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AUTOTOPIC CHARACTERISATION OF RIGHT CHEBAN LOOP

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ABSTRACT. Some fundamental properties of right Cheban loop are established in the present study in relation to the autotopism of right Cheban loop. It is found that the autotopism is a right pseudo-automorphism with companion $c = u\dot{v}u^{-1}$. The autotopism and pseudo-automorphism are characterized in terms of the right and left translation. Thus, necessary and sufficient conditions are established for the Moufang part of right Cheban loops to be an autotopism and pseudo-automorphism with companion $c = (x^3)^{-1} \cdot x^{-1}x^3$. The study also describes the behaviour of the pseudo-automorphism of the right Cheban loops on the nuclei and Moufang part of the right Cheban loops.

1. INTRODUCTION

Let (Q) be a non-empty set. Define a binary operation $(\cdot)$ on Q. If $x \cdot y \in Q$ for all $x, y \in Q$, then the pair $(Q, \cdot)$ is called a groupoid. If the system of equations:

$$a \cdot x = b \text{ and } y \cdot a = b$$

have unique solutions in Q for x and y respectively, then $(Q, \cdot)$ is called a quasigroup. A quasigroup is therefore an algebra having a binary multiplication $x \cdot y$ usually written as xy which satisfies the conditions that for any a, b in the quasigroup the equations:

$$a \cdot x = b \text{ and } y \cdot a = b$$

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have unique solutions for x *and* y. If there exist a unique element $e \in Q$ called the identity element such that for all $x \in Q$

$$x \cdot e = e \cdot x = x$$

$(Q, \cdot)$ is called a loop. Let x be a fixed element in a groupoid $(Q, \cdot)$. The left and right translation maps of Q, L_x and R_x are respectively defined by

$$yL_x = x \cdot y \quad and \quad yR_x = y \cdot x.$$

Some useful information about groupoids, quasigroups and loops can be found in [1, 7, 10] A loop satisfying the identity

$$(z \cdot yx)x = zx \cdot xy \tag{1}$$

is called a *Right Cheban loop* (RChL). *Left Cheban loops* on the other hand are loops satisfying the mirror identity

$$x(xy \cdot z) = yx \cdot xz \tag{2}$$

Loops that are both right and left Cheban are called *Cheban loops*. Cheban loops can also be characterized as those loops that satisfy the identity

$$x(xy \cdot z) = (y \cdot zx)x \tag{3}$$

Right, left and Cheban loops are loops of the generalized Bol-Moufang type. Both Left and Cheban loops were introduced by Cheban in [2]. He showed that Cheban loops are generalized Moufang loops. He also gave an example of a left Cheban loop that was not Moufang. Phillips and Shcherbacov in [9] also carried out a study on the structural properties of left and Cheban loops. They established that left Cheban loops are left conjugacy closed (LCC). Furthermore, they proved that Cheban loops are weak inverse property, power associative and conjugacy closed loops. Construction of right Cheban loops of small order was given in [3] and in [4], holomorph of right Cheban loops was study.
In the present study, we investigate the autotopism of right Cheban loops. It is found that the autotopism is a right pseudo-automorphism with companion $c = u\dot{v}u^{-1}$. The autotopism and pseudo-automorphism are characterised in terms of the right and left translation. Thus, the Moufang part of the right Cheban loops, the autotopism and pseudo-automorphism with companion $c = (x^3)^{-1} \cdot x^{-1}x^3$ are found to be equivalent. The study also describes the behaviour of pseudo-automorphism of the right Cheban loops on nuclei and Moufang part of the right Cheban loops.

2. PRELIMINARIES

In this section, we give definitions of terminologies used throughout this study and some previous results used in the body of this work.

Definition 2.1. *A loop $(Q,\cdot)$ is called a left inverse property loop if it satisfies the left inverse property (LIP) given by: $x^{\lambda}(xy) = y$.*

Definition 2.2. *A loop $(Q,\cdot)$ is called a right inverse property loop if it satisfies the right inverse property (RIP) given by: $(yx)x^{\rho} = y$.*

A loop is called an IP loop if it is both LIP-loop and RIP-loop.

