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ON THE CONNECTEDNESS OF TWO-STEP MARKOV CHAINS AND AN APPLICATION TO PATIENTS BEHAVIOUR AND COMPLIANCE WITH MEDICATIONS

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ABSTRACT. This paper models tablet consumption from a jar containing single and double units using a two-urn Markov process framework. By solving system equations, it employs statistical measures such as the odds ratio and Cohen's (h) index to characterize variability in compliance among individual patients. The findings demonstrate that a patient's behavior significantly affects the risk of unintended underdose and overdose.

1. INTRODUCTION

As often the case with prescribed medications for controlling certain medical conditions, each tablet may have varying strength, measured in milligrams (mg). The initial prescribed daily dosage at the onset of management may be reviewed downwards at a later date as adjudged by the managing physician and as the patient's clinical status dictates. For example, a drug like anhydrous Lisinopril may come in measures of 10mgs or 5mgs or a drug like Furosemide may come in measures of 20mg or 40mg, for which the prescribed dose may be lower than the dispensed strength. For a patient that has purchased Lisinopril with formulated strength of 10mg but now instructed to reduce the dose to 5mg per day, due to improving symptoms or side-effects, rather than discard the already purchased 10 mg daily tablets, it is expedient that the patient begins to physically divide the higher measure into two on daily basis. Such a patient may soon end up having a mixture of 10 mg and 5 mg measures to choose from in the tablet container or case. A

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purchase originally intended for N days will now last $2N$ days, saving some cost. Of interest is the distribution of number X_j of the lower measuring unit available in the container at the end of day j, $(0 \leq j \leq 2N)$. Of course, for the boundary points $X_0 = X_{2N} = 0$ with probability one, being the case at the initiation and completion of taking the lot of tablets; and logically $X_1 = X_{2N-1} = 1$, also with probability one. While at the end of an even $2j$ days of the $2N$ days for taking the lot of tablets, the jar may contain N-j-k of double units and $2k$ of single units; $k = 0, 1, \ldots j$ if $2j \leq N$ or $k = 0, 1, \ldots N - j$ if $N < 2j \leq 2N$, the jar may also, at the end of an odd $2j+1$ days of the $2N$ days of taking the lot of tablets, contain $N-k-1$ of double units and $2k+1$ single units; $k = 0, 1, \ldots j$ if $2j+1 < N$ or $k = 0, 1, \ldots N - j - 1$ if $N < 2j + 1 \leq 2N-1$. With the help of separate system equations and governing recurrence relations for the odd and even days, we arrive at two related two-step Markov Chains. Subsequently the equations give the required joint and conditional probabilities for specific number of single tablets that may be available in the jar on specific days. The main interest in this work is to obtain the distribution of the number of single units X_j at the end of each day j, j=0,1,2,...,2N and to apply this in assessing the relative chance of unintended under-dose or overdose for patients who may need to split purchased medications. This interest stems from the fact that a patient who is organized and not haphazard will ensure keeping track so as to take whatever split half from the previous day and not proceed to split an entirely different double unit on the next day. If this is strictly adhered to the jar must contain no single unit at the end of an even number of days or only one single-unit at the end of an odd number of days of taking the drug. As accurate splitting is not guaranteed, a patient of haphazard behaviour cares less and soon begins to go very close to not taking medications in the prescribed dosage and risks being classified as being non-compliant patient. It is therefore a straightforward exercise to assess the extent to which any haphazard behaviour, as opposed to some conscious balancing strategy on the part of patients, may amount to some periodic overdose or under-dose of prescribed tablets. This is rather in line with the application of two-step Markov Chain for predicting pattern and level of rainfall occurrences, based on certain structured data conditions, and then requiring being able to assess deviations of forecasts from reality (Francis, 2017).

1.0.1. *Notation.* we define set S_j comprising of elements s_j (or related alternative s'_j) as follows:

$$S_{2j+1} = \{s'_{2j+1}\} = \{(N-j-k-1, 2k+1) : k = 0, 1, \ldots j$$

if $2j+1<N$ or $k=0,1,\ldots N-j-1$ if $N<2j+1\leq 2N-1\}$,
$j=0,1,2,\ldots,N-1$.

$$S_{2j}=\{s'_{2j}\}=\{(N-j-k,2k):k=0,1,\ldots j \text{ if } 2j\leq N$$
$$\text{or } k=0,1,\ldots N-j \text{ if } N<2j\leq 2N, j=0,1,2,\cdots,N.$$

where,
$s_j=(j,k)$, state of the system, comprising of number j of days and number k of single units at the end of j days of taking the tablet. $j=0,1,\cdots,2N$
$s'_j=(n_j,k)$, alternative state of the system, comprising of number n_j of double units and number of k of single units at the end of j days of taking the tablet. $j=0,1,\cdots,2N$

