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EXISTENCE OF FIXED POINT FOR A NEW CLASS OF ENRICHED PSEUDOCONTRATIVE MAPPINGS

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ABSTRACT. In this paper, we introduce a new class of mappings known as β-enriched pseudocontractive mappings and proved that the fixed point for such mappings do exist in the setup of uniformly convex Banach spaces. Our result is an extension of comparable results in the existing literature. Some examples are presented to support the main results established in this paper.

1. INTRODUCTION

Let $\mho$ be a Banach space, $\Lambda \subset \mho$ and $\mho^\star$ denotes the topological dual space of $\mho$. Define the mapping $J : \mho \longrightarrow 2^{\mho^\star}$ by

$$J\hbar = \{\hbar^\star \in \mho^\star : \langle \hbar, \hbar^\star \rangle = \|\hbar\|^2 = \|\hbar^\star\|^2, \forall \hbar \in \mho\|^2\}. \tag{1}$$

Then, J is called the normalised duality map on $\mho$. It is worthy to note that if $\mho$ is strictly convex, then J is single-valued. In this paper, we shall denote the single-valued duality map by j. In real Hilbert spaces, J is the identity map.

Let $\Gamma : \Lambda \longrightarrow \mho$ be a nonlinear mapping. Then, is known as pseudocontractive (see [3]) if, for all $\hbar, \eth \in \Lambda$ and for all $r > 0$, the following inequality is satisfied:

$$\|\hbar - \eth\| \leq \|(1+r)(\hbar - \eth) - r(\Gamma\hbar - \Gamma\eth)\|. \tag{2}$$

We note, in passing, that the class of mapping defined by inequality (2) is more general than the class of nonexpansive mappings. Recall that Γ is called nonexpansive if, for all $\hbar, \eth \in \Lambda$, we get

$$\|\Gamma\hbar - \Gamma\eth\| \leq \|\hbar - \eth\|. \tag{3}$$

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In recent times, a lot of interest has been geared toward evaluation of fixed point for nonlinear mappings with a particular interest in pseudocontractions. This captivating interest was found to be rooted in the firm connection which exists between pseudocontractive mappings and an important class of accretive operators.

A mapping B with domain $D(B)$ and range $R(B)$ is known as accretive if, for all $\hbar, \eth \in D(B)$, the inequality below holds:

$$\|\hbar - \eth\| \leq \|\hbar - \eth + (B\hbar - B\eth)\|. \tag{4}$$

As an immediate consequence of a result of Kato [1], it follows from inequality (4) that B is accretive if, for all $\hbar, \eth \in D(B)$, there exists $j(\hbar - \eth) \in J(\hbar - \eth)$ such that

$$\langle B\hbar - B\eth, j(\hbar - \eth)\rangle \geq 0, \tag{5}$$

where $J : \mho \longrightarrow 2^{\mho^\star}$ is the nonlinear duality map on $\mho$. Again, it follows from inequality (4) that B is accretive if and only if $(I + rB)$ is expansive and, as a result, its inverse $(I + rB)^{-1}$ exists and is nonexpansive mapping from $R(I + rB)$ into $D(B)$, where $R(I + rB)$ denotes the range of $(I + rB)$. The concept of accretive mappings was initiated independently in 1967 by Kato [1] and Browder [4]. The introduction of accretive mappings captured the interest of several researchers possibly because of their firm connection with the existence theory of nonlinear evolution equations in Banach spaces of the form

$$\frac{d\hbar}{dt} + B\hbar = 0, \quad \hbar(0) = \hbar_0, \tag{6}$$

where B is an accretive map on a suitable Banach space. At a equilibrium, $\frac{d\hbar}{dt} = 0$ and the solution of the equation

$$B\hbar = 0 \tag{7}$$

provides an answer for the equilibrium points of the evolution equation (6).

In [3], Browder showed that (7) is equivalent to the fixed point problem. He introduced an operator Γ and established the relationship $\Gamma = I - B$, where B is an accretive map, and called Γ a pseudocontraction. It is clear that the fixed point of Γ corresponds to the zeros of B. Dwelling on the above relationship, and employing a highly analytical method, the following theorem due to Browder [3] ensued:

Theorem 1.1. Let $\mho$ be a uniformly convex Banach space, Λ a closed ball in $\mho$, Υ an open set containing Λ. Let Γ be a paeudocontractive mapping of Υ into $\mho$ such that Γ maps the boundary of Λ into Λ. Suppose also that Γ is demicontinuous and that either (a) Γ is uniformly continuous in the strong topology on bounded subsets of $\mho$, or (b) $\mho^\star$ is uniformly convex. Then Γ has a fixed point in Λ.

Subsequently, Kirk [7] gave an elementary geometric proof of Theorem 1.1 by strengthening the assumption of demicontinuity of Γ and discarding the condition of Γ being defined on an open set containing Λ. To be precise, he proved the following theorem:

Theorem 1.2. Let $\mho$ be a uniformly convex Banach space and Λ a closed ball in $\mho$. Let Γ be a Lipschitizian pseudocontractive mapping of Λ into $\mho$ such that Γ also maps the boundary of Λ into Λ. Then Γ has a fixed point in Λ.

