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PRODUCT OF SEMI-TRANSPOSITIONS IN FINITE SEMIGROUP OF INJECTIVE ORDER-PRESERVING TRANSFORMATIONS

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ABSTRACT. Let X_n be the finite totally ordered set $\{1,2,\cdots,n\}$, $\mathscr{IO}_n$ be the semigroup of all injective order-preserving transformation of X_n and $\mathscr{IO}_{n,r} = \{\alpha \in \mathscr{IO}_n : |im(\alpha)| \leq r\}$ for $(1 \leq r \leq n-1)$ be the ideals of injective order-preserving transformations on X_n. The semigroup $\mathscr{IO}_n$ on X_n is an inverse semigroup and so cannot be generated by its idempotents. In a search for generating set for $\mathscr{IO}_n$ in this article, we identify a class of quasi-idempotents (i.e elements α in $\mathscr{IO}_n$ satisfying $\alpha \neq \alpha^2 = \alpha^4$.) which we refer to as semi-transpositions and showed that the ideals $\mathscr{IO}_{n,r}$ are generated by semi-trnaspositions. The semi-transposition rank of $\mathscr{IO}_{n,r}$ (defined to be the minimum of such generating set) is obtained to be $2\binom{n}{r} - 2$.

1. INTRODUCTION

The study of various finite semigroups of transformations makes significant contribution to semigroup theory similar to the contributions made by the study of finite symmetric group theory. The idea of generating semigroups of transformations started with the work of Howie [1] on the full transformation semigroup on a set X. Since the emergence of this article, there follows many more, see for example [2, 3, 5, 6, 9, 13].

Let X_n be the finite totally ordered set $\{1,2,\cdots,n\}$ and let $\mathscr{IO}_n$ be the semigroup of all injective order-preserving transformations on X_n. It is quite known that the semigroup $\mathscr{IO}_n$ (being an inverse semigroup) is not generated by its set of idempotents(i.e elements α in $\mathscr{IO}_n$ such

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that $\alpha = \alpha^2$). The first attempt to generate this semigroup was done by Garba [16] where he consider nilpotents elements(i.e elements α in $\mathscr{I}\mathscr{O}_n$ satisfying $\alpha^m = 0 \; \forall \; m \geq 1$) and charactarized the subsemigroup of $\mathscr{I}\mathscr{O}_n$ that is nilpotent generated. An element $\alpha \in \mathscr{I}\mathscr{O}_n$ is called quasi-idempotent if $\alpha \neq \alpha^2 = \alpha^4$. Quasi-idempotents were first studied by Umar [20] to generate another class partial one-to-one transformation semigroup which was not idempotent generated, namely the semigroup of all partial one-to-one order decreasing transformations of X_n. At present, there are many recent articles on quasi-idempotents used to generate other classes of semigroups of transformations, see for example [10, 12–15, 19].

In this article we generate the ideals of $\mathscr{I}\mathscr{O}_n$ using special type of quasi-idempotents which we refer to as semi-transpositions. We also obtain the minimum number of such elements requared to generate $\mathscr{I}\mathscr{O}_{n,r}$.

2. PRELIMINARIES

Let $\mathscr{I}_n$ be the symmetric inverse semigroup consisting of all partial one-to-one transformations of $X_n = \{1, 2, \dots, n\}$. For $n \geq 3$, and $i, j \in X_n$ with $i \neq j$, let $A \subseteq X_n \setminus \{i, j\}$. Then we denote by $[i, j]_A$ the map $\alpha \in \mathscr{I}_n$ with $\mathrm{dom}(\alpha) = A \cup \{i\}$ and $\mathrm{im}(\alpha) = A \cup \{j\}$ such that

$$x\alpha = \begin{cases} j & \text{if} \quad x = i, \\ x & \text{otherwise.} \end{cases}$$

Such an $\alpha \in \mathscr{I}_n$ will be referred to as semi-trnaspositions. In the next lemma, we present a necessary and sufficient condition for a semi-transpositions to be an order-preserving transformation. Before stating the Lemma we introduce the following notations. For any two distinct points $i, j \in X_n$, we write

$$\overline{ij} = \begin{cases} \{i, i+1, \dots, j\} & \text{if} \quad i < j, \\ \{j, j+1, \dots, i\} & \text{if} \quad j < i. \end{cases}$$

Lemma 2.1: For $n \geq 3, \alpha = [i, j]_A$ is order-preserving if and only if $A \cap \overline{ij} = \emptyset$.

