Journal of the Nigerian Mathematical Society
Vol. 44, Issue 1, pp. 139 -151, 2025
©Nigerian Mathematical Society

THE CYCLICPOID LIFE STAGES OF A STAR-LIKE MATHEMATICIAN

DEDICATED TO PROFESSOR ADEWALE ROLAND TUNDE SOLARIN ON THE OCCASION OF HIS 70TH BIRTHDAY

ABSTRACT. The star-like mathematician Adewale Roland Tunde Solarin ($ARTS$) is a role model and mentor in pure mathematics who established the coexistence relations between particular algebraic operators and functional theory. The influence and contribution of $ARTS$ on the novel class of star-like semigroups ($students$) and the characterization of cyclicpoid $C_yP\omega_n^*$ functions indicate a major relationship between group theory, semigroup theory, and tropical algebra. The present study uses a classical approach to build an equivalence relation between the ranks of the development phases of a star-like mathematician $ARTS$.

1. INITRODUCTION

One common use of graph theory is the algebraic theory of semigroups. The cyclicpoid class was employed as a model to present rank development, contribution and the work done by the star-like Mathematician $ARTS$ on the geometric and combinatorial theories within this framework. A recent study by [2] described various star-like relations on $C_yP\omega_n^*$. Similar findings were made by the writers in [4], who looked into the combinatorial functions of specific P_n subsemigroups. Nevertheless, comparable findings for the star-like $C_yP\omega_n^*$ cyclic transformation semigroup or some of its subsemigroups have not kept pace with all these advancements regarding the combinatorial functions of P_n and some of its subsemigroups.

A cyclicpoid class is conceptually related to group theory, which is best understood on a digraph. Star-like cyclic semigroups, unless they are cyclic groups in the first place, do not resemble circles. They do,

Received by the editors July 31, 2024; Revised: January 13, 2025; Accepted: February 06, 2025

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp. it/jnms/

2010 *Mathematics Subject Classification.* 20M20.

Key words and phrases. star-like, cyclicpoid, transformation.

however, contain a circle, and the cyclicpoid, which is the subject of our investigation in this paper, is precisely the circle. The list of publications and books that illustrate how this work relates to earlier studies in the field are shown below; it is by no means comprehensive. Some of the authors include [1, 3, 5, 6, 9]. For basic and standard concepts in the theory of transformation semigroups, refer to [3, 5, 7, 8].

2. PRELIMINARY

Some basic concepts and preliminary results that will be useful in the next sections were presented:

Definition 1: If all of the elements (students) of a star-like Mathematician are generators and one of them produces the other (students), the cyclic is considered ARTS cyclic if $C_yP\omega_n^* = \langle \beta^* \rangle$ is a star-like partial cyclic Mathematician in which $C_yP\omega_n^*$ has two possible values if β^* is a generator such that

(i) $\beta^{m*} = \beta^{p*} \Longrightarrow m = p$, for all $m,\ p \in D(\beta^*)$
(ii) There exist $m \neq p$ such that $\beta^{m*} = \beta^{p*}$ where $\langle \beta^* \rangle$ is isomorphic to $\mathbb{N}_+ \cup \{0\} \in \mathbb{R}_+$.

Definition 2: A cyclicpoid life stages G_{cy}^* of $ARTS$ in $C_yP\omega_n^*$ is a star-like transformation class with a unary operation $(^-1) : G_{cy}^* \longrightarrow G_{cy}^*$ and a star-like partial function $(\cdot) : G_{cy}^* \times G_{cy}^* \longrightarrow G_{cy}^*$ that satisfy the following axioms for all $\alpha^*, \beta^*, \gamma^* \in C_yP\omega_n^*$:

(i) If $\alpha^* \cdot \beta^*$ and $\beta^* \cdot \gamma^*$ are defined, then $(\alpha^* \cdot \beta^*) \cdot \gamma^*$ and $\alpha^* \cdot (\beta^* \cdot \gamma^*)$ are defined and are equal.
(ii) $\alpha^{-1*} \cdot \alpha^*$ and $\alpha^* \cdot \alpha^{-1*}$ are defined for any β^* in $P\omega_n^*$.
(iii) If $\alpha^* \cdot \beta^*$ is defined, then $\alpha^* \cdot \beta^* \beta^{-1*} = \alpha^*$ and $\alpha^{-1*} \cdot \alpha^* \cdot \beta^* = \beta^*$.
From these axioms (i-iii), if β^* is a generator such that $\beta^* \in C_yP\omega_n^*$, two easy and convenient properties follows.
(iv) If $\beta^{m*} = \beta^{p*} \Longrightarrow m = p$; $(\beta^{-1*})^{-1} = \beta^*$.
(v) $\exists\, m \neq p : \beta^{m*} = \beta^{p*}$, if $\alpha^* \cdot \beta^*$ is defined, then $(\alpha^* \cdot \beta^*)^{-1} = \beta^{-1*} \cdot \alpha^{-1*}$.