Other notations include: $I_j=$ number of single units present in the system at the end of day j of taking the tablet.
$p_j(k)=$ Joint probability that there are k single units present in the system at the end of j days of taking the tablet.
$q_j(k)=$ Conditional probability that there are k single units present in the system at the end of j days of taking the tablet.

$$q_{2j}(2k)=\frac{p_{2j}(2k)}{\sum_{k=0}^{j}p_{2j}(2k)}, \text{ if } 2j\leq N;$$

$$q_{2j}(2k)=\frac{p_{2j}(2k)}{\sum_{k=0}^{N-j}p_{2j}(2k)}, \text{ if } N<2j\leq 2N,$$

conditional probability for even state $(2j,2k)$

$$q_{2j+1}(2k+1)=\frac{p_{2j+1}(2k+1)}{\sum_{k=0}^{j}p_{2j+1}(2k+1)}, \text{ if } 2j+1\leq N$$

$$q_{2j+1}(2k+1)=\frac{p_{2j+1}(2k+1)}{\sum_{k=0}^{N-j-1}p_{2j+1}(2k+1)}, \text{ if } N<2j+1\leq 2N-1,$$

conditional probability for odd state $(2j+1,2k+1)$.

$$\rho_{2j}(2k)=q_{2j}(2k)(1-q_{2j}(2k))^{-1},$$

odds for even state $(2j,2k)$

$$\rho_{2j+1}(2k+1)=q_{2j+1}(2k+1)(1-q_{2j+1}(2k+1))^{-1},$$

odds for odd state $(2j+1,2k+1)$

$$r_{2j}(2k)=\frac{q_{2j}(2k)(1-q_{2j}(0))}{q_{2j}(0)(1-q_{2j}(2k))},$$

conditional probability odds ratio for even state $(2j,2k)$ in relation to $(2j,0)$.

$$r_{2j+1}(2k+1) = \frac{q_{2j+1}(2k+1)(1-q_{2j+1}(1))}{q_{2j+1}(1)(1-q_{2j+1}(2k+1))}$$

, conditional probability odds ratio for odd state $(2j,2k)$ in relation to $(2j+1,1)$.

$$(h_0)_{2j}(2k) = 2(sin^{-1}\sqrt{q_{2j}(2k)} - sin^{-1}\sqrt{q_{2j}(0)}),$$

Cohen's h_0 index for even state $(2j,2k)$.

$$(h_0)_{2j+1}(2k+1) = 2(sin^{-1}\sqrt{q_{2j+1}(2k+1)} - sin^{-1}\sqrt{q_{2j+1}(1)}),$$

Cohen's h_0 index for odd state $(2j+1,2k+1)$.

$$a_{2j}(2(k-1)) = \frac{2k-1}{N-(j-k)},$$

probability that an even number $2(k-1)$ of single units has resulted from a choice of a single unit on day $2j$ with $\frac{1}{2} < k < N+1-j$. Thus, $b_{2j}(2k)$ is the probability that an even number $2\,k$ of single units resulted from a choice of a double unit on day $2j$.

$$b_{2j}(2k) = 1 - a_{2j}(2(k-1)).$$

Also,

$$c_{2j+1}(2(k-1)) = \frac{2k}{N-(j-k)},$$

is the probability that an odd number $2(k-1)$ of single units has resulted from a choice of a single unit on day $2j+1$ with $0 < k < N-j$, and, $d_{2j+1}(2k+1)$ is the probability that an odd number $2k+1$ of single units resulted from a choice of a double unit on day $2j+1$.

$$d_{2j+1}(2k+1) = 1 - c_{2j+1}(2(k-1))$$

which is the probability that an odd number 2k+1 of single units resulted from a choice of a double unit on day 2j+1.

For clarity, in probability theory terminology, the odds for an event is just the probability or chance of the occurrence of an event while odds ratio for an event is the multiplicative relative measure of the probability of occurrence of the event in relation to a baseline event of interest. Similarly, the Cohen's h index for an event is a lateral statistical measure that quantifies the relative distance of the event from a baseline event of interest. These multiplicative and lateral indices adopted here have given some measure of the risk factors of any arbitrary patient behaviour from an organized patient with minimum number of single units on a daily basis and who is least likely to incur under-dose or overdose to the extent of being classified as non-compliant.