Definition 2.3. *A loop $(G,\cdot)$ is called an automorphic inverse property loop if it satisfies the automorphic inverse identity given by $(xy)^{-1} = x^{-1}y^{-1}$.*

Definition 2.4. *A triple (U,V,W) of bijections from a set G onto a set H is called an isotopism of a groupoid $(G,\cdot)$ onto groupoid $(H,\circ)$ provided $xU \circ yV = (xy)W$ for all $x,y \in G$. $(H,\circ)$ is then called an isotope of $(G,\cdot)$, and groupoids $(G,\cdot)$ and $(H,\circ)$ are said to be isotopic to each other.*
An isotopism of $(G,\cdot)$ onto $(G,\cdot)$ is called an autotopism of $(G,\cdot)$.

Definition 2.5. *A bijection U on G is called a right pseudo-automorphism of a quasigroup $(G,\cdot)$ if there exists at least one element $c \in G$ such that*

$$xU \cdot (yU \cdot c) = (xy)U \cdot c \quad for \ all \quad x,y \in G.$$

The element c is then called a companion of U.

Definition 2.6. *The left nucleus of $(Q,\cdot)$ denoted by*

$$N_{\lambda}(Q,\cdot) = \{a \in Q : a \cdot xy = ax \cdot y \quad \forall \quad x,y \in Q\}$$

The right nucleus of $(Q,\cdot)$ denoted by

$$N_{\rho}(Q,\cdot) = \{a \in Q : xy \cdot a = x \cdot ya \quad \forall \quad x,y \in Q\}$$

The middle nucleus of $(Q,\cdot)$ denoted by

$$N_{\mu}(Q,\cdot) = \{a \in Q : xa \cdot y = x \cdot ay \quad \forall \quad x,y \in Q\}$$

The nucleus of $(Q,\cdot)$ denoted by

$$N(Q,\cdot) = N_{\lambda}(Q,\cdot)\bigcap N_{\rho}(Q,\cdot)\bigcap N_{\mu}(Q,\cdot)$$

The centrum of $(Q,\cdot)$ denoted by

$$C(Q,\cdot) = \{a \in G : ax = xa \quad \forall \quad x \in G\}$$

The center of $(Q,\cdot)$ denoted by

$$Z(Q,\cdot) = N(Q,\cdot)\bigcap C(Q,\cdot)$$

Definition 2.7. *A Moufang loop is a loop that satisfies the identity*

$$xy \cdot zx = (x \cdot yz)x \tag{4}$$

Definition 2.8. *Moufang part $M(Q)$ of a loop $(Q,\cdot)$ is defined by*

$$M(Q) = \{x \in Q | (x \cdot yx)z = x(y \cdot xz) \ \forall \ y,z \in Q\} \tag{5}$$

Theorem 2.1 ([5])**.** *Let $(Q,\cdot)$ be a RChL that satisfies automorphic inverse property (AIP) then $(R_x^{-1}, R_xL_x^{-1}, R_x^{-1})$ is an autotopism for all $x \in (Q,\cdot)$.*

Theorem 2.2 ([5])**.** *For a right Cheban loop $(Q,\cdot)$, $x^\lambda = x^\rho = x^{-1}$*

Corollary 2.1 ([5])**.** *If $(Q,\cdot)$ is a RChL, then $x^2 \cdot x = x \cdot x^2 = x^3 = xxx$.*

3. MAIN RESULTS

The heart of the matter is to state without proof the following:

Theorem 3.1. *Let (U,V,W) be an autotopism of RChL $(Q,\cdot)$ and $u^{-1} = 1U$ and $1V = v$ then $A = UR_u$ is a right pseudo-automorphism of $(Q,\cdot)$ with companion $c = u \cdot vu^{-1}$*