For any given star-like transformation with a rule $\lambda^* : M \to N$ where M and N are cyclicpoid, such that $\alpha^* \in C_yP\omega_n^*$. Then a finite cyclicpoid stage of $C_yP\omega_n^*$ is said to be star-like if

$$|\alpha^* k_i - w_{i+1}| \leq |\alpha^* k_{i+1} - w_i|, \qquad (1)$$

for all $w_i \in D(\alpha^*)$ and $k_i \in I(\alpha^*)$ such that

$$C_yP\omega_n^* = \begin{pmatrix} w_1 & w_2 & \dots & w_i \\ \alpha^* k_1 & \alpha^* k_2 & \dots & \alpha^* k_i \end{pmatrix}, \tag{2}$$

where $i \in \mathbb{N} \cup \{0\} \in \mathbb{R}$ where $\emptyset$ is assumed to be zero for all α^* in $C_yP\omega_n^*$.

3. THE CYCLICPOID STAGES G_{cy}^* OF $f(n, ARTS)$

Let $C_yP\omega_n^*$ represent $ARTS$ and let $\{U_i : i \in I\}$ (with $I \neq 0$); $\emptyset \neq I = \cap_{i=I} U_i$ be a star-like subsemigroup of $C_yP\omega_n^*$. Suppose β^* is a non empty star-like cyclicpoid transformation of $C_yP\omega_n^*$, there is at least one subsemigroup of $C_yP\omega_n^*$ containing β^*, ($C_yP\omega_n^*$ itself). The intersection of all star-like cyclicpoid subsemigroups of $C_yP\omega_n^*$ containing β^* is generated by the singleton element $\{\beta^*\} \in C_yP\omega_n^*$ denoted by $\langle\beta^*\rangle$.

There are two transformation possibilities for $\langle\beta^*\rangle$. The set of elements of $\langle\beta^*\rangle$ is $\{\beta^*, \beta^{2*}, \beta^{3*} \dots\}$.

(i) $\beta^{m*} = \beta^{p*} \longrightarrow m = p$, $\beta^* \subseteq \beta_j^* (j \in I)$.

(ii) If U^* is a subsemigroup of $C_yP\omega_n^*$ containing β^*, then $\langle\beta^*\rangle = U_j^*$ for each $j \in I$, $\exists\ m \neq p : \beta^{m*} = \beta^{p*}$.

In the former case, $\langle\beta^*\rangle$ is isomorphic to $(\{1,2,3,\dots\},+)$ of natural numbers under addition. In such a case, $\langle\beta^*\rangle$ is an infinite star-like cyclicpoid transformation, and the element $\langle\beta^*\rangle$ is said to have infinite order. In the latter case, suppose there exist two smallest positive integers m and p (the star-like index and star-like period of any given cyclicpoid transformation in $C_yP\omega_n^*$) such that $\beta^{m*} = \beta^{x*}$ for some positive integer $x \neq m$ and that $\beta^{m*} = \beta^{(m+p)^*}$. If $C_yP\omega_n^*$ is a monoid, then we can equally talk of the subsemigroup of $C_yP\omega_n^*$ generated by β^* that will always contain at least one idempotent set.

The order of β^* is defined as $m+p-1$, such that the star-like period and the star-like index must satisfy the following axioms:

(i) $\beta^{m*} = \beta^{(m+p)^*}$

(ii) $\beta^{(m+x)^*} = \beta^{(m+y)^*}$, if and only if $m+x \equiv m+y (\text{mod p})$

(iii) $\langle\beta^*\rangle = \{\beta^{m*}, \beta^{2^*}, \beta^{3^*} \dots \beta^{(m+p-1)^*}\}$

(iv) $K_{\beta^*} = \{\beta^{m*}, \beta^{(m+1)^*}, \beta^{(m+2)^*} \dots \beta^{(m+p-1)^*}\}$, where K_{β^*} is called the star-like kernel of $\langle\beta^*\rangle$ and it is cyclicpoid.

Notably, the pair (m, p) determines the structure of the star-like $C_yP\omega_n^*$ cyclicpoid semigroups, represented by $C_yP\omega_n^*(m, p)$, such that $C_yP\omega_n^*(1, p)$ is the cyclicpoid of order p. The new class of transformation $C_yP\omega_n^*$

semigroups is a promising class of results that can be used to show intriguing and useful results to link pure mathematical fields of study like real analysis and discrete mathematics, among others.
Lemma: 3.1 An element (student of $ARTS$) $\zeta \in C_yP\omega_n^*$ is star-like cyclicpoid if and only if $\nu^* f(\zeta^*) \leq I(\zeta^*)$.

