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A FURTHER INVESTIGATION ON THE CORE OF MIDDLE BOL LOOPS

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DEDICATED TO PROFESSOR ADEWALE ROLAND TUNDE SOLARIN ON THE OCCASION OF HIS 70TH BIRTHDAY

ABSTRACT. In this paper, further investigation of the core of middle Bol loop relative to the core of right Bol loop, is presented. The efforts revealed that, (i) if T is any permutation of Q, then $xT \otimes yT = (x \oplus y)T$ if and only if $[(x \circ y\backslash\backslash x)T \circ f]I = (xT \circ f)I \circ (yT \circ f)I\backslash\backslash(xT \circ f)I, \forall\, x,y \in Q$ and some $f \in Q$, where $(Q,\oplus)$ and $(Q,\otimes)$ are respectively, the cores of the middle Bol loop $(Q,\circ)$ and its isotope $(Q,*)$. (ii) in particular it was also shown that a middle Bol loop satisfies the identity: $(x \circ f^{-1}\backslash\backslash x) \circ f = (x \circ f) \circ (x \circ f), \forall\, f,x \in Q$ if and only if for each isotope $(Q,*)$ of the middle Bol loop $(Q,\circ)$ given by $x * y = (xR^o_f T \circ yI)R^{o^{-1}}_f I$, then $x \otimes y = x \oplus y, \forall\, x,y \in Q$, where $(Q,\otimes)$ and $(Q,\oplus)$ the cores of $(Q,*)$ and $(Q,\circ)$ respectively . (iii) it was shown also that, the core exhibits some left self symmetry, left self distributive and that, the middle Bol loop is right distributive over its core. It was also remarked that, the core of middle Bol loop exhibits a form of semi-automorphism. New results were obtained and old ones extended.

1. INTRODUCTION

Let Q be a non -empty set. Define a binary operation "$\cdot$" on Q. If $x \cdot y \in Q$ for all $x,y \in Q$, then the pair $(Q,\cdot)$ is called a groupoid. If in addition to this, the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions $x,y \in Q$ for all $a,b \in Q$ then $(Q,\cdot)$ is called a quasigroup. Let $(Q,\cdot)$ be a quasigroup and there exist a unique element $1 \in Q$ called the identity element such that for all $x \in Q, x \cdot 1 = 1 \cdot x = x$, then $(Q,\cdot)$ is called a

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loop. At times, we shall write xy instead of $x\cdot y$ and stipulate that "$\cdot$" has lower priority than juxtaposition among factors to be multiplied. Let $(Q,\cdot)$ be a groupoid and a be a fixed element in Q, then the left and right translations L_a and R_a of a are respectively defined by $xL_a = a\cdot x$ and $xR_a = x\cdot a$ for all $x\in Q$. It can now be seen that a groupoid $(Q,\cdot)$ is a quasigroup if its left and right translation mappings are permutations. Since the left and right translation mappings of a quasigroup are bijections, then the inverse mappings L_x^{-1} and R_x^{-1} exist. Let

$$x\backslash y = yL_x^{-1} = xM_y \qquad \text{and} \qquad x/y = xR_y^{-1} = yM_x^{-1}$$

and note that

$$x\backslash y = z \Longleftrightarrow x\cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z\cdot y = x.$$

Thus, for any quasigroup $(Q,\cdot)$, we have two new binary operations; right division $(/)$ and left division $(\backslash)$ and middle translation M_a for any fixed $a\in Q$. Consequently, $(Q,\backslash)$ and $(Q,/)$ are also quasigroups. Using the operations $(\backslash)$ and $(/)$, the definition of a loop can be restated as follows.

Definition 1: A *loop* $(Q,\cdot,/,\backslash,1)$ is a set G together with three binary operations $(\cdot)$, $(/)$, $(\backslash)$ and one nullary operation e such that

(i): $x\cdot(x\backslash y) = y$, $(y/x)\cdot x = y$ for all $x,y\in Q$,

(ii): $x\backslash(x\cdot y) = y$, $(y\cdot x)/x = y$ for all $x,y\in Q$ and

(iii): $x\backslash x = y/y$ or $1\cdot x = x$ for all $x,y\in Q$.

We also stipulate that $(/)$ and $(\backslash)$ have higher priority than $(\cdot)$ among factors to be multiplied. For instance, $x\cdot y/z$ and $x\cdot y\backslash z$ stand for $x(y/z)$ and $x(y\backslash z)$ respectively.

In a loop $(Q,\cdot)$ with identity element 1, the *left inverse element* of $x\in Q$ is the element $xJ_\lambda = x^\lambda \in Q$ such that $x^\lambda\cdot x = 1$ while the *right inverse element* of $x\in G$ is the element $xJ_\rho = x^\rho\in G$ such that $x\cdot x^\rho = 1$.