Definition 1.3. (see [16]) A mapping $\Gamma : \Lambda \longrightarrow \mho$ is known as Φ_Γ-enriched Lipschitzian (or (β, Φ_Γ)-enriched Lipschitzian) if, for all $\hbar, \eth \in C$, there exist $\beta \in [0,\infty)$ and a continuous nondecreasing function $\Phi_\Gamma : R^+ \longrightarrow R^+$ with $\Phi(0) = 0$ such that

$$\|\beta(\hbar - \eth) + \Gamma\hbar - \Gamma\eth\| \leq (\beta+1)\Phi_\Gamma(\|\hbar - \eth\|). \tag{8}$$

The following special cases are immediate consequence of (8): if $\beta = 0$ in (8), then Γ becomes Φ_Γ- Lipschitzian (see [19] for more details). Also, set $\beta = 0$ and $\Phi_\Gamma(r) = Lr$ for $L > 0$, then Γ reduces to Lipschitzian mapping with L as Lipschitz constant. ln particular, if $\beta = 0$, $\Phi_\Gamma(r) = Lr$ and $L = 1$, then Γ reduces to the class of mappings defined by (3) below. For $\beta \in (0,\infty)$, set $\beta = \dfrac{1}{1+\rho}$ and notice that $0 < \rho < 1$. Substituting this value for β in (8) and simplifying the resulting equation yields

$$\|\Gamma_\rho\hbar - \Gamma_\rho\eth\| \leq \gamma\Phi_\Gamma(\|\hbar - \eth\|), \tag{9}$$

where $\Gamma_\rho = (1-\rho)I + \rho\Gamma$ (I denoting the identity map on Λ) and $\gamma = (\rho+1)\beta$. Again, the mapping Γ_ρ is Φ_Γ-Lipschitzian.

Remark 1.4. Every Lipschitz mapping Γ is Φ_Γ-Lipschitzian but the converse may not be true. It is evident (taking $\beta = 0$) that every Φ_Γ-Lipschitz mapping Γ is automatically $(0, \Phi_\Gamma)$-enriched Lipschitzian. Also, if Φ_Γ is not necessarily nondecreasing and guarantees $\Phi_\Gamma(r) < r$ for $r > 0$, then Γ is called a nonlinear contraction on $\mho$.

Example 1.5. Let $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\Gamma\hbar = \sqrt{|\hbar|}, \quad \forall \hbar \in \mathbb{R}.$$

Consider $\Phi_\Gamma(\ell) = \sqrt{\ell}, \ell \geq 0$. Clearly, Φ_Γ is continuous and nondecreasing. It is not difficult to see that Γ is subadditive. Indeed, for all $\hbar, \eth \in \mathbb{R}$, we have

$$\begin{aligned}(\Gamma(\hbar+\eth))^2 &= |\hbar+\eth| \\ &\leq (\sqrt{|\hbar|}+\sqrt{|\eth|})^2 \\ &= (\Gamma\hbar+\Gamma\eth)^2\end{aligned}$$

Using the subadditivity of Γ, we obtain

$$|\Gamma\hbar-\Gamma\eth| \leq \Gamma(\hbar-\eth) = \Phi_\Gamma|\hbar-\eth|$$

Thus, Γ is Φ_Γ-Lipschitzian (or $(0,\Phi_\Gamma)$-enriched Lipschitzian) mapping with Φ_Γ as the Φ_Γ-function. Now, suppose Γ is Lipschitzian with constant $L>0$. Then, $\forall\hbar,\eth\in\mathbb{R}$ with $\eth=0$ and $\hbar\neq 0$, we get $\Gamma\hbar\leq L|\hbar|$. Hence, $\forall\hbar\neq 0$, $L\geq\dfrac{1}{\sqrt{|\hbar|}}$. Letting $\hbar\to 0$, we obtain a contradiction. Consequently, Γ is not Lipschitzian mapping.

Recently, Berinde [11] introduced the concept of (β,k) -enriched strictly pseudocontractive mappings which was later generalised by Saleem et al. [16].

Definition 1.6. [11] A mapping $\Gamma:\Lambda\longrightarrow\Lambda$ is said to be (β,k)-enriched strictly pseudocontractive if for all $\hbar,\eth\in\Lambda$, there exist $\beta\in[0,+\infty)$ and $k\in[0,1)$ such that

$$\|\beta(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2\leq(\beta+1)^2\|\hbar-\eth\|^2+k\|\hbar-\eth-(\Gamma\hbar-\Gamma\eth)|^2. \tag{10}$$

Some nonlinear mappings that could be recovered from (10) when either β or k assumes the value 0 or 1 are as follows: if $\beta=0$ in inequality (10), we get a class of mapping studied in [4] and if $k=0$, then the inequality (10) reduces to a class of mapping studied in [10]. Hence, the class of (β,k)-enriched strictly pseudocontractive mapping contains the class β-enriched nonexpansive mapping (which in turn contains the class of mappings studied in [2, 3, 5–7], [12]) and the class of k-strictly pseudocontractive mapping (see, for instance, [8, 13–15, 17]).
Setting $\beta=\dfrac{1}{\rho}-1$, for $\rho\in(0,1]$, in inequality (10), we obtain

$$\left\|\frac{1-\rho}{\rho}(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\right\|^2 \leq \frac{1}{\rho^2}\|\hbar-\eth\|^2+k\|\hbar-\eth-(\Gamma\hbar-\Gamma\eth)\|^2,$$

which implies that

$$\begin{aligned}\|(1-\rho)(\hbar-\eth)+\rho\Gamma\hbar-\rho\Gamma\eth\Big\|^2 &\leq \|\hbar-\eth\|^2+k\|\hbar-[(1-\rho)\hbar+\rho\Gamma\hbar] \\ &\quad -\{\eth-[(1-\rho)\eth+\rho\Gamma\eth]\}\|^2.\end{aligned}$$

Thus,

$$\|\Gamma_\rho \hbar - \Gamma_\rho \eth\|^2 \leq \|\hbar - \eth\|^2 + k\|\hbar - \eth - (\Gamma_\rho \hbar - \Gamma_\rho \eth)\|^2, \qquad (11)$$

where $\Gamma_\rho = (1-\rho)I + \rho\Gamma$. We note in passing that the average mapping Γ_ρ is a strict pseudocontraction.

