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BOHR INEQUALITIES FOR SOME GENERALIZED INTEGRAL OPERATORS ON SIMPLY CONNECTED DOMAIN

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ABSTRACT. We obtain the Bohr inequalities for some generalized integral operators of analytic function defined on simply connected domain. Our results are generalizations of some existing results in the literature.

1. INTRODUCTION AND PRELIMINARIES

In the literature, the Bohr inequality deals with finding the largest value of $r \in (0,1)$ such that the majorant series $\sum_{n=0}^{\infty} |a_n| r^n$ is less than or equal to one. For a bounded analytic function $f(z)$ in the unit disk $\mathbb{D} = \{|z| < 1 : |z| \in \mathbb{C}\}$, Bohr's inequality is stated as:

Theorem 1.1. [1] *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is analytic in the unit disk $\mathbb{D}$ and $|f(z)| \le 1$ in $\mathbb{D}$. Then*

$$\sum_{n=0}^{\infty} |a_n| r^n \le 1 \quad \textit{for} \quad r \le \frac{1}{3}, \tag{1.1}$$

where $1/3$ cannot be improved.

The constant $1/3$ is known as Bohr radius. The famous Bohr inequality obtained by the contribution of Harald Bohr [1] has become of interest to several researchers in the field of complex analysis, and various refinements and improvements have also been studied for various classes of analytic functions. For more information (see [2, 8, 11]).

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In [3], Kayumov et al. studied Bohr inequality for Cesàro operator of an analytic function $f(z)$ in unit disk $\mathbb{D}$ and obtained the following:

Theorem 1.2[3]. *Suppose* $f(z)=\sum_{n=0}^{\infty} a_n z^n$ *is analytic in the unit disk* $\mathbb{D}$ *and* $|f(z)|\le 1$ in $\mathbb{D}$. *Then*

$$|C_f(z)| = C_f(r) = \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}|a_k|\right) r^n \le \frac{1}{r}\ln\left(\frac{1}{1-r}\right), \tag{1.2}$$

for $r\le 0.5335...$, *where* $r=|z|$ *and the constant* $0.5335\ldots$ *is the positive root of the equation* $2r=3(1-r)\ln\frac{1}{1-r}$. *The number* $0.5335\ldots$ *is best possible.*

Kayumov et al. [4] established the following Bohr inequality for generalized Cesàro operator which they named as α-Cesàro for an analytic function on the unit disk $\mathbb{D}$.

Theorem 1.3. *Suppose* $f(z)$ *is defined as in Theorem* 1.2*, and if* $\alpha>-1$*, then*

$$C_f^{\alpha}(r) \le (\alpha+1)\sum_{n=0}^{\infty}\frac{r^n}{n+\alpha+1} = \frac{\alpha+1}{r^{\alpha+1}}\int_0^r \frac{t^{\alpha}}{1-t}dt, \tag{1.3}$$

for all $r\le R(\alpha)$*, where* $R(\alpha)$ *is the positive root of the equation*

$$3(\alpha+1)\sum_{n=0}^{\infty}\frac{x^n}{n+\alpha+1}=\frac{2}{1-x} \text{ or } \sum_{n=0}^{\infty}\frac{\alpha+1-2n}{n+\alpha+1}x^n=0.$$

In [5], Shankey and Swadesh obtained another generalization of the Cesàro operator (called β-Cesàro operator). The following Bohr theorem was stated and proved:

Theorem 1.4[5]. *Suppose* $f(z)=\sum_{n=0}^{\infty} a_n z^n \in \mathscr{B}$ *and if* $0<\beta\ne 1$. *Then*

$$\begin{aligned} C_{\beta}[f](r) &= \sum_{n=0}^{\infty}\left(\frac{1}{n+1}\sum_{k=0}^{n}\frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)}|a_k|\right) r^n \\ &\le \frac{1}{r}\left[\frac{1-(1-r)^{1-\beta}}{1-\beta}\right], \end{aligned} \tag{1.4}$$

for $r\le R(\beta)$*, where* $R(\beta)$ *is the positive root of*

$$\frac{3\left[1-(1-x)^{1-\beta}\right]}{1-\beta}-\frac{2\left[(1-x)^{-\beta}-1\right]}{\beta}=0\,.$$