Middle Bol loop was first mentioned in the work of V. D. Belousov [3], while studying loops with universal anti-automorphic inverse property (AAIP), which led to the derivation of the identity (1), now known in the literature as, middle Bol identity.

$$(x/y)(z\backslash x) = x(zy\backslash x) \tag{1}$$

After this beautiful characterization by Belousov and the laying of foundations for a classical study of this structure, Gwaramija in [8] proved that a loop $(Q,\circ)$ is middle Bol if there exist a right Bol loop $(Q,\cdot)$ such that $x \circ y = (y \cdot xy^{-1})y$ for all $x, y, \in Q$. If $(Q, \circ, //, \backslash\backslash)$ is a middle Bol loop and $(Q, \cdot, /, \backslash)$ is the corresponding right Bol loop, he then showed that:

$$x \circ y = y^{-1} \backslash x \qquad \text{and} \qquad x \cdot y = y // x^{-1},\ \forall\, x, y \in Q. \tag{2}$$

Also, if $(Q, \circ, //, \backslash\backslash)$ is a middle Bol loop and $(Q, \cdot, /, \backslash)$ is the corresponding left Bol loop, then

$$x \circ y = x / y^{-1} \qquad \text{and} \qquad x \cdot y = x // y^{-1},\ \forall x, y \in Q. \tag{3}$$

Grecu [5] showed that right multiplication group of a middle Bol loop coincides with the left multiplication group of the corresponding right Bol loop.

Syrbu [21, 22] in 1994 and 1996, obtained series of results on middle Bol loop, using the concept of universal elastic law.

Kuznetsov [17] in 2003, while carrying out study on gyrogroups-a special class of group, established some algebraic properties of middle Bol loop and designed a method of constructing a middle Bol loop from a gyrogroup.

Also in 2010, Syrbu [23] studied the connections between structure and properties of middle Bol loops and that of the corresponding left Bol loops. It was noted that two middle Bol loops are isomorphic if and only if the corresponding left (right) Bol loops are isomorphic, and a general form of the autotopisms of middle Bol loops was deduced. Relations between different sets of elements, such as nucleus, left (right,middle) nuclei, the set of Moufang elements, the center, e.t.c. of a middle Bol loop and left Bol loops were established.

Grecu and Syrbu [6] further proved in 2012, that two middle Bol loops are isotopic if and only if the corresponding right (left) Bol loops are isotopic. Same year with Grecu and Syrbu, Drapal and Shcherbacov [4] rediscovered the middle Bol identities in a new way and showed that, the identity (1) of middle Bol loop has a dual.

Syrbu and Grecu [24], in 2013 continued their exploits on the loop under scrutiny and established a necessary and sufficient condition for the quotient loop of a middle Bol loop and of its corresponding right Bol loop to be isomorphic. This same set of authors in 2014 established that, the commutant (centrum) of a middle Bol loop is an automophic inverse property subloop. They further gave a necessary and sufficient condition for commutant invariance under the existing isostrophy between middle Bol loop and the corresponding right Bol loop[7].

Osoba and Oyebo [18, 26] in 2018, investigated further the multiplication group of middle Bol loop in relation to left Bol loop. For more on quasigroups and loops, see Jaiyéọlá [12], Shcherbacov [20], Pflugfelder [19] and Solarin et al [28].

The core of a loop was originally introduced by R. H. Bruck in connection with invariants of isotopically closed Moufang loops (isotopic Moufang loops that have isomorphic core). More general definition was created by V. D Belousov [3]. The core of middle Bol loop was introduced and initially studied by Gwaramija [8]. Using Bruck's approach for the notion of core, Gwaramija introduced a definition for the core of a middle Bol loop. By using the properties of algebraic 3-nets, Gwaramija remarked that the core of a middle Bol loop is isomorphic to the core of a left Bol loop. Belousov [9] used geometrical approach, namely on evaluation of coordinates of points and lines in the corresponding Bol net to show that cores of left Bol loops, particularly cores of Moufang loops, or groups, are left distributive, left symmetric, and idempotent.

In 2005, Vanžurová [29] used purely algebraic approach to clarify some of the well known results, the relationship between cores and the variety of left symmetric, left distributive, idempotent groupoids or its medial subvariety of Bol loops, Moufang loops or groups.

Adeniran et al. [1], in 2015 carried out a comprehensive study on the core and some isotopic characterisation of generalised Bol loops, it was shown that the set of semi-automorphisms of the generalised Bol loops are the automorphisms of the core. Jaiyéọlá et al. [14], in 2017 studied the holomorphic structure of Middle Bol loops and showed that the holomorph of a commutative loop is a commutative middle Bol loop if and only if the loop is a middle Bol loop and its automorphism group is abelian. Adeniran et al. [2], Jaiyéọlá and Popoola [16] studied generalised Bol loops.