Proof. If (U,V,W) is an autotopism of RChL $(Q,\cdot)$ with identity I and $IU = u^{-1}$ and $IV = v$ then by Theorem 2.1, $(R_u, R_u^{-1}L_u, R_u)$ is an autotopism of $(Q,\cdot)$ and so is the product

$$(A,B,C) = (U,V,W)(R_u, R_u^{-1}L_u, R_u)$$

Then, $A = UR_u, B = VR_u^{-1}L_u, C = WR_u$ Applying (A,B,C) to the product of $a,b \in Q$,we have
$aA \cdot bB = (ab)C$ With $a = 1$, we obtain $1A = 1UR_u = u^{-1}R_u = 1$ and $bB = bC$ *or* $B = C = 1VR_u^{-1}L_u$ with $b = 1$ we get $aA \cdot 1B = aB$, where $1B = 1VR_u^{-1}L_u = u \cdot vu^{-1}$ then, $aA \cdot 1B = aA \cdot u \cdot vu^{-1} = aAR_u \cdot vu^{-1}$
Then, autotopism (A,B,C) now has the form $(UR_u, UR_uR_{u\cdot vu^{-1}}, UR_uR_{u\cdot vu^{-1}})$ which means that $A = UR_u$ is the right pseudo-automorphism with companion $c = u \cdot vu^{-1}$. □

Remark 3.1. *Theorem 3.1 shows that every autotopism of a RChL can be expressed as a product of an autotopism of the type appearing in the definition 2.4*

Theorem 3.2. *If $(Q,\cdot)$ is a RChL, the following statement are equivalent:*
(i) $x \in M(Q)$, Moufang part of Q,
(ii) $(L_{x^3}, R_x^{-1}, R_xL_x)$ is an autotopism of $(Q,\cdot)$

(iii) $L_{x^3}R_{x^3}^{-1}$ *is a pseudo-automorphism of* $(Q,\cdot)$ *with companion* $c = (x^3)^{-1}\cdot x^{-1}x^3$

Proof. Assume (i), then from (5)

$$(x\cdot yx)z = x(y\cdot xz)$$

For all $y,z \in Q$. Let $y = x^{-1}$ and we get $xz = x(x^{-1}\cdot xz)$ for all $z \in Q$
So, $x^{-1}\cdot xz = z$ for all $z \in Q$ and $L_xL_{x^{-1}} = 1$
Consequently, $L_x^{-1} = L_{x^{-1}}$ and Pre-multiplying both sides by L_x^{-1}, we obtain $L_x^{-1}L_{x^{-1}} = 1$
That is,

$$x\cdot x^{-1}z = z \tag{6}$$

For all $z \in Q$. Replacing z by $x^{-1}z$ in (5) and using (6)
We have,

$$(x\cdot yx)(x^{-1}z) = x[y(x\cdot x^{-1}z)] = x\cdot yz$$

for all $y,z \in Q$
So, $\alpha = (R_xL_x, L_x^{-1}, L_x)$ is an autotopism of $(Q\cdot)$. So $\beta = (R_x, R_x^{-1}L_x, R_x)$

$$\beta\alpha = (R_x, R_x^{-1}L_x, R_x)\alpha$$

is an autotopism of $(Q\cdot)$. Thus,

$$\begin{aligned}\beta\alpha &= (R_x, R_x^{-1}L_x, R_x)(R_xL_x, L_x^{-1}, L_x)\\ &= (R_xR_xL_x, R_x^{-1}, R_xL_x)\end{aligned} \tag{7}$$