2. METHODOLOGY

2.1. **Mathematical Model.** We consider a medication jar or purse containing a number N of same drug of possibly different measures of active components but which can be divided into multiple appropriate measures for daily administration. Suppose the case of the N tablets being of double (two) measuring units at the commencement of management. Suppose the desired appropriate measure on a daily basis is single (one) measuring unit such that a chosen tablet is either an already-dosed single- unit measure or a double- unit measure, which then is physically broken into two, for administration. While at the end of an even $2j$ days of the $2N$ days for taking the lot of tablets, the jar may contain $N-j-k$ of double units and $2k$; $k=0,1,\dots j$ if $2j \leq N$ or $k=0,1,\dots N-j$ if $N<2j\leq 2N$, of single units, the jar may also, at the end of an odd $2j+1$ days of the $2N$ days of taking the lot of tablets, contain $N|j-k-1$ of double units and $2k+1$ single units; $k=0,1,\dots j$ if $2j+1<N$ or $k=0,1,\dots N-j-1$ if $N<2j+1\leq 2N-1$. Of interest is the distribution of number of single units at the end of each day k, $k=0,1,2,\dots,2N$. A basic fact is that any resulting recurrence relations will be expected to be markedly different from the basic one found with versions of the gambler's ruin (Stewart, 2009) problem since it is not a question of winning or losing with some constant or varying probability at each play by the gambler. The model can actually be equivalent to an unusual two-urns, (A,B), model which may, at a point, contain a number x units in urn A and a number y units in urn B and such that a unit has to be chosen at random and without replacement in one of the two urns but with the provision that any unit chosen in urn A may only be replaced into urn B but that any unit chosen from urn B is automatically out of the system. Thus the system obviously moves with probability $\frac{x}{x+y}$ from state (x,y), for x doubles and y singles, into state $(x-1,y+1)$ if urn A unit is selected or with probability $\frac{y}{x+y}$ into state $(x,y-1)$ when an urn B unit is selected. Of course nothing is chosen from an empty urn and this has some end point implication and consideration. This interpretation is clearly different from the previous urn models as discussed in Obilade(2004a, 2004b). It also structurally different from the Ehrenfest birth and death chain model (see Hoel, Port and Stone, 1972), for exchange of heat or gas molecules, in that exchange is limited to only one of the boxes Yet another view of the model is as a death process of a gated queueing system of two tandem queues with uniform service time in both service points such that the second queue is essentially a constant arrival and constant departure system except with the provisions that there is no arrival into system in between rounds after some accumulation for next round and, most unusually, service restarts at both queues after one unit has moved from its position. This also makes this model different from the repairman models in Agboola and Obilade(2018, 2022).

2.2. **Definition of States.** Given a number N of double units in the jar at initiation or commencement of treatment, the state of the system can be given by any two of three components namely the number of days from initiation of treatment, the number of double units in the drug jar on the particular day and the number of single units in the jar on the particular day. We will refer to a state with even number of single units as even state and refer a state with odd number of single states as odd state. We choose to use for day $2j$ the state $s_{2j} = (2j, 2k)$ and for day $2j+1$ the state $s_{2j+1} = (2j+1, 2k+1)$ to denote the number of days (either $2j$ or $2j+1$) since commencement as well as the number of single units (either $2k$ or $2k+1$) on the day. These are equivalent to using $s'_{2j} = (N-j-k, 2k)$ and $s'_{2j+1} = (N-j-k-1, 2k+1)$ for number of double units as well as number of single units respectively for even day $2k$ and odd day $2k+1$. For N even, the number of even states can be calculated as $(N+1)^2/4$ and the number of odd states as $N(N+2)/4$. For N odd, the number of even states is $(N+1)(N+3)/4$ and the number of odd states is $(N+1)^2/4$. Thus, for example, if $N = 10$, the number of even states is 36 and the number of odd states is 30. On other hand if $N = 5$, the number of even states is 12 and the number of odd states is 9. The numbers follow by noting that, for even states, $2j = 0, 2 \ldots 2N$ and $k = 0, 1, \ldots j$ if $2j \leq N$ (or $k = 0, 1, \ldots N-j$ if $N < 2j \leq 2N$) and, for odd states, $2j+1 = 1, 3, 2N-1$ and $k = 0, 1, \ldots j$ if $2j+1 \leq N$ (or $k = 0, 1, \ldots N-j-1$ if $N < 2j+1 \leq 2N-1$). The values of k in relation to any day j are guided by the number of some admissible partition of the number $2N-j$ into twos and ones.

2.3. **Transition Equations between States.** A schematic representation of the transition of the system is illustrated for the even or odd days in Fig. 1.

The number of the number of single units in the system at the end of day $j+1$ can be obtained from the number of double units and number of single units at the end of day j as: $I_{j+1} = I_j + 1$ if $2N - j > I_j > 0$ and double unit is selected ; $I_{j+1} = I_j + 1$ if $2N - j > I_j = 0$ and double unit

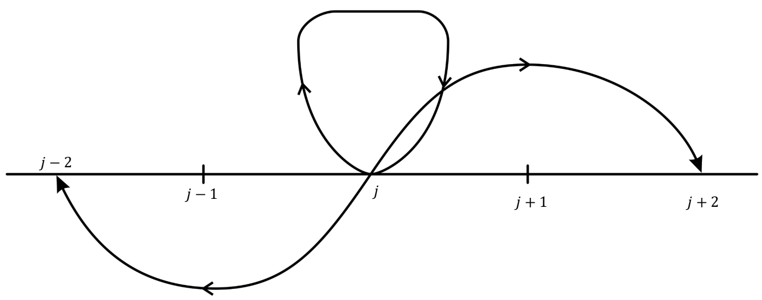

FIGURE 1. Transition Diagram of the System

is selected; $I_{j+1} = I_j - 1$ if $2N - j > I_j > 0$ and single unit is selected; $I_{j+1} = I_j - 1$ if $2N - j = I_j > 0$ and single unit is selected. It is zero otherwise.