The radius $R(\beta)$ *cannot be improved.*

Remark 1.5: If we set $\alpha=0$ in Theorem 1.3 and $\beta\to 1$ in Theorem

1.4, the exact Bohr inequality for Cesàro operator in Theorem 1.2 will be obtained. This shows that the Theorems 1.3 and 1.4 are the generalization of Theorem 1.2.
Shankey and Swadesh [5] also provided sharp Bohr radius for Bernardi and Libera integral operators; these are respectively given as follows:

Theorem 1.6. *Let $k > -m$, if $f(z) = \sum_{n=m}^{\infty} a_n z^n \in \mathscr{B}$. Then*

$$\sum_{n=m}^{\infty} \frac{|a_n|}{n+k} \leq \frac{1}{m+k} r^m \quad for \quad r \leq R(k), \tag{1.5}$$

where $R(k)$ is the positive root of $\frac{x^m}{m+k} - 2\sum_{n=m+1}^{\infty} \frac{x^n}{n+k} = 0$ that cannot improved.

Theorem 1.7 (Libera Operator)[5]. *If $f(z) = \sum_{n=0}^{\infty} a_n z^n \in \mathscr{B}$. Then*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+1} \leq 1 \quad for \quad r \leq R = 0.5828... \tag{1.6}$$

where the number $0.5828...$ is the positive root of $3x + 2\ln(1-x) = 0$.

The Bohr inequality of Cesàro operator for a normalized analytic function is also obtained as follows:

Theorem 1.8[5]. *If $f(z) = \sum_{n=1}^{\infty} a_n z^n \in \mathscr{B}$. Then*

$$\sum_{n=1}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |a_{k+1}| \right) r^{n+1} \leq \ln \frac{1}{1-r}, \tag{1.7}$$

for $r \leq R = 0.5335...$, where the number $0.5335...$ is the positive root of $2x = 3(1-x)\ln\frac{1}{1-x}$.

We refer the reader to [5] for the details of the results on Bohr inequality for other integral operators.

Observe that all the results in Theorems 1.2 to 1.8 were generated for the class of analytic functions in unit disk $\mathbb{D}$.

Now, let $\mathscr{B}$ represents the set of $f(z) = \sum_{n=0}^{\infty} a_n z^n$ analytic in the unit disk $\mathbb{D}$ such that $|f(z)| \leq 1$ with $|a_n| \leq 1 - |a_0|^2 \ \forall\, n \geq 1$. Then $\mathscr{B}(\Omega_\gamma)$ where

$$\Omega_\gamma = \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}, \quad 0 \leq \gamma < 1, \tag{1.8}$$

represents the class of $f(z)$ analytic in Ω_γ. Clearly, disk (1.8) contains the unit disk when $\gamma = 0$ (see [7, 9]).

Allu and Ghosh [10] obtained Bohr type inequality for Cesàro and Bernardi integral operators on simply connected domain (1.8). In fact, they proved following:

Theorem 1.9. *For $\gamma \in [0,1)$, let $f(z) \in \mathscr{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then we have*

$$C_f(r) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} |a_k| \right) r^n \leq \frac{1}{r} \ln \left(\frac{1}{1-r} \right) \quad r \leq R_\gamma, \tag{1.9}$$

where R_γ is the positive root of $(3+\gamma)(1-x)\ln \frac{1}{1-x} = 2x$. The number R_γ is best possible.

Theorem 1.10. *For $\gamma \in [0,1)$, let $f(z) \in \mathscr{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then for $k > 0$,*

$$\sum_{n=0}^{\infty} \frac{|a_n|}{n+k} r^n \leq \frac{1}{k} \quad \text{for } r \leq R_{\gamma,k}, \tag{1.10}$$

where $R_{\gamma,k}$ is the positive root of $\frac{1}{k} = \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^n}{n+k}$. The number $r \leq R_{\gamma,k}$ is best possible.