In a real Banach space (see [20]), inequality (11) is equivalent to

$$\langle \Gamma_\rho \hbar - \Gamma_\rho \eth, j(\hbar - \eth)\rangle \leq \|\hbar - \eth\|^2 - \lambda\|\hbar - \eth - (\Gamma_\rho \hbar - \Gamma_\rho \eth)\|^2, \quad (12)$$

where $\lambda = \dfrac{1}{2}(1-k)$. If I represents the identity mapping on Λ, then inequality (12) is equivalent to

$$\langle (I-\Gamma_\rho)\hbar - (I-\Gamma_\rho)\eth, j(\hbar - \eth)\rangle \geq \lambda\|x - y - (\Gamma_\rho \hbar - \Gamma_\rho \eth)\|^2. \quad (13)$$

As in above, the average mapping Γ_ρ is still strictly pseudocontractive. The concept of (β, k)-enriched strictly pseudocontractive mapping was initiated by Berinde [11] as a generalization of the concept of k-strictly pseudocontractive mappings (recall that a mapping $\Gamma : \Lambda \longrightarrow \Lambda$ is called k-strictly pseudocontractive if for all $\hbar, \eth \in \Lambda$, we can find $k \in [0,1)$, we have $\|\Gamma\hbar - \Gamma\eth\|^2 \leq \|\hbar - \eth\|^2 + k\|\hbar - \eth - (\Gamma\hbar - \Gamma\eth)\|^2$. If $k = 1$, then we have a pseudocontraction. The class of strictly pseudocontractive mappings, defined in the setup of a real Hilbert space, was introduced in 1967 by Browder and Petryshym [4] as an intermediary class of mappings between the class of nonexpansive mappings and the class of Lipshitz pseudocontractive mappings. Note that while the class of Lipshitz pseudocontractive mappings are generally not continuous, the class of strictly pseudocontractive mappings inherit Lipshitz property from their definitions). For more information concerning this class of mappings, interested reader should see [16].

Now, in the light of Theorems [1.1, 1.2], a natural question arises on whether there exists a class of nonlinear mappings, that is more general than the class of mappings considered in both theorems, for which the results still remain valid.

Inspired and motivated by the results in [6] and [11], in this paper, we first introduced a new class of nonlinear mappings called β-enriched pseudocontractive mappings which is more general than the class of mappings studied in [6], [11] and [16]. Following similar approach used in [6], we proved the existence of fixed point for this new class of mappings in the setup of uniformly convex Banach spaces. Furthermore, we gave some examples to support our main result.

2. RESULTS AND DISCUSSION

We begin this section with the following definition and examples.

Definition 2.1. Let $\mho$ be a Banach space and $\Lambda \subset \mho$. A mapping $\Gamma : \Lambda \longrightarrow \mho$ is known as enriched pseudocontractive if for all $\hbar, \eth \in \Lambda$ and $r > 0, \exists \beta \in [0, \infty)$ such that the following inequality holds:

$$(\beta+1)\|\hbar-\eth\| \leq \|(1+\beta+r)(\hbar-\eth)-r(\Gamma\hbar-\Gamma\eth)\|. \tag{14}$$

It is important to note that 0-enriched pseudocontractive mappings is pseudocontractive with $\beta = 0$. Hence, inequality (14) reduces to inequality (2) when $\beta = 0$.

Remark 2.2. Set $\beta = \dfrac{1}{\rho} - 1 \in [0, \infty)$ for $\rho \in (0, 1]$, then it follows from inequality (14) that

$$\begin{aligned} \frac{1}{\rho}\|\hbar-\eth\| &\leq \|(1+\frac{1}{\rho}-1+r)(\hbar-\eth)-r(\Gamma\hbar-\Gamma\eth)\| \\ &= \|\frac{1+r\rho}{\rho}(\hbar-\eth)-r(\Gamma\hbar-\Gamma\eth)\|. \end{aligned}$$

The above inequality implies that

$$\begin{aligned} \|\hbar-\eth\| &\leq \|(1+r\rho)(\hbar-\eth)-r\rho(\Gamma\hbar-\Gamma\eth)\| \\ &= \|(\hbar-\eth)-r[(1-\rho)(\hbar-\eth)-(\hbar-\eth)+\rho(\Gamma\hbar-\Gamma\eth)]\| \\ &= \|(1+r)(\hbar-\eth)-r[(1-\rho)\hbar+\rho\Gamma\hbar-((1-\rho)\eth+\rho\Gamma\eth)]\| \\ &= \|(1+r)(\hbar-\eth)-r(U_\rho\hbar-U_\rho\eth)\| \\ &= \|(\hbar-\eth)+r[(I-U_\rho)\hbar-(I-U_\rho)\eth]\|, \end{aligned} \tag{15}$$

which, as an immediate consequence of a result of Kato [1], yields

$$\langle (I-U_\rho)\hbar-(I-U_\rho)\eth, j(\hbar-\eth)\rangle \geq 0, \quad \forall j(\hbar-\eth) \in J(\hbar-\eth), \tag{16}$$

where $U_\rho = (1-\rho)I + \rho\Gamma$.