Examining the system at the end of each day $2j$ or $2j+1$, the system equations for the probability transitions from one day to the next are given as follows::

$$p_{-1}(k) = p_{2N+1}(k) = p_{2j}(-1) = p_{2j+1}(-1) = 0 \tag{1}$$

$$q_0(0) = q_1(1) = q_{2N-1}(1) = q_{2N}(0) = 1 \tag{2}$$

for day $2j+1$, that is, $1, 3, 5, \cdots, 2N-1$

$$p_{2j+1}(2k+1) = p_{2j}(2k)d_{2j}(2k) + p_{2j}(2k+2)(1 - d_{2j}(2k+2)) \tag{3}$$

with $2k+1 = 1, 3, 5, \cdots 2j+1 \leq N$ and $2k+1 = 1, 3, \cdots, 2j+1$ if $2j \leq N$ and $2k+1 = 1, 3, \cdots, 2N-(2j+1)$ if $2j > N$

$$p_{2j}(2k) = p_{2j-1}(2k-1)b_{2j-1}(2k-1) + p_{2j-1}(2k+2)(1 - b_{2j-1}(2k+2)) \tag{4}$$

with $2k = 0, 2, 4, \cdots 2j \leq N$ and $2k = 0, 2, \cdots, 2j$ if $2j \leq N$ and $2k+1 = 0, 2, \cdots, 2N-2j$ if $2j > N$ We remark that, apart from the more complex notations, Equations (3) and (4) are comparable to system equations connecting probability $\Pr\{i, j, m+c-i, N+m-c-j\}$ to probabilities $\Pr\{i-1, j, m+c-i+1, N+m-c-j\}$ and $\Pr\{i, j-1, m+$

$c-i, N+m-c-j+1\}$ in Obilade(2004a) for underlying distribution of balls of a particular colour for two related negative hypergeometric distributions. Substituting Equations (3) into Equations (4) and vice-versa, we obtain, $p_{2j}(2k) = p_{2j-2}(2k-2)(d_{2j-2}(2k-2).b_{2j-1}(2k-1)) + p_{2j-2}(2k)((1-d_{2j-2}(2k)).b_{2j-1}(2k-1) + d_{2j-2}(2k).(1-b_{2j-1}(2k+1))) + p_{2j-2}(2k+2)(1-d_{2j-2}(2k+2))(1-b_{2j-1}(2k+1))$ and, $p_{2j+1}(2k+1) = p_{2j-1}(2k-1)(d_{2j}(2k).b_{2j-1}(2k-1)) + p_{2j-1}(2k+1)((1-d_{2j}(2k+2)).b_{2j-1}(2k+1) + d_{2j}(2k).(1-b_{2j-1}(2k+1))) + p_{2j-1}(2k+3)(1-d_{2j-2}(2k+2))(1-b_{2j-1}(2k+3))$ The boundary conditions governing Equations (2.3) include, for the early and latter days:

$$\begin{cases} p_2(0) = \frac{1}{N} p_0(0), \\ p_2(2) = \frac{N-1}{N} p_0(0), \\ p_{2N-2}(0) = \frac{1}{2} p_{2N-4}(0) + \frac{1}{3} p_{2N-4}(2), \\ p_{2N-2}(2) = \frac{1}{2} p_{2N-4}(0) + \frac{2}{3} p_{2N-4}(2) + p_{2N-4}(4), \\ p_{2N}(0) = p_{2N-2}(0) + p_{2N-2}(2). \end{cases} \tag{5}$$

Similarly, the boundary conditions governing Equations (2.3) include, for the early and latter days:

$$\begin{cases} p_3(1) = \frac{3N-2}{N^2} p_1(1), \\ p_3(3) = \frac{(N-1)(N-2)}{N^2} p_1(1), \\ p_{2N-3}(1) = \frac{7}{9} p_{2N-5}(1) + \frac{1}{2} p_{2N-5}(3), \\ p_{2N-3}(3) = \frac{2}{9} p_{2N-5}(1) + \frac{1}{2} p_{2N-5}(3) + p_{2N-5}(5), \\ p_{2N-1}(1) = p_{2N-3}(1) + p_{2N-3}(3) \end{cases} \tag{6}$$