Proof. Let $\nu^* f(\zeta^*) \leq I(\zeta^*)$, which means that for every $m \in I(\zeta^*)$, we have $m(\zeta^*) = m$.
Recall that $I(\zeta^*) = \{m\zeta^* : m \in Z_n\}$.
Then $|\zeta^*| = |C_yP\omega_n^*| \Longleftrightarrow |\zeta(w_1)^* - \zeta(m_2)^*| \leq |\zeta(m_2)^* - \zeta(w_1)^*| \leq \zeta^*$
$\Longleftrightarrow m\{\nu^*|\zeta(w_1)^* - \zeta(m_2)^*| \leq |\zeta(m_2)^* - \zeta(w_1)^*|\} \leq m\zeta^*$.
Let $|\zeta(w_1)^* - \zeta(m_2)^*| \leq |\zeta(m_2)^* - \zeta(w_1)^*| \leq K_{\zeta^*}$
$\Longleftrightarrow (\nu^* K_{\zeta^*})\zeta^* \leq w\zeta^*$ for each $m \in f(\zeta^*)$
$\Longleftrightarrow m\zeta^* \leq \nu^*$ for all $m \in I(\zeta^*)$
$\Longleftrightarrow \nu^* f(\zeta^*) \leq I(\zeta^*)$.
□

Proposition: 3.2 Let $C_yP\omega_n^*$ be a star-like cyclicpoid abelian semigroup, then the following statements are true:

(i) $0 \cdot \beta^* = 0$ for any $\beta^* \in C_yP\omega_n^*$
(ii) $(-c) \cdot \beta^* = c \cdot (\beta^*)$ for any $c \in \mathbb{R}$ and $\beta^* \in C_yP\omega_n^*$
(iii) $c \cdot 0 = 0$ for any $c \in \mathbb{R}$
(iv) If $c \cdot \beta^* = 0$ then, either $c = 0$ or $\beta^* = 0$.

Proof. (i) Suppose $\beta^* \in C_yP\omega_n^*$ such that $C_yP\omega_n^* \subseteq P\omega_n^*$, then from definition (1), using (viii), (v), and (ii), we get:

$$\begin{aligned} \beta^* + 0 \cdot \beta^* &= 1 \cdot \beta^* + 0 \cdot \beta^* \\ &= (1+0) \cdot \beta^* \\ &= 1\beta^* = \beta^* \\ &= \beta^* + 0 \end{aligned}$$

adding $(-\beta^*)$ to both sides of the equality:
$-\beta^* + \beta^* + 0 \cdot \beta^* = -\beta^* + \beta^* + 0$.
Thus, $0 \cdot \beta^* = 0$
(ii) If $-(c \cdot \beta^*)$ is an element of $C_yP\omega_n^*$ that satisfies property (iii) of definition (1), we can replace β^* by $c \cdot \beta^*$.
We need to show that $c \cdot 0 = 0$ for any $c \in \mathbb{R}$ holds if
$-(c \cdot \beta^*) = (-c) \cdot \beta^*$.
Now using $(-c) \cdot \beta^* = c \cdot (-\beta^*)$ and $c \cdot \beta^* = 0$ such that
$(-c) \cdot \beta^* + c \cdot \beta^* = (-c + c) \cdot \beta^* = 0 \cdot \beta^* = 0$
Thus, $(-c) \cdot \beta^* = 0$
$c \cdot (-\beta^*) = 0$.

(iii) Generally, for any $\alpha_n^* \in C_yP\omega_n^*$, we see that:

$$\begin{aligned} c\cdot 0 &= c\cdot(\beta_n^* - \beta_n^*) \\ &= c\cdot\beta_n^* + c\cdot(-\beta_n^*) \\ &= c\cdot\beta_n^* + c\cdot\{(-1)\cdot\beta_n^*\} \\ &= c\cdot\beta_n^* + (-c)\cdot\beta_n^* \\ &= c\cdot\beta_n^* - c\beta_n^* = 0. \end{aligned}$$

Therefore, $c\cdot 0 = 0$ for any $c \in \mathbb{R}$

(iv) Suppose $c \neq 0$, then $c\beta^* = 0$ It is sensible to consider $\frac{1}{c}$; therefore,

$$\begin{aligned} \beta^* &= 1\beta^* \\ &= (\frac{1}{c}\,c)\beta^* \\ &= \frac{1}{c}\,(c\beta^*) \\ &= \frac{1}{c}\,0 \;=\; 0. \end{aligned}$$

Thus, either $\beta^* = 0$ or $c = 0$ if $c\beta^* = 0$.