In a real Hilbert space $\mathscr{H}$, inequalities (15) and (16) is equivalent to

$$\|U_\rho\hbar-U_\rho\eth\|^2 \leq \|\hbar-\eth\|^2 + \|(I-U_\rho)\hbar-(I-U_\rho)\eth\|^2. \tag{17}$$

It is worth mentioning in passing that the average operator U_ρ in inequalities (15), (16) and (17) is a pseudocontractive mapping whenever Γ in (14) is a β-enriched pseudocontaction (that is, $\forall \hbar, \eth \in \Lambda, \exists \beta \in [0, \infty)$ such that the inequality $\|\beta(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2 \leq (\beta+1)^2\|\hbar-\eth\|^2 + \|(I-\Gamma)\hbar-(I-\Gamma)\eth\|^2$ holds). If $k = 1$ in (11), then we have a pseudocontraction (and if in addition $\beta = 0$, then we have the class of mapping studied in [18]). Thus, the class of (β, k)-strictly pseudocontractive mappings and the class of pseudocontractive mappings are subclasses of the class of β-enriched pseudocontractive mappings .(see Example 2.5 and Example 2.6 below for more details).

Remark 2.3. Observe from (14) that

$$\begin{aligned}(\beta+1)\|\hbar-\eth\| &\leq \|(1+\beta+r)(\hbar-\eth)-r(\Gamma\hbar-\Gamma\eth)\| \\ &= \|(1+\beta)(\hbar-\eth)+r[(I-\Gamma)\hbar-(I-\Gamma)\eth]\| \\ &= \|(1+\beta)(\hbar-\eth)+r(B\hbar-B\eth)\|, \qquad (18)\end{aligned}$$

where $B=I-\Gamma$. The last inequality is called β-enriched accretive operator. Note that (18) becomes (4) when $\beta=0$. Also, since the average operator U_ρ is pseudocontractive, using the result of Browder [2] (see also [4]) (i.e., $B=I-\Gamma$ whenver B is an accretive operator and Γ a pseudocontraction), it follows from inequality (15) and inequality (16) that

$$\|\hbar-\eth\| \leq \|(\hbar-\eth)+r(B_\rho\hbar-B_\rho\eth)\| \qquad (19)$$

and

$$\langle B_\rho\hbar-B_\rho\eth, j(\hbar-\eth)\rangle \geq 0, \quad \forall j(\hbar-\eth)\in J(\hbar-\eth), \qquad (20)$$

where $B_\rho=I-U_\rho=\rho-\rho\Gamma=\rho-\rho(I-B)=\rho B$. Again, inequality (19) and inequality (20) are accretive mappings.

The example below shows that the existence of class of of (β,L) - enriched Lipschitzian pseudocontractive mappings.

Example 2.4. Let $\mathscr{H}=\mathbb{R}$, where $\mathbb{R}$ denotes the reals with the usual norm. Define $\Gamma:\mathbb{R}\longrightarrow\mathbb{R}$ by

$$\Gamma\hbar=\begin{cases}\hbar+\hbar^2, & \text{if} \quad \hbar\in[-2,0]\\ \hbar, & \text{if} \quad \hbar\in(0,1].\end{cases}$$

Clearly, $F(\Gamma)=0$. It is interesting to know that Γ is 0-enriched pseudocontractive mapping. To see this, observe that if $\hbar,\eth\in[-2,0]$, then, we get

$$|\beta(\hbar-\eth)+\Gamma\hbar-\Gamma\eth|^2=|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth|^2=|1+\hbar+\eth|^2|\hbar-\eth|^2$$

and

$$|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)|^2=|\hbar-(\hbar+\hbar^2)|^2=|-(\hbar^2-\eth^2)|^2=|\hbar^2-\eth^2|^2=(\hbar-\eth)^2(\hbar+\eth)^2$$

Thus,

$$\begin{aligned}(0+1)^2|\hbar-\eth|^2+|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)|^2 &= (\hbar-\eth)^2+(\hbar-\eth)^2(\hbar+\eth)^2 \\ &= (1+(\hbar+\eth)^2)|\hbar-\eth|^2 \\ &\geq |1+\hbar+\eth|^2|\hbar-\eth|^2 \\ &= |0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth|^2\end{aligned}$$

If $\hbar,\eth\in(0,1]$, then

$$|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth|^2=|\hbar-\eth|^2=|\hbar-\eth|^2+0=(0+1)^2|\hbar-\eth|^2+|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)|^2$$

Lastly, if $\hbar \in [-2,0]$ and $\eth \in (0,1]$, then

$$|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth|^2 = |\hbar+\hbar^2-\eth|^2 = |\hbar-\eth+\hbar^2|^2$$

and

$$|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)|^2 = \hbar^2.$$

Thus,

$$\begin{aligned}(0+1)^2|\hbar-\eth|^2+|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)|^2 &= |\hbar-\eth|^2+\hbar^2 \geq |\hbar-\eth+\hbar^2|^2\\ &= |0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth|^2,\end{aligned}$$

which shows that the mapping 0-enriched pseudocontractive.

Now, to show that Γ is 0-enriched Lipschitzian with the Lipschitz constant $L=5$, we proceed as follows: If $\hbar,\eth \in [-2,0]$, then we have

$$|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth| = |\hbar+\hbar^2-\eth-\eth^2| = |(\hbar+\eth)+1||\hbar-\eth| \leq (0+1)3|\hbar-\eth|$$

If $\hbar,\eth \in (0,1]$, then we get

$$|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth| = (0+1)|\hbar-\eth|. \tag{21}$$

If $\hbar \in [-2,0]$ and $\eth \in (0,1]$, then we have

$$\begin{aligned}|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth| &= |\hbar-\eth+\hbar^2| = |\hbar-\eth+\hbar^2-\hbar^2+\hbar^2|\\ &= |\hbar-\eth+\hbar^2-\hbar^2|+\hbar^2\\ &\leq |\hbar+\eth+1||\hbar-\eth|+|\eth-\hbar| = |2+\hbar+\eth||\hbar-\eth|\\ &\leq (0+1)3|\hbar-\eth|\end{aligned}$$