Equations (2.3) together with the boundary conditions give the governing equations for a two-step Markov chain involving only the even days. Equations (2.3) together with the boundary conditions give the corresponding governing equations for a two-step Markov chain involving only the odd days. It should be stated that governing equations indicate that the joint probabilities $p_{2j}(2k)$, for the even states, can all be expressed as fractions of the joint probability $p_0(0)$ for the first even state. In the same way, the governing equations indicate that the joint probabilities $p_{2j+1}(2k+1)$, for the odd states, can all be expressed as fractions of the joint probability $p_1(1)$ for the first odd state. This is not unexpected since states $s_0 = (0,0)$ and $s_1 = (1,1)$ are the nodes on top of the trees for the two-step Markov processes and similar to the case with all such hierarchies like in the gambler's ruin problem (see Stewart 2009). For illustration, Table 1 gives the admissible states and transitions for the 36 even states for case $N = 10$ number of days while Table 2 gives the corresponding list for the 30 odd states. Needless to add the

TABLE 1. Even State and Possible Two-Step States of Departure by even days $2j$; $\{j = 0, 1, \cdots 10\}$

Day $2j$	Origin state	Departure states $s_{2j} = (2j, 2k)$		
0	(0,0)	(2,0)	(2,2)	-
1	(2,2)	(4,4)	(4,2)	(4, 0)
2	(2,0)	(4,2)	(4,0)	-
4	(4,4)	(6,6)	(6,4)	(6,2)
4	(4,2)	(6,4)	(6,2)	(6,0)
4	(4,0)	(6,2)	(6,0)	-
6	(6,6)	(8,8)	(8,6)	(8,4)
6	(6,4)	(8,6)	(8,4)	(8,2)
6	(6,2)	(8,4)	(8,2)	(8,0)
6	(6,0)	(8,2)	(8,0)	-
8	(8,8)	(10,10)	(10,8)	(10,6)
8	(8,8)	(10,8)	(10,6)	(10,4)
8	(8,8)	(10,6)	(10,4)	(10,2)
8	(8,8)	(10,4)	(10,2)	(10,0)
8	(8,8)	(10,2)	(10,0)	-
10	(10,10)			(12,8)
10	(10,8)		(12,8)	(12,6)
10	(10,6)	(12,6)	(12,4)	(12,2)
10	(10,4)	(12,4)	(12,2)	(12,0)
10	(10,10)	(12,2)	(12,0)	-
12	(12,8)			(14,6)
12	(12,6)		(14,6)	(14,4)
12	(12,4)	(14,6)	(14,4)	(14,2)
12	(12,2)	(14,4)	(14,2)	(14,0)
12	(12,0)	(14,2)	(14,0)	-
14	(14,6)			(16,4)
14	(14,4)		(16,4)	(16,2)
14	(14,2)	(16,4)	(16,2)	(16,0)
14	(14,0)	(16,2)	(16,0)	-
16	(16,4)			(18,2)
16	(16,2)		(18,2)	(18,0)
16	(16,2)	(18,2)	(18,0)	-
18	(18,2)			(20,0)
18	(18,0)		(20,0)	-
20	(20,0)		(20,0)	-

TABLE 2. Odd State and Possible Two-Step States of Departure by odd days $2j+1$, $\{j=0,1,\cdots 9\}$

Day $2j+1$	Origin state	Departure states $s_{2j+1}=(2j+1,2k+1)$		
1	(1,1)	(3,3)	(3,1)	-
3	(3,3)	(5,5)	(5,3)	(5, 1)
3	(3,1)	(5,3)	(5,1)	-
5	(5,5)	(7,7)	(7,5)	(7,3)
5	(5,3)	(7,5)	(7,3)	(7,1)
5	(5,1)	(7,3)	(7,1)	-
7	(7,7)	(9,9)	(9,7)	(9,5)
7	(7,5)	(9,7)	(9,5)	(9,3)
7	(7,3)	(9,5)	(9,3)	(9,1)
7	(7,1)	(9,3)	(9,1)	-
9	(9,9)		(11,9)	(11,7)
9	(9,7)	(11,9)	(11,7)	(11,5)
9	(9,5)	(11,7)	(11,5)	(11,3)
9	(9,3)	(11,5)	(11,3)	(11,1)
9	(9,1)	(11,3)	(11,1)	-
11	(11,9)			(13,7)
11	(11,7)		(13,7)	(13,5)
11	(11,5)	(13,7)	(13,5)	(13,3)
11	(11,3)	(13,5)	(13,3)	(13,1)
11	(11,1)	(13,3)	(13,1)	-
13	(13,7)			(15,5)
13	(13,5)		(15,5)	(15,3)
13	(13,3)	(15,5)	(15,3)	(15,1)
13	(13,1)	(15,3)	(15,1)	-
15	(15,5)			(17,3)
15	(15,3)		(17,3)	(17,1)
15	(15,1)	(17,3)	(17,1)	-
17	(17,3)			(19,1)
17	(17,1)		(19,1)	
17	(19,1)			-

two two-step Markov chains are connected by the necessary condition $p_0(0)=p_1(1)$.