The aim of this paper is to generalize the recent work of Allu and Ghosh [10] in Theorems 1.9 and 1.10, and then obtain Bohr inequality for some other integral operators on simply connected domain (1.8).
For analysis of the proof of our main theorems, the following Lemma will be considered.

Lemma 1.11[9]. *For $\gamma \in [0,1)$, let*

$$\Omega_\gamma = \left\{ z \in \mathbb{C} : \left| z + \frac{\gamma}{1-\gamma} \right| < \frac{1}{1-\gamma} \right\}$$

and let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ be analytic in Ω_γ, bounded by 1 in the unit disk $\mathbb{D}$. Then $|a_n| \leq \frac{1-|a_0|^2}{1+\gamma}$.

2. MAIN RESULTS

Theorem 2.1. *For $\gamma \in [0,1)$, let $f(z) \in \mathscr{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Then for* $0 < \beta \neq 1$, $|a_0| \in [0,1)$

$$C_\beta[f](r) = \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} |a_k| \right) r^n \leq \frac{1-(1-r)^{1-\beta}}{r(1-\beta)}, \tag{2.1}$$

for $r \leq R_{\beta,\gamma}$, where $R_{\beta,\gamma}$ is the positive root of

$$\frac{(3+\gamma)\left[1-(1-r)^{1-\beta}\right]}{1-\beta} - \frac{2\left[(1-r)^{-\beta}-1\right]}{\beta} = 0.$$

The radius $R_{\beta,\gamma}$ cannot be improved.

Proof. Observe that

$$\begin{aligned} C_\beta[f](r) &= \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \sum_{k=0}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} |a_k| \right) r^n \\ &= |a_0| \sum_{n=0}^{\infty} \left(\frac{1}{n+1} \frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)} \right) r^n \\ &\quad + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} \sum_{k=1}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} |a_k| \right) r^n. \end{aligned}$$

Let $|a_0| = a$, by Lemma 1.11, we have

$$\begin{aligned} C_\beta[f](r) &\leq a \sum_{n=0}^{\infty} \frac{1}{n+1} \left(\frac{\Gamma(n+\beta)}{\Gamma(n+1)\Gamma(\beta)} \right) r^n \\ &\quad + \frac{1-a^2}{1+\gamma} \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\sum_{k=1}^{n} \frac{\Gamma(n-k+\beta)}{\Gamma(n-k+1)\Gamma(\beta)} \right) r^n \\ &= \frac{a}{r} \int_0^r \frac{dt}{(1-t)^\beta} + \frac{1-a^2}{r(1+\gamma)} \int_0^r \frac{t}{(1-t)^{\beta+1}} dt \\ &= \frac{a^2 + a(1+\gamma) - 1}{r(1+\gamma)} \int_0^r \frac{dt}{(1-t)^\beta} + \frac{1-a^2}{r(1+\gamma)} \int_0^r \frac{dt}{(1-t)^{\beta+1}} \\ &= P(a), \end{aligned}$$

where

$$P(a) = \frac{1}{r(1+\gamma)} \left[\frac{\left(a^2 + a(1+\gamma) - 1\right)\left(1-(1-r)^{1-\beta}\right)}{1-\beta} + \frac{(1-a^2)\left((1-r)^{-\beta}-1\right)}{\beta} \right]. \tag{2.2}$$

Then differentiating twice *w.r.t* a yields

$$P'(a) = \frac{1}{r(1+\gamma)}\left[\frac{(2a+1+\gamma)\left(1-(1-r)^{1-\beta}\right)}{1-\beta} - \frac{2a\left((1-r)^{-\beta}-1\right)}{\beta}\right]$$

and

$$P''(a) = \frac{2}{r(1+\gamma)}\left[\frac{1-(1-r)^{1-\beta}}{1-\beta} - \frac{(1-r)^{-\beta}-1}{\beta}\right].$$

Clearly, $P''(a) \leq 0$ for all $r \in (0,1)$, $\gamma \in [0,1)$ and $|a_0| = a \in [0,1)$. Since $a < 1$, it means that $P'(a) \geq P'(1)$. That is,

$$\frac{1}{r(1+\gamma)}\left[\frac{(3+\gamma)\left(1-(1-r)^{1-\beta}\right)}{1-\beta} - \frac{2\left((1-r)^{-\beta}-1\right)}{\beta}\right] \geq 0. \tag{2.3}$$