If $\eth \in [-2,0]$ and $\hbar \in (0,1]$, then we obtain

$$\begin{aligned}|0(\hbar-\eth)+\Gamma\hbar-\Gamma\eth| &= |\hbar-(\eth+\eth^2)| = |\hbar-\eth-\eth^2+\hbar^2-\hbar^2|\\ &= |\hbar-\eth+(\hbar-\eth)(\hbar+\eth)-\hbar^2|\\ &\leq |1+\hbar+\eth||\hbar-\eth|+\hbar^2\\ &\leq |1+\hbar+\eth||\hbar-\eth|+(\hbar^2-\eth)^2\\ &\leq [|1+\hbar+\eth|+|\hbar-\eth|]|\hbar-\eth|\\ &\leq (0+1)5|\hbar-\eth|.\end{aligned}$$

Thus, we have that Γ is a $(0,5)$-enriched Lipschitzian pseudocontraction with $L=5$.

Also, it is interesting to note that the class of β-enriched pseudocontractive mappings is larger than the class of pseudocontractive mappings as evident in the following example.

Example 2.5. Define $\Gamma : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$\Gamma\hbar = \begin{cases} 0, \ if \ \hbar \in (-\infty, 2] \\ \\ 1, \ if \ \hbar \in (2, \infty). \end{cases}$$

Then, for all $\hbar, \eth \in (-\infty, 2]$ and $\beta = 1$, we have

$$\begin{aligned} (\beta+1)^2|\hbar-\eth|^2 + |\hbar - \Gamma\hbar - (\eth - \Gamma\eth)|^2 &= 5|\hbar-\eth|^2 \\ &> |\hbar-\eth|^2 = |\beta(\hbar-\eth) + \Gamma\hbar - \Gamma\eth|^2. \end{aligned}$$

Further, for all $\hbar, \eth \in (2, \infty)$ and $\beta = 1$, we get

$$\begin{aligned} (1+1)^2|\hbar-\eth|^2 + |\hbar - \Gamma\hbar - (\eth - \Gamma\eth)|^2 &= 5|\hbar-\eth|^2 \\ &> |\hbar-\eth|^2 = |\beta(\hbar-\eth) + \Gamma\hbar - \Gamma\eth|^2 \end{aligned}$$

and for $\hbar \in (-\infty, 2]$ and $\eth \in (1, \infty)$, we obtain

$$\begin{aligned} (1+1)^2|\hbar-\eth|^2 + |\hbar - \Gamma\hbar - (\eth - \Gamma\eth)|^2 &= 4|\hbar-\eth|^2 + |\hbar - (\eth - 1)|^2 \\ &= 4|\hbar-\eth|^2 + |\hbar - \eth + 1|^2 \\ &> |\hbar - \eth - 1|^2 = |\beta(\hbar-\eth) + \Gamma\hbar - \Gamma\eth|^2. \end{aligned}$$

Thus, for all $\hbar, \eth \in \mathbb{R}$ and $\beta = 1$, we obtain

$$\|\beta(\hbar-\eth) + \Gamma\hbar - \Gamma\eth\|^2 \leq (\beta+1)^2\|\hbar-\eth\|^2 + \|(I-\Gamma)\hbar - (I-\Gamma)\eth\|^2$$

and Γ is β-enriched pseudocontractive. However, Γ is not pseudocontractive since for $\hbar = 2$ and $\eth = \frac{5}{2}$, we get

$$|\Gamma\hbar - \Gamma\eth|^2 > \frac{1}{2} = |\hbar-\eth|^2 + |\hbar - \Gamma\hbar - (\eth - \Gamma\eth)|^2.$$

Again, it is worth mentioning that the class of (β, k)-enriched strictly pseudocontractive mappings is a subclass of the class of β -enriched pseudocontractive mappings as shown in the example below.

Example 2.6. Let $\mathbb{R}^2$ denote the 2-dimensional Euclidean plane endowed with the norm

$$\|(\hbar+\eth)\|_2 = \|\hbar_1+\eth_1, \hbar_2+\eth_2\|_2 = \sqrt{\sum_{i=1}^{2}(\hbar_i+\eth_i)^2}, \tag{22}$$

where $\hbar = (\hbar_1, \hbar_2)$ and $\eth = (\eth_1, \eth_2)$. Define $\Gamma : R^2 \longrightarrow R^2$ by

$$\begin{aligned} \Gamma\hbar &= (\hbar_1, \hbar_2) + (\hbar_2, -\hbar_1) \\ &= (\hbar_1+\hbar_2, \hbar_2-\hbar_1), \forall \hbar = (\hbar_1, \hbar_2) \in R^2. \end{aligned}$$

Then, for $\hbar=(\hbar_1,\hbar_2), \eth=(\eth_1,\eth_2)\in R^2$ and $\beta=2$, we get

$$\begin{aligned}\|2(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2 &= \|2((\hbar_1,\hbar_2)-(\eth_1,\eth_2))+(\hbar_1+\hbar_2,\hbar_2-\hbar_1)-(\eth_1+\eth_2,\eth_2-\eth_1)\|^2\\ &= \|2((\hbar_1-\eth_1),(\hbar_2,\eth_2))+(\hbar_1+\hbar_2,\hbar_2-\hbar_1)-(\eth_1+\eth_2,\eth_2-\eth_1)\|^2\\ &= \|2((\hbar_1-\eth_1),(\hbar_2-\eth_2))+(\hbar_1-\eth_1+\hbar_2-\eth_2,\hbar_2-\eth_2-(\hbar_1-\eth_1))\|^2\\ &= \|(3(\hbar_1-\eth_1)+\hbar_2-\eth_2,3(\hbar_2-\eth_2)-(\hbar_1-\eth_1))\|^2\\ &= (3(\hbar_1-\eth_1)+(\hbar_2-\eth_2))^2+(3(\hbar_2-\eth_2)-(\hbar_1-\eth_1))^2\\ &= 10[(\hbar_1-\eth_1)^2+(\hbar_2-\eth_2)^2]\\ &= 10\|\hbar-\eth\|^2=(2+1)^2\Phi_\Gamma(\|\hbar-\eth\|).\end{aligned}$$

Hence, Γ is 2-enriched Φ_Γ-Lipshitz mapping with $\Phi_\Gamma(p)=\dfrac{10p^2}{9}$.