3. NUMERICAL ILLUSTRATION

For the case of a jar that contains a number of drugs N=10 which have to be broken into two to be used for 20 days, we present in Table 3 from a Maple 17 run, the results for the joint probabilities $p_{2j}(2k)$, conditional probabilities $q_{2j}(2k)$ and conditional probability odds $\rho_{2j}(2k)$ for $2k$ single drugs in the jar on each even day $2j$, (j=0,...N). Also provided are some system odds ratio r and index h_0, respectively, for some patient's conditional probability odds ratio as well as the Cohen's in relation to if the patient is so meticulous, organized and rational in the breaking of tablets that there are no singles $(k = 0)$ at the end of each even day as baseline. The baseline represents the best daily strategy for balancing overdose or underdose resulting from unequal and inaccurate breaking of the tablets into two. The corresponding case for odd singles in relation to the unavoidable case of one single unit on odd days is presented on Table 4. These tables can be combined to give a complete picture for the numerical illustration.The combined data give a odds ratio ranging from .025043 to a maximum 81.0000; and for the Cohen h ,the range is -0.536471 to 1.854600 .The combined data also reveal a non-normal population with mean 11.72505421 with higher standard error 18.787465551 and right skewness value of 2.380 for odd rates. The corresponding values for Cohen's h are .4265950, 0.542020370 and 0.434 for, respectively, mean, higher standard error and right skeweness value. This is aptly captured in Figure 2 and Figure 3 for the $P-P$ plots for odd rates and Cohen's h respectively. Needless to mention, the performance measures are highly correlated, giving a Pearson value of 0.947 for example, for the odd days. We note that the odds ratio for an event is the multiplicative relative measure of the probability of occurrence of the event in relation to a baseline event of interest and that the Cohen's h index for an event is a lateral statistical measure that quantifies the relative distance of the event from a baseline event of interest. It follows that the multiplicative and lateral indices adopted here have given some clear measure of the risk factors of any arbitrary patient behaviour in relation to an organized patient with minimum number of single units on a daily basis and who is least likely to incur an under-dose or overdose to the extent of being classified as non-compliant. Furthermore the inference from the P-P plot that fails to be fitted to a straight line, as expected for the very symmetric normal distribution, is that the large multiplicative and lateral indices of distance from a compliant patient cannot be merely explained by random chance. The wide non-normal range for the measures suggests a wide difference in individual behaiour and compliance to treatment which can probably be a reflection of the organized or haphazard nature of the patient and this can be useful in some classification of patients. This will complement such studies as Bagbe, Obilade and Olatayo (2022).

4. CONCLUSION

This study considered a drug bottle or jar containing a mixture of double and single measuring unit from which a patient may randomly pick a tablet on a daily basis and may consume, as a single unit, or break into two to consume one half, if double, and drop the other back into the jar for another day. The jar initially containing N double-units takes $2N$ days to be exhausted as drug is taken at the prescribed dosage of single unit per day. We choose to use for day $2j$ the even state $=(2j,2k)$ and for day $2j+1$ the odd state $=(2j+1,2k+1)$ to denote the number of days (either $2j$ or $2j+1$) since commencement as well as the number of single units (either $2k$ or $2k+1$) on the day. It is determined that for N even, the number of even states can be calculated as and the number of odd states as $N(N+2)/4$. For N odd, the number of even states is and the number of odd states is $\frac{(N+1)(N+3)}{4}$ and the number of odd state is $(N+1)^2/4$. It is noted that the model is equivalent to a two-urn model with some unusual provision for exchange of balls between and from the urns. With the help of system equations and governing recurrence relations for two related two-step Markov Chains, the required probabilities are obtained. Also found useful are odds ratio and Cohens h index for some respectively multiplicative and additive lateral comparisons of the conditional probabilities for single units on the different days using the case no single unit for even days (or odd case one single unit for odd days) as baseline. The baseline represents the best daily strategy for balancing overdose or under-dose resulting from unequal and inaccurate breaking of the tablets into two. Numerical values for odd ratios and Cohen's h index are provided for different days in the case of N=10 double units. Also provided are P-P plot for the combined indices. It is clear that the multiplicative and lateral indices adopted here have given some clear measure of the risk factors of any arbitrary patient behaviour in relation to an organized patient with minimum number of single units on a daily basis and who is least likely to incur an under-dose or overdose to the extent of being classified as non-compliant. Similarly the P-P plots showed marked distinction from fitted straight lines, as otherwise expected for the very symmetric normal distribution. We affirm that the large multiplicative and lateral indices of distance from a compliant patient cannot be merely explained by random chance. We concluded that our measures provide some lateral

