [[9], Lemma 2.1]. For any given constant $\lambda > 0$, let $S(\lambda)$ denote the surface: $x^2+y^2+z^2 = \lambda$. Since V is continuous in x, y, z and tends to $+\infty$ as $x^2+y^2+z^2 \longrightarrow \infty$, there is a constant $\Phi_{11} > 0$, depending on Φ_9 as well as on b, k, m, η_0 and β, such that

$$\min_{(x,y,z)\in S(\Phi_{11})} V(x,y,z) > \max_{(x,y,z)\in S(\Phi_9)} V(x,y,z). \tag{28}$$

It can be seen from (26) and (28) that

$$x^2(t)+y^2(t)+z^2(t) < \Phi_{11},\ for\, t \geq T. \tag{29}$$

Suppose on the contrary that there is a $t > T$ such that

$$x^2(t)+y^2(t)+z^2(t) \geq \Phi_{11}. \tag{30}$$

Then, by (26) and by the continuity of the quantities $x(t)$, $y(t)$, $z(t)$ in the argument displayed, there exist t_1, t_2, $T < t_1 < t_2$ such that

$$x^2(t_1)+y^2(t_1)+z^2(t_1) = \Phi_9, \tag{31}$$

$$x^2(t_2)+y^2(t_2)+z^2(t_2) = \Phi_{11} \tag{32}$$

and such that

$$\Phi_9 \leq x^2(t)+y^2(t)+z^2(t) \leq \Phi_{11},\ t_1 \leq t \leq t_2. \tag{33}$$

But, stating $V \equiv V(x(t),y(t),z(t))$, since $\Phi_9 \geq \Phi_3$, by (33) this implies [in view of inequality (12)] that

$$V(t_2) < V(t_1)$$

and this contradicts the conclusion from (28), (31) and (32)

$$V(t_2) > V(t_1).$$

Thus, inequality (29) holds. This completes the proof of (4), and the theorem follows.

5. CONCLUSION

Clearly, we have been able to establish the ultimate boundedness result of (1) which is an improvement on the one considered by Ademola et al [1]. In particular, from our theorem we see that (iv) assumed in Theorem 3.1 [1] has been replaced by $\dfrac{\phi(x,y,z)}{z} \geq b$ for all $z \neq 0$ and b a positive constant.

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