Inequality (2.3) holds for $r \leq R_{\beta,\gamma}$ where $R_{\beta,\gamma}$ is the positive root of

$$\frac{(3+\gamma)\left(1-(1-r)^{1-\beta}\right)}{1-\beta} - \frac{2\left((1-r)^{-\beta}-1\right)}{\beta} = 0.$$

Also $P(a)$ is an increasing function of a, and since $a < 1$ then $P(a) \leq P(1)$. That is,

$$C_\beta[f](r) = P(a) \leq P(1) = \frac{1-(1-r)^{1-\beta}}{r(1-\beta)}.$$

This completes the proof.

Remark 2.2: The following well known results will be obtained if values are given to β and γ.

i. If $\beta \to 1$ in Theorem 2.2, we obtain the exact result of Allu and Ghosh [10] in Theorem 1.9.

ii. If $\beta \to 1$ and $\gamma = 0$ in Theorem 2.2, it gives the result in Theorem 1.2.

The next theorem establishes the Bohr radius for a Cesàro operator defined on a normalized analytic function in the disk (1.8) (see [5] for definition of a normalied function):

Theorem 2.3. *Suppose* $0 \leq \gamma < 1$ *and* $f(z) \in \mathscr{B}(\Omega_\gamma)$ *with* $f(z) = \sum_{n=1}^{\infty} a_n z^n$. *Then for* $|a_1| \in [0,1)$

$$C_f'(r) = \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\sum_{k=0}^{n} |a_{k+1}|\right) r^{n+1} \leq \ln\frac{1}{1-r} \text{ for } r \leq R_{1,\gamma}, \tag{2.4}$$

where $R_{1,\gamma}$ *is the positive root of* $2r + (3+\gamma)(1-r)\ln(1-r) = 0$. *The radius* $R_{1,\gamma}$ *cannot be improved.*

Proof. Suppose $|a_1| = a < 1$, then from Lemma 1.11, we have

$$\begin{aligned} C_f'(r) &= \sum_{n=0}^{\infty} \frac{1}{n+1}\left(\sum_{k=0}^{n} |a_{k+1}|\right) r^{n+1} \\ &= ar\sum_{n=0}^{\infty} \frac{r^n}{n+1} + \sum_{n=1}^{\infty} \frac{1}{n+1}\left(\sum_{k=1}^{n} |a_{k+1}|\right) r^{n+1} \end{aligned}$$

$$\leq ar\sum_{n=0}^{\infty}\frac{r^n}{n+1}+\frac{1-a^2}{1+\gamma}\sum_{n=1}^{\infty}\frac{n}{n+1}r^{n+1}$$
$$=ar\left(\frac{1}{r}\ln\frac{1}{1-r}\right)+\frac{1-a^2}{1+\gamma}\left[\frac{r^2+r(1-r)+(1-r)\ln(1-r)}{1-r}\right]$$
$$=a\ln\frac{1}{1-r}+\frac{1-a^2}{1+\gamma}\left[\frac{r}{1-r}-\ln\frac{1}{1-r}\right]=P(a).$$

Then differentiating $P(a)$ *w.r.t.* a twice, gives

$$P'(a)=\ln\frac{1}{1-r}-\frac{2a}{1+\gamma}\left[\frac{r}{1-r}-\ln\frac{1}{1-r}\right] \text{ and}$$
$$P''(a)=-\frac{2}{1+\gamma}\left[\frac{r}{1-r}-\ln\frac{1}{1-r}\right].$$