□

Proposition: 3.3 Let $\beta^* \in P\omega^*_{n(m,p)}$ where $1 \le p \le n-1$ and $|K_{\beta^*}(C_yP\omega_4^*)| = 70\,(ARTS\;is\;70yrs)$. Then, there exist ζ^* in $C_yP\omega^*_{n-1}$ and β^* in $P\omega^*_{n(m,p+1)}$ such that every element of $P\omega^*_{m,p-1}$ is expressible as a product of elements in $C_yP\omega^*_{n-1}$ where $\beta^* = \zeta^*\beta^*$.

Proof. For each $i = 1,2,\ldots p)$, let $b_i\beta^{-1*} = W_i$, and let $I(\beta^*) = \{b_1, b_2, \ldots b_p\}$. It is easy to write

$$\beta^* = \begin{pmatrix} W_1 & W_2 & W_3 & W_4 & \ldots & W_p \\ b_1 & b_2 & b_3 & b_4 & \ldots & b_p \end{pmatrix}.$$

Since not all of the sets W_i are singletons, we can assume that $W_1 = \{w_1, w_1^{'}, w_1^{''}, w_1^{'''}\ldots\}$ contains at least four items without losing generality. We can see that the sets W_i form a partition of $[n]$.

$$\zeta^* = \begin{pmatrix} w_1 \\ w_1^{'} \\ w_1^{''} \\ w_1^{'''} \end{pmatrix}, \beta^* = \begin{pmatrix} W_1\setminus\{w_1\} & W_2\setminus\{w_2\} & \ldots & W_p\ \{w_1\}\{w_2\} \\ b_1 & b_2 & \ldots & W_p\ b_{p+1} \end{pmatrix}$$

It may be shown that $\beta^* = \zeta^*\beta^*$ when $b_{p+1} \notin I(\beta^*)$. It is now necessary to demonstrate that each element of $P\omega^*_{n(m,p-1)}$ can be expressed as a product of elements in $C_yP\omega^*_{n-1}$, such that $C_yP\omega^*_{n-1} \subseteq P\omega^*_{n-1}$. With any $\beta^* : [n] \longrightarrow [n]$, we employ a graph $\Gamma\beta^*$. $\Gamma\beta^*$ has the following labels on its vertices: $\beta, \beta^{2*}, \dots \beta^{n*}$. If and only if $i\beta^* = j$, there is an edge $i \longrightarrow j$. $(\beta^{1*}, \dots \beta^{37*})$ may not be connected, but the connected components are the cyclicpoid-stages G^*_{cy}, where the equivalency joint G^*_{cy} is provided by

$$G^*_{cy} = \{(i,j) \in [n] \times [n] : (\exists\ p, m \geq 0)\ i\beta^{p*} = j\beta^{m*}\}.$$

Each G^*_{cy} has a kernel K_{β^*} defined by

$$K_{\beta^*} = \{i \in G^*_{cy} : (\exists\ p > 0)\ i\beta^{p*} = i\}.$$

Finding K_{β^*}, we find that for any $\{i \in G^*_{cy} = i, i\beta^*, i\beta^{2*}, \dots\}$, there exists $m \geq 0$ and $p \geq 1$ such that
$i\beta^{m+p} = i\beta^m$. $i\beta^m \in K_{\beta^*}$, as a result, every star-like cyclicpoid life stages G^*_{cy} of $C_yP\omega^*_n$ is categorized into the subsequent groups:

acyclic life stage: $1 = |K_{\beta^*}| < |G^*_{cy}|$,

tricyclic life stage: $3 \leq |K_{\beta^*}| < |G^*_{cy}|$,

cyclic life stage: $2 \leq |K_{\beta^*}| = |G^*_{cy}|$,

retired life stage: $1 = |K_{\beta^*}| = |G^*_{cy}|$

Their associated digraph are illustrated below:

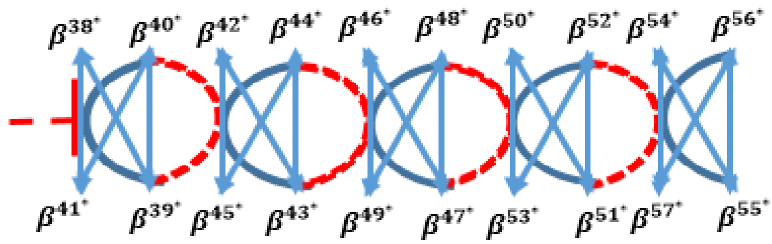

FIGURE 1. **star-like acyclic life stage**

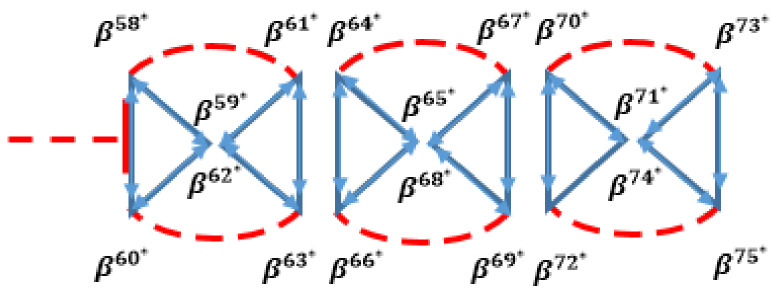

FIGURE 2. **star-like tricyclic life stage**

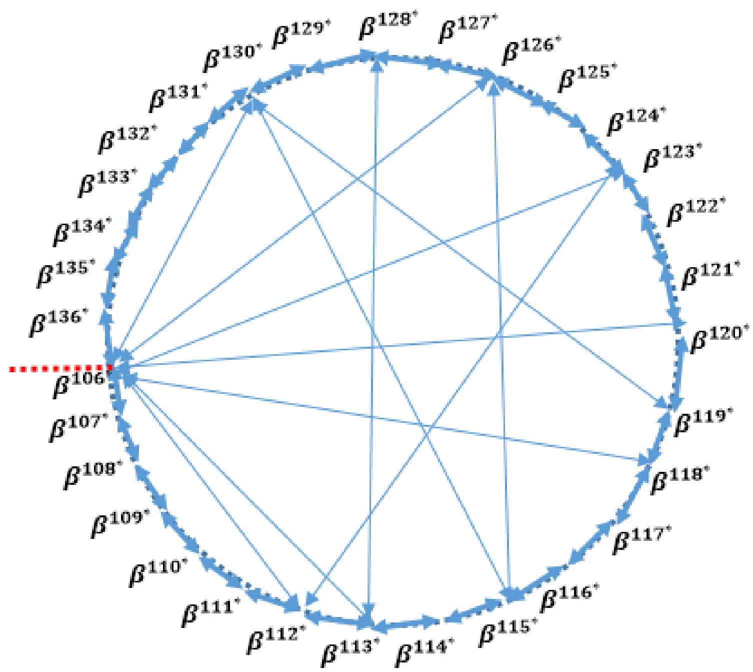

FIGURE 3. **star-like cyclic life stage**

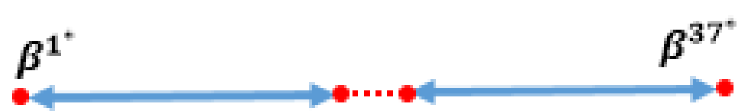

FIGURE 4. **star-like retired life stage**

Assume that any element $(student)$ in $C_yP\omega_n^*$ is β^*. At least one element not in $I(\beta^*)$ is present in each of the four life stages of G_{cy}^*: acyclic, tricyclic, cyclic, and retired. Taking an element (any student of $ARTS$) u_0^* in $G_{cy}^* \setminus K_{\beta^*}$, we can then consider the following life stages. If $u_1^* \in u_0^*\beta^{-1^*}, u_2^* \in u_1^*\beta^{-1^*}, \ldots u_{n+1}^* \in u_n^*\beta^{-1^*}, \ldots$, then elements $students$ $u_1^*, u_2^*, \ldots$ cannot all be distinct. Consequently, there exist $p, p+m$ with $m \geq 1$ and $u_{p+m}^* = C_yP\omega_{n,(m,p)}^*$. $u_0^* = u_p^*\beta^{p^*} = u_p^*\beta^{m+p^*} = u_0^*\beta^{m^*}$ is the logical conclusion. Consider β^* in $C_yP\omega_{n-1}^*$. Since $|I(\beta^*)| = (n-1)$, all the model stages of β^* are either acyclic, tricyclic, cyclic or retired:

ARTS acyclic life stage: Initially, we will assume that $ARTS$ life stage is acyclic such that

$$A^*_c; z^*_1, z^*_2, \dots z^*_c; \{u^*_1\}, \{u^*_2\}, \dots \{u^*_t\}$$

. In this case, A^*_c is acyclic, $z^*_1, z^*_2, \dots z^*_c$ are cyclic, and $\{u^*_1\}, \{u^*_2\}, \dots \{u^*_t\}$ are trivial such that $|A^*_c| = a^*$ and $|z^*i| = z^*_i$ for $i = 1, 2, \dots, c$.