If the autotopism in equation (7) acts on the product ab we have,
$aR_xR_xL_x\cdot bR_x^{-1}$ = $(ab)R_xL_x$ for all $a,b \in Q$
For $a = 1$ we have,

$$(x\cdot x^2)\cdot bR_x^{-1} = bR_xL_x \Longrightarrow (x^3)\cdot bR_x^{-1} = bR_xL_x \text{ (By Corollary 2.1)}$$

$$bR_x^{-1}L_{x^3} = bR_xL_x$$

$$R_x^{-1}L_{x^3} = R_xL_x$$

$$R_xR_x^{-1}L_{x^3} = R_xR_xL_x(\text{Pre-multiply by } R_x).$$

$$L_{x^3} = R_xR_xL_x \tag{8}$$

Putting (8) into (7) we have,

$$\beta\alpha = (L_{x^3}, R_x^{-1}, R_xL_x) = (R_xR_xL_x, R_x^{-1}, R_xL_x)$$

Now, assume (ii) then we show that (ii) $\Longrightarrow$ (i)
Let the autotopism in (ii) act on ab

$$\alpha = (R_x, R_x^{-1}L_x, R_x)$$

$$\alpha^{-1} = (R_x^{-1}, L_x^{-1}R_x, R_x^{-1})$$
$$\alpha^{-1}\beta\alpha = (R_x^{-1}, L_x^{-1}R_x, R_x^{-1})(L_{x^3}, R_x^{-1}, R_xL_x)$$
$$= (R_x^{-1}, L_x^{-1}R_x, R_x^{-1})(R_xR_xL_x, R_x^{-1}, R_xL_x) \quad by\ (8)$$
$$= (R_xL_x, L_x^{-1}, L_x) \in AUT(Q\cdot) \tag{9}$$

If the last autotopism takes on the product ab as an argument, we have:

$$aR_xL_x \cdot bL_x^{-1} = (ab)L_x \Longrightarrow (x \cdot ax)bL_x^{-1} = x \cdot ab$$
$$(x \cdot ax)b = x(a \cdot bL_x) \quad (\text{setting} \quad b = bL_x)$$
$$\Longrightarrow (x \cdot ax)b = x(a \cdot bx) \Longrightarrow x \in M(Q).$$

(ii) $\Longleftrightarrow$ (iii)
Suppose (ii) holds, that is,

$$(L_{x^3}, R_x^{-1}, R_xL_x) \in AUT(Q\cdot)$$

then

$$1L_{x^3} = x^3$$
$$1R_x^{-1} = 1/x$$

But $(1/x)x = 1 \Longrightarrow 1/x = x^\rho = x^\lambda = x^{-1}$ (By Theorem 2.2)
Therefore,

$$1R_x^{-1} = 1/x = x^\rho = x^\lambda = x^{-1}.$$

Hence, by applying Theorem (3.1), with $u^{-1} = x^3$ and $v = x^{-1}$ we have that
$(L_{x^3}, R_x^{-1}, R_xL_x) \in AUT(Q\cdot)$ if and only if $A = L_{x^3}R_{(x^3)^{-1}}$ is a pseudo-automorphism of $(Q\cdot)$ with companion $c = (x^3)^{-1} \cdot x^{-1}x^3$ $\square$

Theorem 3.3. *Let $(Q,\cdot)$ be a RChL, for each pseudo-automorphism P of $(Q,\cdot)$ $N_\lambda P = N_\lambda$, $N_\rho P = N_\rho$ and P is an automorphism on N_λ and N_ρ. Infact, $aP \cdot xP = (ax)P$ and $xP \cdot bP = (xb)P$ for all $a \in N_\lambda, b \in N_\rho$ and $x \in Q$.*

Proof. Let c be a companion for the pseudo-automorphism P. For $a \in N_\lambda$ *and* $x, y \in Q$,

$$aP \cdot [xP \cdot (yP \cdot c)] = aP \cdot [(xy)P \cdot c]$$
$$(a \cdot xy)P \cdot c = (ax \cdot y)P \cdot c = (ax)P \cdot yP \cdot c$$
$$aP \cdot [xP \cdot yP \cdot c] = (ax)P \cdot (yP \cdot c) \tag{10}$$