Jaiyéọlá et al. [13], in 2015 further studied new algebraic identities of middle Bol loop, where they obtained a necessary and sufficient conditions for a bi-variate mapping of a middle Bol loop to have right inverse property (RIP), left inverse property (LIP), right alternative inverse property (RAIP), left alternative inverse property (LAIP) and flexible property. Additional algebraic properties of middle Bol loop were announced in 2021 by Jaiyéọlá et al. [15]. Osoba and Jaiyéọlá et al, [25] in 2021, studied the algebraic connections between right and middle Bol loops using their cores. It was shown that the core of a right Bol loop is elastic and right idempotent. The core of a right Bol loop was shown to be alternative (or left idempotent) if and only if its corresponding middle Bol loop is right symmetric. It was furthered revealed that If a middle Bol loop is right (left) symmetric, then the core of its corresponding right Bol loop is medial (semimedial).

In this current research, we furthered the investigation of the cores of middle Bol loops relative to the cores of right Bol loops using the concept of principal isotopes.

2. PRELIMINARIES

In this section, existing definitions and Lemmas related to concepts under scrutiny are systematically outlined.

Definition 2: A groupoid (quasigroup) $(G,\cdot)$ is said to have the

(1) left inverse property (LIP) if there exists a mapping $J_\lambda : x \mapsto x^\lambda$ such that $x^\lambda \cdot xy = y$ for all $x,y \in G$.
(2) right inverse property (RIP) if there exists a mapping $J_\rho : x \mapsto x^\rho$ such that $yx \cdot x^\rho = y$ for all $x,y \in G$.
(3) inverse property (IP) if it has both the LIP and RIP.
(4) left symmetric or left key law if $x \cdot xy = y$ for all $x,y \in Q$.
(5) cross inverse property (CIP) if there exist mapping $J_\lambda : x \mapsto x^\lambda$ or $J_\rho : x \mapsto x^\rho$ such that $xy \cdot x^\rho = y$ or $x \cdot yx^\rho = y$ or $x^\lambda \cdot yx = y$ or $x^\lambda y \cdot x = y$ for all $x,y \in G$.

Definition 3: A groupoid $(Q,+)$ is called a: left self distributive (LSD) if $x+(y+z) = (x+y)+(x+z)$ for all $x,y,z, \in Q$.

Definition 4: Let $(Q,\cdot)$ be any loop. Then $(Q,\cdot)$ is called:

(1) a Moufang loop if and only if the identity $xz \cdot yx = x(zy \cdot x)$ is satisfied in $(Q,\cdot)$ for all $x,y,z \in Q$.
(2) a right Bol loop if and only if the identity $(zx \cdot y)x = z(xy \cdot x)$ is satisfied in $(Q,\cdot)$ for all $x,y,z \in Q$.

Definition 5: The triple (A,B,C) of bijections of a loop $(Q,\cdot)$ is called autotopism, if $xA\cdot yB=(x\cdot y)C,\forall\ x,y\in Q$, such triples form a group $AUT(Q,\cdot)$ called the autotopism group of $(A,\cdot)$. Furthermore, if $A=B=C$ then A is called an automorphism of $(Q,\cdot)$, also such bijections form a group $AUM(Q,\cdot)$ known as automorphism group of $(Q,\cdot)$.

Definition 6: [3] Let $(Q,\cdot)$ be a loop. For all $x,y\in Q$, define $x+y=xy^{-1}\cdot x$. The groupoid $(Q,+)$ is called the core of $(Q,\cdot)$.

Lemma 1: Let $(Q,\cdot)$ be any loop and $\theta\in AUM(Q,\cdot)$, then (i) $L_{x\theta}=\theta^{-1}L_x\theta$ (ii) $R_{x\theta}=\theta^{-1}R_x\theta$ (iii) $M_{x\theta}=\theta^{-1}M_x\theta$.

3. MAIN RESULTS

We now discuss the core of middle Bol loop relative to that of the right Bol loop. But before we do that we give the following lemmas:

Lemma 2: Let $(Q,\cdot)$ be any loop. Then $(Q,\cdot)$ is a middle Bol loop if and only if (IM_x^{-1},IM_x,IM_xL_x) is an autotopism of Q.

Proof: From the identity of a middle Bol loop, the result follows. □

Lemma 3: Let $(Q,\cdot)$ be a middle Bol loop. Then $(A,B,C)\in Atp(Q,\cdot)$ if and only if $(IBI,IAI,ICI)\in Atp(Q,\cdot)$

Proof: Using the fact that, $\forall\,x,y\in Q,(x\cdot y)I=yI\cdot xI$, the result follows. □

Lemma 4: Let $(Q,\cdot)$ be any loop. Then $(Q,\cdot)$ is a middle Bol loop if (i) $L_{(x/z)}=M_x^{-1}R_zM_xL_x$ (ii) $R_{(y\backslash x)}=M_xL_yM_xL_x$ (iii) $M_xL_xM_xL_x=i$ (iv) $M_x^{-1}R_xM_xL_x=i$.