Proof: Let $\alpha = [i, j]_A$. Suppose that $A \cap \overline{ij} \neq \emptyset$ and let $x \in A \cap \overline{ij}$. Then, since $i, j \notin A$, either $i < x < j$ or $j < x < i$. And so, in the former, $i < x$ with $i\alpha = j > x = x\alpha$ and in the latter, we have $x < i$ with $x\alpha = x > j = i\alpha$. Thus, in both situations, α is not an order-preserving map.

Conversely, suppose $A \cap \overline{ij} = \emptyset$. Then for each $x \in A$ either $i < x \implies i\alpha = j < x = x\alpha$ or $x < i \implies x\alpha = x < j = i\alpha$. Thus, in both cases, α is an order-preserving map.

A general characterization of quasi-idempotents in the semigroup $\mathscr{IO}_{n,r}$. is documented in [12] Lemma 1, which we record in the next Lemma.

Lemma 2.2:[12]
For $n \geq 3$ and $2 \leq r \leq n-1$, $\alpha \in \mathscr{IO}_{n,r}$ is a quasi idempotent if and only if $\text{dom}\alpha \cap \text{im}\alpha = Fix(\alpha) = \{x \in X_n : x\alpha = x\}$.

It follows immediately from Lemmas 1 and 2 that, for each $n \geq 3$ and each $2 \leq r \leq n-2$ the set of all semi-transpositions in $\mathscr{IO}_{n,r}$ consists, precisely of all quasi-idempotents $\alpha \in \mathscr{IO}_{n,r})$ of shift1, i.e $|\{x \in X_n : x\alpha \neq x\}| = 1$. Under our notation, a semi-transposition $\alpha = [i,j]_A \in \mathscr{IO}_{n,r}$ is decreasing (in the sense that $x\alpha \leq x \ \forall x \in \text{dom}(\alpha)$) if $i > j$ and increasing (in the sense that $x\alpha \geq x \ \forall x \in \text{dom}(\alpha)$ if $i < j$.

Of interest to us are semi-transpositions $\alpha = [i,j]_A$ in $\mathscr{IO}_{n,r}$ satisfying $\overline{ij} = \{i,j\}$. Consequently, if α is increasing, then $j = i+1$ and if α is decreasing then $j = i-1$. The set of all such semi-transpositions will be denoted by QE_1. Thus

$$QE_1 = \left\{ [i,i+1]_A,\ [i+1,i]_A \ : \ 1 \leq i \leq n-1,\ A \subseteq X_n \setminus \{i,i+1\} \text{ and } |A| \leq r-1 \right\}. \tag{1}$$

3. PRODUCTS OF SEMI-TRANSPOSITIONS

For any semigroup S, and $A \subseteq S$, A is said to generate S if every element in S is expressed as a finite product of elements of A, written $\langle A \rangle = S$.

The main object of this section is to show that the ideals $\mathscr{IO}_{n,r}$ are generated by the subset QE_1 of semi-transpositions. For this, we have the following theorem:

Theorem 3.1: For $n \geq 2$ and $1 \leq r \leq n-1$, $\langle QE_1 \rangle = \mathscr{IO}_{n,r}$.

Proof: Since $\mathscr{IO}_{n,r}$ can be genrated by elements of height $n-1$ (see Madu [12]), it suffices to express only elements of height r as a product of semi-transposition in QE_1. Let $\alpha = \begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \cdots & b_r \end{pmatrix}$. Then α can be decomposed into a product of semitranspositions in QE_1 as

$$\alpha \ = \ \beta_r \beta_{r-1} \cdots \beta_1, \qquad (1 \leq i \leq r)$$

where

$$\beta_i = \begin{cases} [a_i, a_i+1]_{A_i}[a_i+1, a_i+2]_{A_i} \cdots [b_i-1, b_i]_{A_i} & \text{if } a_i < b_i \\ [a_i, a_i-1]_{A_i}[a_i+1, a_i+2]_{A_i} \cdots [b_i+1, a_i]_{A_i} & \text{if } a_i > b_i \end{cases}$$

with

$$A_i = \begin{cases} \mathrm{dom}(\alpha) \setminus \{a_r\} & \text{if} \quad i = r, \\ \mathrm{im}[b_{i+1}-1, a_i+1]_{A_i+1} \setminus \{a_i\} & \text{if} \quad i < r \quad \text{and} \quad a_i < b_i, \\ \mathrm{im}[b_{i+1}, b_i]_{A_{i+1}} \setminus \{a_i\} & \text{if} \quad i < r \quad \text{and} \quad a_i > b_i. \end{cases}$$

Before we proceed, we pause to exemplify the decomposition of $\alpha \in \mathscr{I}\mathscr{O}_{n,r}$ described in this theorem.