It is easy to see that, for all $r\in(0,1)$ and $\gamma\in[0,1)$, $P''(a)\leq 0$. Hence

$$\begin{aligned}P'(a)\geq P'(1)&=\ln\frac{1}{1-r}-\frac{2}{1+\gamma}\left[\frac{r}{1-r}-\ln\frac{1}{1-r}\right]\\&=\frac{3+\gamma}{1+\gamma}\ln\frac{1}{1-r}-\frac{2r}{(1-r)(1+\gamma)}\geq 0,\end{aligned} \tag{2.5}$$

which is true if $r\leq R_{1,\gamma}$, where $R_{1,\gamma}$ is the positive root of

$$\frac{3+\gamma}{1+\gamma}\ln\frac{1}{1-r}-\frac{2r}{(1-r)(1+\gamma)}=0 \text{ or } (3+\gamma)(1-r)\ln(1-r)+2r=0.$$

Also, observe that $P'(a)>0$ for $r\in(0,1)$ and $a\in[0,1)$, thus, $P(a)$ is an increasing function of a. Hence, $P(a)\leq P(1)$ since $a<1$. Therefore,

$$C_f^{'}(r)=P(a)\leq P(1)=\ln\frac{1}{1-r}.$$

Hence the proof is complete.

Remark 2.4: Theorem 2.3 yields the result of Theorem 1.9 if we set $\gamma=0$.

Before stating the next theorem on α-Cesàro operator, the following definition is necessary

For analytic function $f(z)$ on unit disk $\mathbb{D}$, the α-Cesàro operator of f is given by [4] as

$$C^{\alpha}f(z)=\sum_{n=0}^{\infty}\left(\frac{1}{A_n^{\alpha+1}}\sum_{k=0}^{n}A_{n-k}^{\alpha}a_k\right)z^n=(\alpha+1)\int_0^1\frac{f(tz)(1-t)^{\alpha}}{(1-tz)^{\alpha+1}}dt$$

where $A_k^{\alpha}=\dfrac{(\alpha+1)_n}{(1)_n}$, $\dfrac{1}{A_n^{\alpha+1}}\sum_{k=0}^{n}A_{n-k}^{\alpha}=1$ and $(m)_n=\dfrac{\Gamma(m+n)}{\Gamma(m)}$ is the Pochhammer symbol.

We now proceed to state and prove the Bohr inequality for α-Cesàro operator on simply connected domain (1.8).

Theorem 2.5. *For $\gamma \in [0,1)$ and $f(z) \in \mathscr{B}(\Omega_\gamma)$ with $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\alpha > -1$. Then for $|a_0| \in [0,1)$*

$$C^\alpha f(r) \leq (\alpha+1) \sum_{n=0}^{\infty} \frac{r^n}{n+\alpha+1} \text{ for all } r \leq R_{\alpha,\gamma}, \tag{2.6}$$

where $R_{\alpha,\gamma}$ is the positive root of the equation

$$(1+\gamma)(1+\alpha) \sum_{n=0}^{\infty} \frac{x^n}{n+\alpha+1} = 2 \sum_{n=1}^{\infty} \frac{nx^n}{n+\alpha+1}.$$

Proof. Let $|a_0| = a$ then by Lemma 1.11 we have

$$\begin{aligned} C^\alpha f(r) &= \sum_{n=0}^{\infty} \left(\frac{1}{A_n^{\alpha+1}} \sum_{k=0}^{n} A_{n-k}^{\alpha} |a_k| \right) r^n \\ &\leq a \sum_{n=0}^{\infty} \frac{r^n}{A_n^{\alpha+1}} A_n^{\alpha} + \frac{1-a^2}{1+\gamma} \sum_{n=1}^{\infty} \left(\frac{1}{A_n^{\alpha+1}} \sum_{k=1}^{n} A_{n-k}^{\alpha} \right) r^n. \end{aligned} \tag{2.7}$$

Now, $\frac{1}{A_n^{\alpha+1}} A_n^{\alpha} = \frac{n!\Gamma(\alpha+2)}{\Gamma(\alpha+n+2)} . \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+1)} = \frac{1+\alpha}{n+\alpha+1}.$

Since $\frac{1}{A_n^{\alpha+1}} \sum_{k=0}^{n} A_{n-k}^{\alpha} = 1$, then $\frac{1}{A_n^{\alpha+1}} A_n^{\alpha} + \frac{1}{A_n^{\alpha+1}} \sum_{k=1}^{n} A_{n-k}^{\alpha} = 1$. So that

$$\frac{1}{A_n^{\alpha+1}} \sum_{k=1}^{n} A_{n-k}^{\alpha} = \frac{n}{n+\alpha+1}.$$