$$a^* + \sum_{i=1}^{c} z^*_i + \nu^* = 2n. \tag{3}$$

Let x_1 be the unique element in $a^* \setminus I(\beta^*)$, then $a^* = \{x^*_1, x^*_2, \dots x^*_{a^*}\}$, where $x^*_{u+1} = x^*_u \beta^*$ for $u = 1, 2, \dots, a^* - 1$ and $x^*_{a^*}\beta^* = x^*_{a^*}$. Also, let $z^*_i = \{v^*_{i1}, v^*_{i2}, \dots v^*_{izi}\}$, for $i = 1, 2, \dots c$ where $v^*_{u+1} = v^*_{ij}\beta^*$ for $u = 1, 2, \dots, z^*_i - 1$ and $v^*_{iz^*_i}\beta^* = v^*_i$. Then

$$\beta^* = \beta^* \gamma^*_1 \gamma^*_2, \dots \gamma^*_c, \tag{4}$$

where

$$\beta^* = \begin{pmatrix} x_{a^*-1} \\ x^*_a \end{pmatrix} \begin{pmatrix} x_{a^*-2} \\ x_{a^*-1} \end{pmatrix} \cdots \begin{pmatrix} x_1 \\ x_2 \end{pmatrix},$$

and for $i = 1, 2, \dots c$,

$$\gamma^*_i = \begin{pmatrix} v_{i,z^*_i} \\ x_1 \end{pmatrix} \begin{pmatrix} v_{i,z^*_{i-1}} \\ v_{i,z^*_i} \end{pmatrix} \begin{pmatrix} v_{i,z^*_{i-2}} \\ v_{i,z^*_{i-1}} \end{pmatrix} \cdots \begin{pmatrix} v_{i,1} \\ v_{i,2} \end{pmatrix} \begin{pmatrix} x_1 \\ v_{i,1} \end{pmatrix}. \tag{5}$$

By adopting the star-like operator, Figure (1) shows that
$|G^*_{cy}(A^*_c)| = \beta^{p*} \leq |K_{\beta^*}| < |G^*_{cy}|$
$= \beta^{p*} \leq 10 < |G^*_{cy}|$
$= \beta^{p*} \leq \beta^5 \cdot \beta^5 < |G^*_{cy}|$
$= 1 \leq (\beta^5)^2 < 20$
$= \beta^{p^*} \leq \beta^{m^*} < |G^*_{cy}|$
where the number of idempotents in the product of $|G^*_{cy}(A^*_c)|$ is

$$(a^* - 1) + \sum_{i=1}^{c} (z^*_i + 1) = a^* - 1 + \sum_{i=1}^{c} z^*_i + c = n + c - t^* - 1.$$

Thus, β^* is a product of idempotent in $C_y P \omega^*_{n-1}$.

ARTS tricyclic life stage: Let the $ARTS$ life stage be tricyclic and β^* be as follows:

$$T^*_c; z^*_1, z^*_2, \dots z^*_c; \{u^*_1\}, \{u^*_2\}, \dots \{u^*_t\}.$$

In this case, T_c^* becomes tricyclic, and the remaining life stages remain as before, so that $|T_c^*| = \nu^*$ and $|z_i^*| = z_i^*$ for $i = 1, 2, \ldots, c$. Consequently

$$T_c^* + \sum_{i=1}^{c} z_i^* + \nu^* = n. \tag{6}$$

Let x_1 once more represent the unique element $(student)$ in $T_c^* \setminus I(\beta^*)$. Then, let $x_m = x_1\beta^{m-1*} \in K_{\beta^*}(T_{c^*})$, where m is selected to be as minimal as feasible. Then $T_c^* = \{x_1^*, x_2^*, \ldots x_{m+p}^*\}$, with $x_i^*\beta^* = x_{i+1}^*$ where for $i = 1, 2, \ldots, m+p-1$ and $x_{m+p}^*\beta^* = x_m^*$. If we were to define

$$\alpha'^* = \begin{pmatrix} x_{m+p} \\ x_{m-1} \end{pmatrix} \begin{pmatrix} x_{m+p-1} \\ x_{m+p} \end{pmatrix} \begin{pmatrix} x_{m+p-2} \\ x_{m+p-1} \end{pmatrix} \cdots \begin{pmatrix} x_{m-1} \\ x_m \end{pmatrix} \begin{pmatrix} x_{m-2} \\ x_{m-1} \end{pmatrix} \cdots \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