[b]0.5

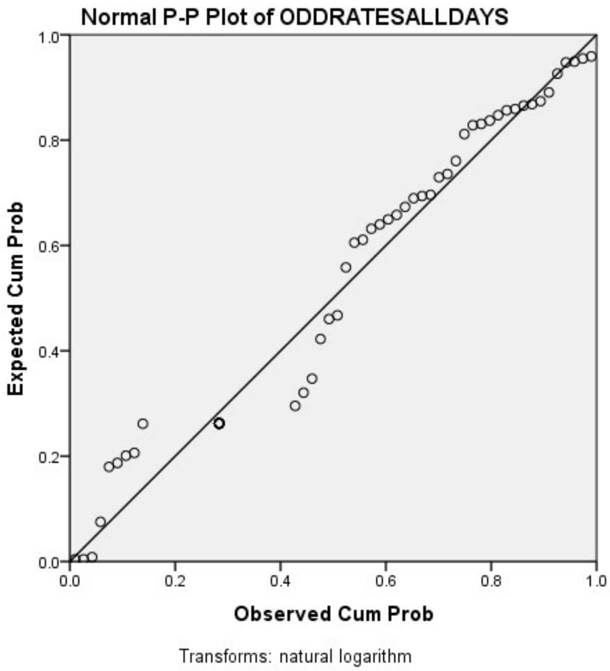

FIGURE 2. Normal P-P plot odd rates all day

[b]0.5

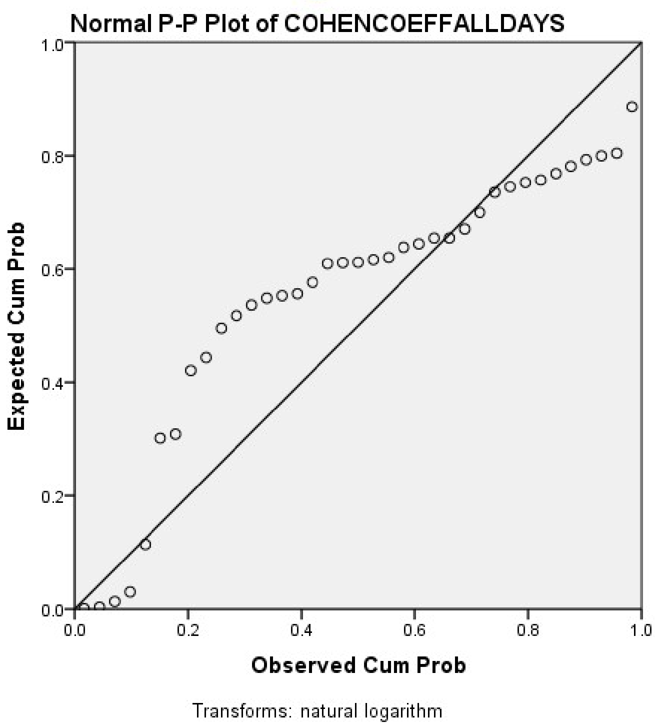

FIGURE 3. P-P plot for Cohen's h_0.

FIGURE 4. Normal P-P Plots for the Numerical Illustration

TABLE 3. ODDS RATIO AND COHENS INDEX FOR EVEN DAYS $2j$, $\{j = 0, 1, \cdots 10$

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Even days $2j$	N$\underline{o}$ of Singles $2k$	Joint Probability $p_{2j}(2k)$	Conditional Probability $q_{2j}(2k)$	Odds $\rho_{2j}(2k)$	Odd Ratio $r_{2j}(2k)$	Cohen's h $(h_0)_{2j}(2k)$
0	0	0.90909	1.00000	-	-	0.00000
2	2	0.0818182	0.90000	9.000000	81.0000	1.85460
2	0	0.0909091	0.10000	0.111111	1.00000	0.00000
4	4	0.0458182	0.504000	1.01613	31.6453	1.22418
4	2	0.0422626	0.464889	0.868771	27.0560	1.14589
4	0	0.00282828	0.031111	-	0.0321101	0.00000
6	6	0.0137455	0.151200	0.178134	10.4235	0.538774
6	4	0.0478776	0.526653	1.11262	65.1049	1.36415
6	2	0.0277585	0.305344	0.439562	25.7210	0.91090
6	0	0.00152750	0.0168025	0.0170896	1.00000	0.00000
8	8	0.00164945	0.0181440	0.0184793	1.37037	0.039010
8	6	0.0193358	0.212694	0.270154	20.0338	0.727452
8	4	0.0456700	0.502370	1.00953	74.8633	1.34433
8	2	0.0230442	0.253487	0.339561	25.1808	0.824018
8	0	0.00120959	0.0133055	0.0134849	1.00000	0.000000
10	10	0.000032989	0.00036288	0.000363012	0.025043	-0.201541
10	8	0.00202242	0.0222466	0.0227528	1.56965	0.059782
10	6	0.0194364	0.213800	0.271941	18.7604	0.721724
10	4	0.0447828	0.492611	0.970876	66.9778	1.31637
10	2	0.0233355	0.256691	0.345335	23.8236	0.82938
10	0	0.00129894	0.0142884	0.144955	1.00000	0.000000
12	8	0.00108980	0.0119878	0.0121333	0.595431	-0.064166
12	6	0.0154733	0.179206	0.205118	10.060	0.566942
12	4	0.00446950	0.491645	0.967129	47.4610	1.27050
12	2	0.0278356	0.306191	0.4411319	21.6574	0.89166
12	0	0.00181549	0.0199704	0.0203713	1.00000	0.000000
14	6	0.00799379	0.0879317	0.0964091	2.261082	0.222424
14	4	0.0420668	0.462735	0.861278	23.3240	1.11651
14	2	0.0376111	0.413722	0.705675	19.1101	1.01768
14	0	0.00323743	0.0356117	0.0369267	1.00000	0.00000
16	4	0.0295219	0.324741	0.480913	5.46920	0.636126
16	2	0.0540395	0.594435	1.46570	16.6687	1.18427
16	0	0.00734766	0.0808242	0.0879312	1.0000	0.0000
18	2	0.0692221	0.761443	3.19187	12.0419	1.17012
18	0	0.0216870	0.209526	0.265063	1.000	0.0000
20	0	0.090909	1.0000	-	-	0.0000