Now, observe that $\|2(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2=10\|\hbar-\eth\|^2, \eth-\Gamma\eth=-(\eth_2,-\eth_1)$ and $\hbar-\Gamma\hbar=-(\hbar_2,-\hbar_1)$ so that

$$\begin{aligned}\|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)\|^2 &= \|-(\hbar_2,-\hbar_1)-(-(\eth_2,-\eth_1))\|^2\\ &= \|-[(\hbar_2,-\hbar_1)-(\eth_2,-\eth_1)]\|^2\\ &= \|(\hbar_2-\eth_2),-(\hbar_1-\eth_1)\|^2\\ &= (\hbar_1-\eth_1)^2+(\hbar_2-\eth_2)^2\\ &= \|\hbar-\eth\|^2.\end{aligned}$$

Consequently,

$$\begin{aligned}(2+1)^2\|\hbar-\eth\|^2+\|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)\|^2 &= 9\|\hbar-\eth\|^2+\|\hbar-\eth\|^2\\ &= 10\|\hbar-\eth\|^2\\ &= \|2(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2\end{aligned}$$

Therefore, Γ is Φ_Γ-Lipshitz 2-enriched pseudocontractive mapping. However, Γ is not $(2,k)$-enriched strictly pseudocontractive since for $\hbar=(1,1), \eth=(-1,-1)\in R^2$, we have $\|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)\|^2=8, \|\hbar-\eth\|^2=8$ and $\|2(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2=80$, so that

$$(2+1)^2\|\hbar-\eth\|^2+k\|\hbar-\Gamma\hbar-(\eth-\Gamma\eth)\|^2=72+8k<80=\|2(\hbar-\eth)+\Gamma\hbar-\Gamma\eth\|^2, \forall k\in(0,1).$$

Theorem 2.7. Let $\mho$ be a uniformly convex Banach space and Λ a closed ball in $\mho$. Let Γ be Φ_Γ-enriched Lipschitzian and β-enriched pseudocontractive mapping of Λ into $\mho$ such that Γ also maps the boundary of Λ into Λ. Then, Γ has a fixed point in Λ.

Proof. Without loss of generality, assume that Λ is a ball centered at the origin with radius τ. Let $\partial\Lambda$ denote the boundary of Λ. For each $r>0, \beta\in[0,\infty)$ and $\hbar,\eth\in\Lambda$, we have

$$(\beta+1)\|\hbar-\eth\|\le\|(1+\beta+r)(\hbar-\eth)-r(\Gamma\hbar-\Gamma\eth)\|.$$

Setting $\beta = \frac{1}{\rho} - 1 \in [0, \infty)$ for $\rho \in (0,1]$ and simplifying, we get

$$\|\hbar - \eth\| \leq \|(\hbar - \eth) + r[(I - U_\rho)\hbar - (I - U_\rho)\eth]\|, \tag{23}$$

where the average operator U_ρ is as in Remark 2.1 and is pseudocontractive.

Now, letting $\alpha = \frac{r}{1+r}$, we obtain from inequality (23) that

$$\begin{aligned}(1-\alpha)\|\hbar - \eth\| &\leq \|(\hbar - \eth) - \alpha(U_\rho \hbar - U_\rho \eth)\| \\ &= \|(I - \alpha U_\rho)\hbar - (I - \alpha U_\rho)\eth\| \\ &= \|\Gamma_\alpha \hbar - \Gamma_\alpha \eth\|, \end{aligned} \tag{24}$$

where $\Gamma_\alpha = I - \alpha U_\rho$.

Since Γ is $(0, \Phi_\Gamma)$-enriched Lipschitizian, there exists a constant $Q > 0$ and strictly nondecreasing and continuous function $\Phi : R^+ \longrightarrow R^+$ with $\Phi(0) = 0$ and $\Phi(r) = Qr$ such that

$$\begin{aligned}\|U_\rho \hbar - U_\rho \eth\| &\leq (1-\rho)\|\hbar - \eth\| + \rho\|\Gamma\hbar - \Gamma\eth\| \\ &\leq (1-\rho)\|\hbar - \eth\| + \rho\Phi_\Gamma(\|\hbar - \eth\|) \\ &= (1-\rho)\|\hbar - \eth\| + \rho Q\|\hbar - \eth\| \\ &= (1-\rho+Q\rho)\|\hbar - \eth\|. \end{aligned} \tag{25}$$

Let

$$\eth^\star \in \Lambda_1 = \{\hbar \in \mho : \|\hbar\| \leq (1-\alpha)\tau\} \tag{26}$$

and define U_α by

$$U_\alpha = \alpha U_\rho + \eth^\star.$$

If $\alpha > 0$ is choosing such that $\alpha(1-\rho+Q\rho) < 1$ and $\alpha < 1$, then

$$\|U_\alpha \hbar - U_\alpha \eth\| = \alpha\|U_\rho \hbar - U_\rho \eth\| \leq \alpha(1-\rho+Q\rho)\|\hbar - \eth\|. \tag{27}$$

Also, since $\|\Gamma\hbar\| \leq \tau$ if $\hbar \in \partial\Lambda$, then it follows that

$$\|U_\alpha \hbar\| \leq \alpha[(1-\rho)\|\hbar\| + \rho\|\Gamma\hbar\|] + \|\eth^\star\| \leq \alpha\tau + (1-\alpha)\tau = \tau,$$

so that U_α maps the boundary Λ into Λ. From the above fact, and following the same reasoning as in [6], the mapping $\eth : \Lambda \longrightarrow \Lambda$ defined by

$$\eth = \frac{I + U_\alpha}{2}$$

maps Λ into Λ.