For all $a \in N_\lambda$ *and all* $x, y \in Q$.
Setting $y = (c^{-1})P^{-1}$ *in* (10) we have,

$$aP \cdot [xP \cdot (c^{-1})P^{-1}P \cdot c] = (ax)P \cdot (c^{-1})P^{-1}P \cdot c$$
$$aP \cdot xP = (ax)P \tag{11}$$

For all $a \in N_\lambda$ *and all* $x \in Q$, using (11) and (10) we get,

$$aP \cdot [xP \cdot (yP.c)] = (ax)P \cdot (yP \cdot c) \tag{12}$$

$$aP \cdot xP = (ax)P \tag{13}$$

using (13) in (12) we have,

$$aP \cdot [xP \cdot (yP \cdot)] = [(aP \cdot xP)(yP \cdot c)] \tag{14}$$

The last equation (14) implies that $aP \in N_\lambda$,but aP is an arbitrary element in $N_\lambda P$ therefore, $aP \in N_\lambda P \subseteq N_\lambda$. The fact that P^{-1} is also a pseudo-automorphism implies that $N_\lambda P^{-1}$ is also a pseudo-automorphism and $N_\lambda PP^{-1} \subseteq N_\lambda P^{-1}$ This implies that ,

$$N_\lambda \subseteq N_\lambda P^{-1} \tag{15}$$

Setting $P = P^{-1}$ in (12) we have,

$$aP^{-1} \cdot [xP^{-1} \cdot (yP^{-1} \cdot c)] = (aP^{-1} \cdot xP^{-1})(yP^{-1} \cdot c) \tag{16}$$

But ((14)) implies that $aP^{-1} \in N_\lambda$ which implies that

$$N_\lambda P^{-1} \subseteq N_\lambda \tag{17}$$

Combining (15) and (17) we have, $N_\lambda = N_\lambda P^{-1} \Longrightarrow N_\lambda P = N_\lambda$ which is the required result.
Similarly, we can proof that $N_\rho P = N_\rho$.
Since, $N_\rho = N_\mu$ so we have,

$$(xb \cdot y)P \cdot c = xP \cdot (by)P \cdot c$$

$$(xb)P \cdot (yP \cdot c) = xP \cdot [bP \cdot .(yP \cdot c)]. \tag{18}$$

Setting, $(c^{-1})P^{-1}$

$$(xb)P \cdot (c^{-1}P^{-1}P \cdot c) = xP \cdot [bP \cdot (c^{-1})P^{-1}P \cdot c]$$

$$(xb)P = xP \cdot bP. \tag{19}$$

Using (19) in (18) we have ,

$$[xP \cdot bP](yP \cdot c) = xP \cdot [bP \cdot (yP \cdot c)] \tag{20}$$

The last equation (20) implies that $bP \in N_\rho$ but bP is an arbitrary element in $N_\rho P$. Therefore, $bP \in N_\rho P \subseteq N_\rho$. The fact that P^{-1} is also a pseudo-automorphism follows that $N_\rho P^{-1}$ is also a pseudo-automorphism and $N_\rho\ PP^{-1} \subseteq N_\rho P^{-1}$. This implies that

$$N_\rho \subseteq N_\rho P^{-1} \tag{21}$$

Setting $P = P^{-1}$ in 18 we have,

$$(xb)P^{-1} \cdot (yP^{-1} \cdot c) = xP^{-1} \cdot [bP^{-1} \cdot .(yP^{-1} \cdot c)] \tag{22}$$

But (20) implies that $bP^{-1} \in N_\rho$ which implies that

$$N_\rho P^{-1} \subseteq N_\rho \tag{23}$$

Combining (21) and (23) we have, $N_\rho = N_\rho P^{-1} \Longrightarrow N_\rho P = N_\rho$ which is the required result. □

Remark 3.2. *Theorem 3.3 describes the behaviour of pseudo-automorphism on nuclei and Moufang part of a RChL.*

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