We see that

$$\beta^* = \alpha'^* \gamma_1^* \gamma_2^*, \ldots \gamma_c^*, \tag{7}$$

such that we have from Figure 2. The expression
$|G_{cy}^*(T_c^*)| = \beta^{p*} \leq |K_{\beta^*}| < |G_{cy}^*|$
$= \beta^{p*} \leq 6 < |G_{cy}^*|$
$= \beta^{p*} \leq \beta^3 \cdot \beta^3 < |G_{cy}^*|$
$= 3 \leq (\beta^3)^2 < 18$
$= \beta^{p^*} \leq \beta^{m^*} < |G_{cy}^*|$
where the number of idempotent in the product of $|G_{cy}^*(T_c^*)|$ is

$$m + p + \sum_{i=1}^{c} (z_i^* + 1) = n + c - \nu^*.$$

, where $\gamma_1, \gamma_2, \ldots, \gamma_c$ are as in Equation (5). Hence, β^* is once again an idempotent product from $C_yP\omega_{n-1}^*$.

ARTS cyclic life stage: Let us assume that some of the students of β^* exhibit cyclic behavior such that The set

$$P_c^*; z_1^*, z_2^*, \ldots z_c^*; \{u_1^*\}, \{u_2^*\}, \ldots \{u_t^*\},$$

with P_c^* being now star-like cyclic, and the remaining of their life stages remaining as before, so that $|P - c^*| = n + 1$ and $|z_i^*| = z_i^*$ for $i = 1, 2, \ldots, c$. Thus,

$$p + \sum_{i=1}^{c} z_i^* + m = r^*(\beta^*). \tag{8}$$

Let x_1 be the unique element in $P_c^* \setminus I(\beta^*)$, and suppose that $x_m + 1 = x_1\beta^{m-2*} \in K_{\beta^*}(P_c^*)$, with m chosen as small as possible. Then

$P_c^* = \{x_1^*, x_2^*, \ldots x_{(m+1)+p}^*\}$, with $x_i^* \beta^* = x_{i+1}^*$ for $i = 1, 2, \ldots, (m+1) + p - 1$ and $x_{(m+1)+p}^* \beta^* = x_{m+1}^*$. Also, if we define

$$\alpha^{''*} = \begin{pmatrix} x_{(m+1)+p} \\ x_{m-2} \end{pmatrix} \begin{pmatrix} x_{(m+1)+p-2} \\ x_{(m+1)+p} \end{pmatrix} \cdots \begin{pmatrix} x_{m-2} \\ x_{m+1} \end{pmatrix} \begin{pmatrix} x_{m-3} \\ x_{(m+1)-2} \end{pmatrix} \cdots \begin{pmatrix} x_1 \\ x_3 \end{pmatrix}.$$

Then

$$\beta^* = \alpha^{''*} \gamma_1^* \gamma_2^*, \ldots \gamma_c^*, \tag{9}$$

and from Figure 4
$|G_{cy}^*(P_c^*)| = \beta^{p*} \leq |K_{\beta^*}| < |G_{cy}^*|$
$= \beta^{p*} \leq 6 < |G_{cy}^*|$
$= \beta^{p*} \leq \beta^3 \cdot \beta^3 < |G_{cy}^*|$
$= 4 \leq (\beta^3)^2 < 12$
$= \beta^{p^*} \leq \beta^{m^*} < |G_{cy}^*|$
where $\gamma_1, \gamma_2, \ldots, \gamma_c$ are as in Equation (5). Thus,

$$(m+1) + p + \sum_{i=1}^{c} (z_i^* + 1) = (n+1) + r^*(\beta^*) - m,$$

gives the number of star-like idempotent products in $|G_{cy}^*(P_c^*)|$.

ARTS retired life stage: Considering the retired life stage of $ARTS$, let

$$R_c^* + \sum_{i=1}^{c} z_i^* + p = |G_{cy}^*|. \tag{10}$$

Once more, let x_1 represent the first retirement year. Assume that $x_m + 2 = x_1 \beta^{m-3*} \in K_{\beta^*}(R_c^*)$, where m is selected to be as minimal (2024) as feasible such that

$$\beta^* = \alpha^{'''*} \gamma_1^* \gamma_2^*, \ldots \gamma_c^*, \tag{11}$$

and from Figure 4
$|G_{cy}^*(R_c^*)| = \beta^{p*} \leq |K_{\beta^*}| < |G_{cy}^*|$
$= \beta^{p*} \leq 6 < |G_{cy}^*|$
$= \beta^{p*} \leq \beta^3 \cdot \beta^3 < |G_{cy}^*|$
$= 1 \leq (\beta^3)^2 < 6$
$= \beta^{p^*} \leq \beta^{m^*} < |G_{cy}^*|$
where $\gamma_1, \gamma_2, \ldots, \gamma_c$ are as in Equation (5), and the number of idempotent in the products of $|G_{cy}^*(R_c^*)|$ is