Example 3.1: Let $n = 10$, $r = 4$ and $\alpha = \begin{pmatrix} 3 & 4 & 6 & 8 \\ 1 & 7 & 8 & 9 \end{pmatrix} \in \mathscr{I}\mathscr{O}_{10,4}$. Then α is decomposed as follows:

$$\alpha \quad = \quad \beta_4\beta_3\beta_2\beta_1,$$

where,

$$\begin{aligned} \beta_4 &= [3,2]_{A_4}[2,1]_{A_4}, & A_4 = \{7,8,9\}, \\ \beta_3 &= [4,5]_{A_3}[5,6]_{A_3}[6,7]_{A_3}, & A_3 = \{3,4,9\}, \\ \beta_2 &= [6,7]_{A_2}[7,8]_{A_2}, & A_2 = \{3,4,9\}, \\ \beta_1 &= [8,9]_{A_1}, & A_1 = \{3,4,6\}. \end{aligned}$$

4. SEMI-TRANSPOSITION RANK

If a semigroup S is generated by a subset $A \subseteq S$, the rank of S with respect to A is defined to be the minimum number of elements in A requared to generate S, i.e

$$min_{B \subseteq A}\{|B| : \langle B \rangle \quad = \quad S\}.$$

In this section, we find the rank of $\mathscr{I}\mathscr{O}_{n,r}$ with respect to QE_1 which we call the semi-transpositions rank of $\mathscr{I}\mathscr{O}_{n,r}$.

Lemma 4.1: For $n \geq 2$ and $r \geq 1$, $|QE_1 \cap J_r| = 2r\binom{n-1}{r}$

Proof: From equation one, we see that each element of $QE_1 \cap J_r$ is either of the form $[i, i+1]_A$ or of the form $[i+1, i]_A$ for $1 \leq r \leq n-1$ and a subset $A \subseteq X_n \setminus \{i, i+1\}$ with $|A| = r-1$. There are equal number of such elements of either form. Thus,

$$
\begin{aligned}
|QE_1 \cap J_r| &= 2|\{[i,i+1]_A : 1 \le i \le n-1, A \subseteq X \setminus \{i,i+1\}, |A| = r-1\}| \\
&= 2(n-1)\binom{n-2}{r-1} \\
&= 2r\binom{n-1}{r}.
\end{aligned}
$$

We immediately have the following corollary for the case when $r = n-1$.

Corollary 4.1: For each $n \ge 2$, we have $|QE_1 \cap J_{n-1}| = 2(n-1)$.

Having established that $\mathscr{I}\mathscr{O}_{n,r}$ is generated by $|QE_1 \cap J_r|$ (see Corollary 1) it follows by Lemma 3 that the semi-transpositions rank in $\mathscr{I}\mathscr{O}_{n,r}$ is less or equal to $2r\binom{n-1}{r}$. In the next Lemma we obtain a lower bound for the rank.

Lemma 4.2: For $n \ge 2$ and $1 \le r \le n-1$, the semi-transposition rank of $\mathscr{I}\mathscr{O}_{n,r}$ is at least $\binom{n}{r}$.

Proof: This is based on a simple observation that $\alpha \in \mathscr{I}\mathscr{O}_{n,r}$ of height r is a product of

$$\alpha = \xi_1\xi_2\cdots\xi_k$$

of semi-transpositions in $|QE_1 \cap J_r|$, then $dom(\alpha) = dom(\xi_1)$ and $im(\alpha) = im(\xi_k)$. Thus, each possible choice for $dom(\alpha)$ and $im(\alpha)$ of cardinality r must be represented in a generating set of elements in $|QE_1 \cap J_r|$ for $\mathscr{I}\mathscr{O}_{n,r}$. Since there are exactly $\binom{n}{r}$ such choices for the $dom(\alpha)$ and $im(\alpha)$ the result follows.

The lower bound obtained in Lemma 4 can be improved, that is the semi-transposition rank for $\mathscr{I}\mathscr{O}_{n,r}$ is potentially larger than the lower bound $\binom{n}{r}$. To show this we require the following definitions.