Further, since $\|U_\alpha \hbar - U_\alpha \eth\| = \alpha\|U_\rho \hbar - U_\rho \eth\|$, it follows from (27) that U_α is strictly contractive and, as a consequence, $\eth$ is as well strictly contractive. Hence, it follows immediately from Banach Contaction Principle that $\eth$ admits a fixed point $\hbar^\star \in \Lambda$; that is, $\eth\hbar^\star = \hbar^\star = U_\alpha \hbar^\star$.

Therefore, $\alpha U_\rho \hbar^\star + \eth^\star = \hbar^\star$. From $\Gamma_\alpha = I - \alpha U_\rho$, we obtain $\hbar^\star - \Gamma_\alpha \hbar^\star + \eth^\star = \hbar^\star$ so that $\Gamma_\alpha \hbar^\star = \eth^\star$. Hence, we have shown that

$$\Gamma_\alpha[\Lambda] \supset \Lambda_1 \textit{ and } \Gamma_\alpha^{-1}[\Lambda_1] \subset \Lambda.$$

Therefore, $(1-\alpha)\Gamma_\alpha^{-1} : \Lambda_1 \longrightarrow \Lambda_1$. By (24), $(1-\alpha)\Gamma_\alpha^{-1}$ is nonexpansive. Since, $(1-\alpha)\Gamma_\alpha^{-1}$ is nonexpansive, it follows from Theorem 1 of Browder [3] (see also Kirk [5]) that $(1-\alpha)\Gamma_\alpha^{-1}$ admits a fixed point $\xi \in \Lambda_1$. Setting $\xi^\star = \dfrac{\xi}{1-\alpha}$, we obtain $\Gamma_\alpha \xi^\star = \xi$. Using the definition of Γ_α, it follows that $U_\rho \xi^\star = (1-\rho)\xi^\star + \rho\Gamma\xi^\star = \xi^\star$. Therefore, $\Gamma\xi^\star = \xi^\star$. This completes the proof. □

Now, we consider another theorem for β-enriched accretive mappings which stemmed out of the intimate connection between this class of mappings and the class of pseudocontractive mapping.

Remark 2.8. Recall that if the average operator U_ρ is pseudocontractive and the operator B_ρ is accretive, then $U_\rho = I - B_\rho$ (where I is the identity map on Λ).

Theorem 2.9. Let $\mho$ be a real Banach space and Λ a closed ball in $\mho$ centred at origin. Let Υ be a β-enriched accretive mapping of Λ into $\mho$ and suppose Υ is also Φ_Υ-Lipschitzian. If Υ maps the boundary of Λ into Λ, then there is an element $\hbar \in \Lambda$ such that $\hbar + 2\beta\hbar + \Upsilon\hbar = 0$.

Proof. By Remark 2.3, the mapping $U_\rho = I - B_\rho$ is pseudocontractive. Let

$$\Gamma_r = (1+r)I - rU_\rho, r > 0,$$

so that from inequality (15), we obtain (on simplifying) that

$$\|\Gamma_r\hbar - \Gamma_r\eth\| \geq \|\hbar - \eth\|, \quad (\hbar, \eth \in \Lambda). \tag{28}$$

Since B_ρ is Φ_Υ-Lipshitizian with $\Phi_\Upsilon(s) = Ls$, we can choose $r > 0$ to be very small that $\partial r = -rB_\rho$ is strictly contractive. Suppose $r < 1$ and let

$$\eth^\star \in \Lambda_1 = \{\hbar \in \mho : \|\hbar\| \leq (1-\alpha)\tau\}, \tag{29}$$

where τ represents the radius of Λ. As in above, the mapping $\bar{\partial}r$ defined by $\partial_r\hbar + \eth^\star, \hbar \in \Lambda$, is a strict contraction on $\mho$ taking the boundary of Λ into itself. Hence, for some $\hbar \in \Lambda$, we get $\partial_r\hbar^\star = \hbar^\star = \partial_r\hbar^\star + \eth^\star$. From the relation

$$\partial_r = rB_\rho = -r + rU_\rho = I - \Gamma_r,$$

we obtain $\Gamma_r\hbar^\star = \eth^\star$, which in turn yields

$$\Gamma_r[\Lambda] \supset \Lambda_1.$$

As a consequence, $(1-r)\Gamma_r^{-1}:\Lambda_1 \longrightarrow \Lambda_1$. From the strict contraction of $(1-r)\Gamma_r^{-1}$, since Γ_r^{-1} is nonexpansive (see Theorem 2.1 above), we obtain from Banach contraction Principle that $(1-r)\Gamma_r^{-1}$ admits a fixed point $\xi \in \Lambda_1$. By setting $\xi^\star = \dfrac{\xi}{1-r}$, we obtain

$$\xi = \Gamma_r \xi^\star = (1+r)\xi^\star - U_\rho \xi^\star,$$

which on simplification yields $U_\rho \xi^\star = 2\xi^\star$. Since $U_\rho = I - \rho\Upsilon$, it follows that $2\xi^\star = \xi^\star - \rho\Upsilon\xi^\star$. Thus, $(1+\beta)\xi^\star + \Upsilon\xi^\star = 0$ as required. □

Note that unlike Theorem 2.1, no assumptions on the Banach space $\mho$ are necessary in Theorem 2.2. This was possible due to the fact that $(1-r)\Gamma_r^{-1}$ is a strict contraction in Theorem 2.2 as compared to the nonexpansivity of $(1-r)\Gamma_r^{-1}$ in Theorem 2.1.