Hence, (2.7) becomes

$$C^\alpha f(r) \leq a(1+\alpha) \sum_{n=0}^{\infty} \frac{r^n}{n+\alpha+1} + \frac{1-a^2}{1+\gamma} \sum_{n=1}^{\infty} \frac{nr^n}{n+\alpha+1}. \tag{2.8}$$

Inequality(2.8) can also be expressed as

$$\begin{aligned} C^\alpha f(r) \leq a(1+\alpha)\Phi(r,1,\alpha+1) &+ \frac{(1-a^2)r}{(1-r)(1+\gamma)} \\ &- \frac{(1-a^2)(1+\alpha)r}{(1+\gamma)} \Phi(r,1,\alpha+2) \end{aligned} \tag{2.9}$$

where $\Phi(z,s,k) = \sum_{n=0}^{\infty} \frac{z^n}{(n+k)^s}$ is the Lerch transcendent function.

Now, let $P(a) = a(1+\alpha) \sum_{n=0}^{\infty} \frac{r^n}{n+\alpha+1} + \frac{1-a^2}{1+\gamma} \sum_{n=1}^{\infty} \frac{nr^n}{n+\alpha+1}$, then

$$P'(a) = (1+\alpha) \sum_{n=0}^{\infty} \frac{r^n}{n+\alpha+1} - \frac{2a}{1+\gamma} \sum_{n=1}^{\infty} \frac{nr^n}{n+\alpha+1} \text{ and}$$

$$P''(a) = -\frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{nr^n}{n+\alpha+1}.$$

Obviously $P''(a) \leq 0$ for $\alpha, \gamma \in [0,1)$ and $r \in (0,1)$. Therefore

$$P'(a) \geq p'(1) = (1+\alpha) \sum_{n=0}^{\infty} \frac{r^n}{n+\alpha+1} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{nr^n}{n+\alpha+1}. \tag{2.10}$$

(2.10) will be greater than or equal to zero only if $r \leq R_{\alpha,\gamma}$ where $R_{\alpha,\gamma}$ is the minimum positive root of

$$(1+\alpha)\sum_{n=0}^{\infty}\frac{r^n}{n+\alpha+1}-\frac{2}{1+\gamma}\sum_{n=1}^{\infty}\frac{nr^n}{n+\alpha+1}=0.$$

Further simplification provides

$$\sum_{n=0}^{\infty}\frac{(1+\gamma)(1+\alpha)-2n}{n+\alpha+1}x^n=0. \tag{2.11}$$

Also, $P(a)$ is an increasing function of a, and since $a<1$ then $P(a)\leq P(1)$. Hence

$$C_f^{\alpha}(r)=P(a)\leq P(1)=(\alpha+1)\sum_{n=0}^{\infty}\frac{r^n}{n+\alpha+1}.$$

The proof of Theorem 2.5 is complete.

Remark 2.6: We remark that with Theorem 2.5, the following existing results will be obtained.

i. When $\gamma=0$ in (2.11) we have $\sum_{n=0}^{\infty}\frac{\alpha+1-2n}{n+\alpha+1}x^n=0$ which gives exact Bohr radius obtained by [4] in Theorem 1.3.

ii. Setting $\gamma=0$ and $\alpha=0$ in Theorem 2.5, the Bohr radius will be obtained from $\sum_{n=0}^{\infty}\frac{1-2n}{n+1}x^n=0$ in which further simplification yields $2x=3(1-x)\ln\frac{1}{1-x}$, the Bohr radius by Kayumov et al. [3] in Theorem 1.2.

iii. If $\alpha=0$ in Theorem 2.5, the work of Allu and Ghosh [10] in Theorem 1.9 will be obtained.

We now proceed to state and prove the Bohr inequality for the generalized Bernardi's operator [5] in the simply connected domain (1.8).