Definition 4.1: By an r-subset of X_n we mean a subset $A \subseteq X_n$ of cardinality r. For any two r-subsets $A = \{a_1, \dots, a_r\}$ and $B = \{b_1, \dots, b_r\}$ of X_n, we call the non-negative integer $\sum_{i=1}^{r} |a_i - b_i|$ the distance between A and B and denote it by $d(A,B)$.

Definition 4.2: A list of r-subsets $A_1, \dots, A_m$ (where $m = \binom{n}{r}$) of X_n will be called quasi-orthogonally labelled list if it has the following properties:

(i) $d(A_i, A_{i+1}) = 1 \quad (1 \le r \le m-2)$,
(ii) $d(A_{m-1}, A_m) = 2$,

(iii) $d(A_{m-2}, A_m) = 1$.

Before establishing the semi-transposition rank of $\mathscr{IO}_{n,r}$, we first prove the existence of quasi-orthogonally labelled list for each $n \geq 3$ and for each $(1 \leq r \leq n-1)$.

Conjecture 4.1: For $n \geq 3$ and $1 \leq r \leq n-1$ there is a quasi-orthogonally labelled list of r-subsets of X_n.

Under the assumption the conjecture is valid, we prove the following result.

Theorem 4.2: For $n \geq 3$ and $1 \leq r \leq n-1$, the semi-transposition rank of $\mathscr{IO}_{n,r}$ is $2\binom{n}{r} - 2$.

Proof: Consider a quasi-orthogonally labelled list $A_1, A_2, \ldots, A_{m-1}, A_m (m = \binom{n}{r})$ or r subsets of X_n. Define for each $i = 2, 3, \ldots, m-1$, ξ_i and η_i to be the unique partial order-preserving transformation from A_i to A_{i+1} and from A_{i+1} to A_i respectively.

Then clearly, $A_1, A_2, \ldots, A_m$ is an orthogonally labelled list each ξ_i is an increasing semi-transposition and each η_i is a decreasing semi-transposition. Let $\alpha \in J_r$ with $dom(\alpha) = A_j$ and $im(\alpha) = A_i$. Then, we distinguish the following cases :

Case *i*: If α is an increasing transformation and $dom(\alpha) = A_j$ and $im(\alpha) = A_i$ with $j \leq i+2$, then α can be expressed as

$$\alpha = \xi_j \xi_{j-1} \cdots \xi_i.$$

Case *ii*: If α is an decreasing transformation and $dom(\alpha) = A_j$ and $im(\alpha) = A_i$ with $i \leq j+2$, then α can be expressed as

$$\alpha = \eta_j \eta_{j-1} \cdots \eta_i.$$

Case *iii*: If α is an increasing transformation and $dom(\alpha) = A_j$ and $im(\alpha) = A_i$ with $i = j+1$, then α can be expressed as

$$\alpha = \eta_{m-2} \xi_{m-1}.$$

Case *iv*: If α is an decreasing transformation and $dom(\alpha) = A_j$ and $im(\alpha) = A_i$ with $j = i+1$, then α can be expressed as

$$\alpha = \eta_{m-1} \xi_{m-2}.$$

Case *v*: If α is $dom(\alpha) = A_j$ and $im(\alpha) = A_i$ with $i = j$, then α is an idempotend and can be expressed as

$$\alpha = \xi_i \eta_j.$$

The semi-transpositions ξ_i and η_i covered both the $\mathscr{L}$ and $\mathscr{R}$-classes in $J_r(\mathscr{I}\mathscr{O}_{n,r})$. Hence no smaller number of semi-transpositions will suffice.

5. CONCLUDING REMARKS

This article has been able to identify semi-transpositions as a new generating system of $\mathscr{I}\mathscr{O}_{n,r}$ of a finite set. In fact , this work is a generalization of the work of [12]. We gave necessary and sufficient condition for an injecive transformation to be an injective order-preserving and also described procedures to express each transformation in $\mathscr{I}\mathscr{O}_{n,r}$ as a product of semi-transpositions in $\mathscr{I}\mathscr{O}_{n,r}$ respectively. In an attempt to find the rank of semi-transpositions in $\mathscr{I}\mathscr{O}_{n,r}$, we introduced the concept of orthogonally labelled list and conjuctured that it exists for each n and r. Under the assumption we are able to show that the rank is $2\binom{n}{r}-2$. The validity of the conjecture is still an open problem.

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