TABLE 4. ODDS RATIO AND COHENS INDEX FOR ODD DAYS $2j+1, \{j = 0, 1, \cdots 9\}$

Odd days $2j+1$	No of Singles $2k+1$	Joint Prob. $p_{2j+1}(2k+1)$	Conditional Prob. $q_{2j+1}(2k+1)$	Odds $\rho_{2j+1}(2k+1)$	Odd Ratio $r_{2j+1}(2k+1)$	Cohen's h $(h_0)_{2j+1}(2k+1)$
1	1	0.90909	1.00000	-	-	0.00000
3	3	0.0654545	0.7200	2.57143	6.61224	0.911182
3	1	0.0254545	0.2800	0.38889	1.00000	0.00000
5	5	0.0274909	0.302400	0.433486	2.79138	0.413738
5	3	0.0511982	0.563180	1.28927	8.30213	0.94672
5	1	0.0122200	0.134420	0.155294	1.0000	0.00000
7	7	0.00549818	0.0604800	0.0643733	0.626784	0.123312
7	5	0.0348459	0.383305	0.621548	6.05183	0.71497
7	3	0.0420978	0.463076	0.862462	8.39753	0.876624
7	1	0.00846713	0.0931385	0.102704	1.00000	0.00000
9	9	0.000329891	0.00362880	0.00364202	0.0388403	-0.47375
9	7	0.00776483	0.0854132	0.0933899	0.995956	-0.001134
9	5	0.0357255	0.392981	0.647395	6.90414	0.760788
9	3	0.0392952	0.432247	0.761329	8.11919	0.840568
9	1	0.00779366	0.0857302	0.0937691	1.00000	0.000000
11	9	0.00257702	0.00283473	0.00284279	0.0256272	-0.536471
11	7	0.00665681	0.0732249	0.0790104	0.712265	-0.094972
11	5	0.0337699	0.371469	0.591012	5.32788	0.667814
11	3	0.0411472	0.452619	0.826882	7.45420	0.832884
11	1	0.00907744	0.0998519	0.110928	1.0000	0.0000
13	7	0.00330027	0.0363030	0.0376705	0.226783	-0.390608
13	5	0.0281611	0.309772	0.448797	2.70183	0.406492
13	3	0.046498	0.511478	1.04699	6.30305	0.819742
13	1	0.0129497	0.142447	0.166108	1.0000	0.0000
15	5	0.016407	0.180479	0.220224	0.68802	-0.152176
15	3	0.052459	0.577049	1.36434	4.26244	0.69579
15	1	0.022043	0.242473	0.320084	1.0000	0.0000
17	3	0.0475351	0.522886	1.09593	1.20107	0.09157
17	1	0.0433740	0.0477114	0.912463	1.0000	0.0000
19	1	0.090909	1.0000	-	-	0.00000

comparisons on the different days and that the results provide a natural classification procedure for assessing the behaviour and compliance of patients to treatment. The significance of the result is that we have given a model of a system that can produce two separate but connected

Markov chains. This is quite different from the direct use of Markov chains for weather prediction as in such studies as Vijayalakshmi et. al (2024). We have also demonstrated the applicability of our model to the problem of non-compliance of patients to treatment. Non compliance can be defined as the failure to adhere to medical advices or instructions (Bagbe and Akanbi,(2024)). Such practices may subsequently lead to disease progression, disabilities, premature death and higher healthcare cost as expanciated in Naghavi et, al. (2019).

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