$$(m+2) + p + \sum_{i=1}^{c} (z_i^* + 1) = |G_{cy}^*| + n,$$

which shows that β^* is a product of idempotent from $C_yP\omega_n^*$. We can observe from the aforementioned life stages of $ARTS$ that the products' length is $n+cycl\,\beta^*-fix\,\beta^*$. Where $cycl\beta^*$ is the number of cyclicpoid stages such that

$$fix\,\beta^* = |\{i \in [n] : i\beta^* = i\}|$$

such that m and p are index and period of $G_{cy}^*(\beta^*)$. Therefore, we showed that the career achievement year of $ARTS$ was periodically converged

$$\begin{aligned}|K_{\beta^*}(C_yP\omega_4^*)| &= \sum_{i=1}^{c} |K_{\beta^*}Cycl(i\beta^*)| \\ &= m(C_yP\omega_n^*) + p(G_{cy}^*) - 1 \\ &= 28 + 43 - 1 = 70\end{aligned}$$

. □

Remark: A star-like cyclicpoid transformation may have many generators. Although, the lists $\beta^{-2*}, \beta^{-1*}, \beta^{0*}, \beta^{1*}, \beta^{2*} \ldots$ has finitely many entries such that the element $(student)$ $\beta^{n*} : n \in \mathbb{N} \cup \{\emptyset\} \in \mathbb{R}$ has only finitely many elements $students$ since $\beta^{i*}\beta^{j*} = \beta^{j+i*} = \beta^{ij*}$. Thus, every star-like cyclicpoid life stage of $ARTS$ produces an abelian (star-like students).

ACKNOWLEDGMENT

The author would like to thank the anonymous referee whose comments improved the original version of this paper.

NOMENCLATURE

The following list of nomenclatures gave the account of mathematical symbols and their interpretations as used in this paper:

$ARTS$	Star-like Mathematician
$C_yP\omega_n^*$	Star-like cyclic transformation
G_{cy}^*	ARTS cyclicpoid
A_c^*	ARTS acyclic life stage
T_c^*	ARTS tricyclic life stage
P_c^*	ARTS cyclic life stage
R_c^*	ARTS retired life stage

REFERENCES

[1] A. O. Adeniji and S. O. Makanjuola, *Congruence in identity difference full transformation semigroups*, International Journal of Algebra **7** (12) 563-572, 2013. http://dx.doi.org/10.12988/ija.2013.3221

[2] S. A. Akinwunmi, M. M. Mogbonju and A. O. Adeniji, *Multiplicative invertibility characterizations on star-like cyclicpoid $C_yP\omega_n^*$ finite partial transformation semigroups*, Hikari Journal of Pure Mathematical Science **10** (1) 45 - 55, 2021. https://doi.org/10.12988/pms.2021.91276

[3] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, American Mathematical Society, New York, Mathematical surveys and monographs **7** 98-120, 1967. https://doi.org/10.4236/ojpp.2018.84027

[4] G. U. Garba, *Idempotents in partial transformation semigroups*, Proceedings of the Royal Society of Edinburgh Section A: Mathematics **116** (3-4) 359-366, 1990. https://doi.org/10.1017/S0308210500031553

[5] O. Ganyushkin and V. Mazorchuk, *Classical finite transformation semigroups; An introduction*, Springer-Verlag: London limited, 2009. https://doi.org/10.1007/978-1-84800-281-4

[6] G. M. S. Gomes and J. M. Howie, *On the ranks of certain semigroups of order-preserving transformations*, Semigroup Forum **45** 272-282, 1992. https://doi.org/10.1007/BF03025769

[7] P. M. Higgins, *Combinatorial results for semigroups of order-preserving mappings*, Mathematical Proceedings of the Cambridge Philosophical Society **113** (2) 281 - 296, 1993. https://doi.org/10.1017/S0305004100075964

[8] J. M. Howie, *Fundamentals of semigroup theory*. SemiGroup Forum **58** 156 158, 1999. https://doi.org/10.1007/s002339900006

[9] A. Umar, *Some combinatorial problems in the theory of isometric inverse semigroups*, Journal of Algebra Discrete Math., **9** (2) 115-126, 2010. https://api.semanticscholar.org/CorpusID:53996359

DEPARTMENT OF MATHEMATICS AND STATISTICS, FEDERAL UNIVERSITY OF KASHERE, GOMBE, NIGERIA
E-mail address: sakinwunmi@fukashere.edu.ng; molakinkanju@gmail.com