Theorem 2.7. *If $\gamma\in[0,1)$ and $f(z)\in\mathscr{B}(\Omega_\gamma)$ with $f(z)=\sum_{n=m}^{\infty}a_nz^n$ and $k>-m$. Then for $|a_0|\in[0,1)$*

$$B_\gamma[f](r)=\sum_{n=m}^{\infty}\frac{|a_n|}{n+k}r^n\leq\frac{1}{m+k}r^m \quad \text{for} \quad r\leq R_4(\gamma), \tag{2.12}$$

where $R_4(\gamma)$ is the positive root of $\frac{1+\gamma}{m+k}=2\sum_{n=1}^{\infty}\frac{x^n}{n+m+k}$ that cannot be improved.

Proof. Let $|a_m|=a$. By (2.12) and using Lemma 1.11, we have

$$B_\gamma[f](r)=\sum_{n=m}^{\infty}\frac{|a_n|}{n+k}r^n\leq\frac{ar^m}{m+k}+\frac{1-a^2}{1+\gamma}\sum_{n=m+1}^{\infty}\frac{r^n}{n+k}$$

$$=\frac{ar^m}{m+k}+\frac{(1-a^2)r^{m+1}}{1+\gamma}\Phi(r,1,m+k+1)$$

$$= \frac{ar^m}{m+k} + \frac{1-a^2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^{m+n}}{n+m+k}.$$

Let $P(a) = \frac{ar^m}{m+k} + \frac{1-a^2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^{m+n}}{n+m+k}$, then $P(a) < P(1)$ since $P(a)$ is an increasing function of a for all $m \geq 0$, $a, \gamma \in [0,1)$, $k > -m$ and $r \in (0,1)$.

$$\therefore B_\gamma[f](r) \leq P(a) < P(1) = \frac{1}{m+k} r^m.$$

Also, differentiating $P(a)$ yields

$$P'(a) = \frac{r^m}{m+k} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^{m+n}}{n+m+k} \text{ and}$$
$$P''(a) = -\frac{2a}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^{m+n}}{n+m+k}.$$

Observe that $P''(a) \leq 0$ for all $\gamma \in [0,1)$ and $r \in (0,1)$. Hence,

$$P'(a) \geq P'(1) = \frac{r^m}{m+k} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^{n+m}}{n+m+k}. \tag{2.13}$$

(2.13) will be greater than or equal to zero only if $r \leq R_4(\gamma)$ where $R_4(\gamma)$ is the real positive root of

$$\frac{r^m}{m+k} - \frac{2}{1+\gamma} \sum_{n=1}^{\infty} \frac{r^{n+m}}{n+m+k} = 0 \text{ or } \frac{1+\gamma}{m+k} = 2 \sum_{n=1}^{\infty} \frac{x^n}{n+m+k}.$$

Hence the proof is complete.

Remark 2.8: Theorem 2.7 yields the following if we assigned values to γ, m and k.
(i) If $m = 0$ in Theorem 2.7, exact Bohr inequality and radius for Theorem 1.10 will be obtained which is the recent work of Allu and Ghosh [10]. This shows that Theorem 2.7 generalizes the Theorem 1.10 for simply connected domain.
(ii) If also $\gamma = 0$ in Theorem 2.7, it will yield the Bohr radius obtained in Theorem 1.6 by [5].

Corollary 2.9. *If* $0 \leq \gamma < 1$ *and* $f(z) \in \mathscr{B}(\Omega_\gamma)$ *with* $f(z) = \sum_{n=0}^{\infty} a_n z^n$. *Then*

$$L_f(r) = \sum_{n=0}^{\infty} \frac{|a_n|}{n+1} r^n \leq 1 \qquad \text{for } r \leq R_1, \tag{2.14}$$

where the constant R_1 *is the positive root of* $(3+\gamma)x + 2\ln(1-x) = 0$.