Remark 2.10. If $\beta = 0$ in Theorem 2.2, then we obtain the following result of Theorem 2 due to Kirk [6]:

Theorem 2.11. Let $\mho$ be a uniformly convex Banach space and Λ a closed ball in $\mho$. Let Γ be a Lipschitzian pseudocontractive mapping of Λ into $\mho$ such that Γ also maps the boundary of Λ into Λ. Then, Γ has a fixed point in Λ.

3. Conclusion

The class pseudocontractive mappings are very important class of nonlinear mappings due to their intimate connection with another class of nonlinear mappings called accrative mappings. In this paper, we introduced a more general class of β-enriched pseudocontractive mappings which contains, as a subclass, the class of pseudocontracive mappings. We proved the existence of fixed point for this new class of nonlinear mappings in the setup of a real Banach space. Several examples are given to support our results. Unlike several works in the literature, we do not require strict assumptions on the space or the operator in order to obtain our results. Indeed, this results represent the vanguard for other results in this direction. For interested readers, results obtained in this study can be extended to enriched hemicontractive mappings, and also to other spaces higher than the Banach spaces. It is hoped that the introduction of this new class of nonlinear mappings will provide a new direction for the evaluation of fixed points, and some practical application of the results, in operator theory.

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References

[1] Kato, I.: Nonlinear semigroups and evolution equations, *J. Math. Soc. Jpn.*. 19, 508-520(1967).

[2] Browder, F.E.: Nonlinear mappings of nonexpansive and accretive type in Banach spaces, *Bull. Am. Math. Soc.* 73, 875-882(1969).

[3] Browder, F.E.: Nonexpansive nonlinear operators in a Banach space, *Acad. Sci. U.S.A. Proc. Nat.* 54, 1041-1044(1965).

[4] Browder, F.E., Petryshyn, W.V.: Construction of fixed points of nonlinear mappings in Hilbert space, *J. Math. Anal. Appl.* 20, 197-228(1967).

[5] Kirk, W.A.: A fixed point theorem for mappings which do not increase distances, *Amer. Math. Monthly*. 72, 1004-1006(1965).

[6] Kirk, W.A.: Remarks on pseudocontractive mappings, *Proc. of the Amer. Math. Soc.* 820-823(1970).

[7] Belluce, L P., Kirk, W.A.: Nonexpansive mappings and fixed-points in Banach spaces, *Illinois J. Math.* 11, 474-479(1967).

[8] Igbokwe, D.I.: Construction of fixed points of strictly pseudo-contractive mappings of Brouwder-Petryshyn-type in arbitrary Banach space,
Advance in Fixed Point Theory and Appl. 4, 137-147(2003).

[9] Chidume, C.E.: Strong convergence and stability of Picard iteration sequences for a general class of contractive-type mappings, *Fixed Point Theory and Applications*. 2014, 233(2014).

[10] Berinde, V.: Approximating fixed points of enriched nonexpansive mappings by Krasnolselkii iteration in Hilbert spaces, *Carpathian J. Math.* 3(35), 277-288(2019).

[11] Berinde, V.: Weak and strong convergence theorems for the Kreasnolselkij iteration algorithm in the class of enriched strictly pseudocontractive operators,
Analele Universitatii deVest, Timisoara, Seria Mathematica – Informatica. LVI(2), 13-37(2018).

[12] Shehu, Y., Ugwunnadi, G.C.: Approximation of fixed oints of non-expansive mappings by modified Krasnoselkii-Mann iterative algorithm in Banach spaces,
Thai J. Math.., (2013), 1-15.

[13] Igbokwe, D.I.: Weak and srtong convergence theorems for the iterative approximation of fixed points of strictly pseudocontractive

maps in arbitrary Banach spaces,
J. Inequalities Pure Applied Math.. 5(1), 67-75(2002).

[14] Acedo, G.L.,Xu, H.K.: Iteration methods for strict pseudocontractions in Hilbert space, *Nonlinear Analysis*. 67(7), 2258-2271V.

[15] Liu, L.: Approximation of fixed points of a strictly pseudocontrative mapping, *Proceeding of Amer. Math. Soc.*. 125(5), 1363-1366(1997).

[16] Saleem et al., N.: Strong convergence theorems for a finite family of (b,k)-enriched strictly pseudocontractive mappings and Φ_T-Enriched Lipschitizian mappings using a new modified mixed-type Ishikawa iteration scheme with error, *Symmetry*, 14(2022). https://doi.org/10.3390/sym14051032.

[17] Lion, Y.: Computing the Fixed points of strictly pseudo-contractive mappings by the implicit and explicit iterations, *Abstract and Applied Analysis*. 2012, 13(2012).

[18] Agwu, I.K.: A new iteration scheme for approximating common fixed points of two pseudocontractive mappings in uniformly convex Banach space. *Canad. J. Appl. Math.* 2(1), 111-123(2020).

[19] Jeribi, A., Krichen, B.: Functional Analysis in Banach spaces and Banach algebras Construction of fixed points of strictly pseudo-contractive mappings of Brouwder-Petryshyn-type in arbitrary Banach space, CRC Press-Taylor and Francis group, 2016:120-146.

[20] Agwu, I. K., Igbokwe, D. I.: Convergence theorems and demi-closedness principle for enriched strictly pseuocontractive mappings in real Banach spaces. *Int. J. Nonlinear Anal. Appl.*, 1-11(2023).

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