Remark 2.10: Corollary 2.9 is the Bohr inequality for Libera operator on simply connected domain and is a special case of Bohr inequality for Bernardi operator (2.12) with $m = 0$ and $k = 1$ and so, the proof corollary 2.9 follows from the proof Theorem 2.7.

corollary 2.11. *If* $0 \le \gamma < 1$ *and* $f(z) \in \mathscr{B}(\Omega_\gamma)$ *with* $f(z) = \sum_{n=1}^{\infty} a_n z^n$. *Then*

$$A_f(r) = \sum_{n=1}^{\infty} \frac{|a_n|}{n} r^n \le r \qquad \text{for } r \le R_2, \tag{2.15}$$

where the constant R_2 *is the positive root of* $(3+\gamma)x + 2\ln(1-x) = 0$ *and cannot be improved.*

Remark 2.12: Corollary 2.11 provides the Bohr inequality and radius for Alexander operator on simply connected domain which also a special case of Bernardi's operator of Theorem 2.7 with $m = 1$ and $k = 0$.
We now state and prove Bohr inequality and radius for a new integral operator.

Theorem 2.13. *Suppose* $\gamma \in [0,1)$ *and* $f(z) \in \mathscr{B}(\Omega_\gamma)$ *with* $f(z) = \sum_{n=1}^{\infty} a_n z^n$. *Then for* $|a_0| \in [0,1)$

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n+k} r^n \le \frac{r}{k+1} \qquad \text{for } r \le R_3(\gamma), \tag{2.16}$$

where $R_3(\gamma)$ *is the positive root of*

$$(1+\gamma) = 2x(k+1) \sum_{n=0}^{\infty} \frac{1}{n+k+2} x^n.$$

Proof. Let $|a_1| = a$, then by Lemma 1.11, we have

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n+k} r^n \le \frac{ar}{k+1} + \frac{1-a^2}{1+\gamma} \sum_{n=2}^{\infty} \frac{1}{n+k} r^n$$

$$= \frac{ar}{k+1} + \frac{1-a^2}{1+\gamma} \sum_{n=0}^{\infty} \frac{1}{n+k+2} r^{n+2}.$$

Now, let $P(a) = \frac{ar}{k+1} + \frac{1-a^2}{1+\gamma} \sum_{n=0}^{\infty} \frac{1}{n+k+2} r^{n+2}$.
Easy calculations show that $P(a)$ is an increasing function of a for all $a, \gamma \in [0,1)$ and $r \in (0,1)$. Hence $P(a) < P(1)$, that is,

$$\sum_{n=1}^{\infty} \frac{|a_n|}{n+k} r^n \le P(a) < P(1) = \frac{r}{k+1}.$$

Also, $P'(a) = \frac{r}{k+1} - \frac{2a}{1+\gamma} \sum_{n=0}^{\infty} \frac{1}{n+k+2} r^{n+2}$ and

$$P''(a) = -\frac{2}{1+\gamma} \sum_{n=0}^{\infty} \frac{1}{n+k+2} r^{n+2}.$$

Then clearly, $P''(a) \le 0$ for all $\gamma \in [0,1)$ and $r \in (0,1)$ and so $P'(a) \ge P'(1)$. Thus,

$$P'(a) \ge P'(1) = \frac{r}{k+1} - \frac{2}{1+\gamma} \sum_{n=0}^{\infty} \frac{1}{n+k+2} r^{n+2} \ge 0. \tag{2.17}$$

Inequality (2.17) holds only if $r \leq R_3(\gamma)$ where the constant $R_3(\gamma)$ is the positive root of

$$\frac{r}{k+1} - \frac{2}{1+\gamma}\sum_{n=0}^{\infty}\frac{1}{n+k+2}r^{n+2} = 0. \tag{2.18}$$

Simplifying (2.18), we get

$$(1+\gamma) = 2r(k+1)\sum_{n=0}^{\infty}\frac{1}{n+k+2}r^{n}. \tag{2.19}$$

This completes the proof of the theorem.

Remark 2.14: The following are concluded from Theorem 2.13:
i. Theorem 2.13 is a generalization of the Bohr inequality for Alexander operator in corollary 2.11 since if we set $k = 0$ in Theorem 2.13, the Bohr inequality and radius of corollary 2.11 will be obtained.
ii. Setting $\gamma = k = 0$ in Theorem 2.13 provides the Bohr inequality for Alexander operator in the unit disk (see [5]).

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