Proof: Using the identity of the middle Bol loop, the results follows. □

Lemma 5: Let $(Q,\cdot)$ be any loop. Then $(Q,\cdot)$ is a middle Bol loop if (i) $x\cdot z\backslash x=x/(x\backslash z)$ (ii) $(zx)\backslash x=x\backslash(y\backslash x)$.

Proof: Using the identity expressions (iii) and (iv) in the Lemma 3 above respectively, we obtain the identities (i) and (ii). □

Lemma 6: Let $(Q,\cdot)$ be the right Bol loop and $(Q,\circ)$ its corresponding middle Bol loop. If $(Q,+)$ is the core of $(Q,\cdot)$ defined for all $x,y\in Q$ as $x+y=xy^{-1}\cdot x$, then $(Q,\oplus)$ is the core of $(Q,\circ)$ correspondingly defined for $x,y\in Q$ as $x\oplus y=x\circ y\backslash\backslash x$, where $\backslash\backslash$ is the left division in $(Q,\circ)$.

Proof: Now applying the translation equation (2) from right Bol loop into middle Bol loop and the Definition 6 of core $(Q,+)$ of right Bol loop given by Belousov [3] and Lemma 5, we obtain for $x,y\in Q$

$$\underbrace{x+y}_{(Q,+)}=\underbrace{xy^{-1}\cdot x}_{(Q,\cdot)}=\underbrace{x//(x\backslash\backslash y)=x\circ y\backslash\backslash x}_{(Q,\circ)}=\underbrace{x\oplus y}_{(Q,\oplus)}$$

where $//$ and $\backslash\backslash$ are respective right and left divisions in the middle Bol loop $(Q,\circ)$. Therefore, the core $(Q,\oplus)$ of middle Bol loop $(Q,\circ)$ relative to that of right Bol loop $(Q,\cdot)$ is given by

$$x\oplus y=x\circ y\backslash\backslash x,\forall\, x,y\in Q. \tag{4}$$

□

Lemma 7: Let $(Q,\cdot)$ be the right Bol loop and $(Q,\circ)$ its corresponding middle Bol loop. If $(Q,*)$ is the principal isotope of $(Q,\cdot)$ defined for all $x,y\in Q$ as $x*y=xR_f\cdot yL_f^{-1}$, for $f\in Q$, then $(Q,\otimes)$ is the principal isotope of $(Q,\circ)$ correspondingly defined for all $x,y\in Q$ as $x\otimes y=(xR_f^{\circ}I\circ yI)R_f^{\circ-1}$, where R° is the right translation map in $(Q,\circ)$

Proof: From Definition 2, the isotope $(Q,*)$ of $(Q,\cdot)$ is defined by $x*y=xR_f\cdot yL_f^{-1},\forall\, x,y\in Q$. Using equation (2) on the isotope of $(Q,\cdot)$ we can obtain the isotope $(Q,\otimes)$ of the middle Bol loop $(Q,\circ)$ relative to that of right Bol loop as follows:

$$\underbrace{x*y}_{(Q,*)}=\underbrace{xR_f\cdot yL_f^{-1}}_{(Q,\cdot)}=\underbrace{(xR_f^{\circ}I\circ yI)IR_f^{\circ-1}}_{(Q,\circ)}=\underbrace{x\otimes y}_{(Q,\otimes)}$$

Thus, the above calculus showed that, the isotope $(Q,\otimes)$ of middle Bol loop $(Q,\circ)$ can be defined relative to the isotope $(Q,*)$ of right Bol loop $(Q,\cdot)$ as

$$x \otimes y = (xR_f^{\circ} I \circ yI)IR_f^{\circ -1}, \forall\, x,y \in Q \tag{5}$$

where R° is the right translation map in $(Q,\circ)$. □

Theorem 1: If $(Q,\cdot)$ is a right Bol loop and $(Q,\circ)$ is the corresponding middle Bol loop with the core $(Q,\oplus)$, then

(i) $(Q,\oplus)$ satisfies $x \oplus (x \oplus y) = y$ – the left symmetric law

(ii) $(Q,\oplus)$ is left distributive if $\theta = M_x L_x$ is an automorphism of Q. That is, the law $x \oplus (y \oplus z) = (x \oplus y) \oplus (x \oplus z)$ is satisfied

Proof:

(i) Using the corresponding core definition for the middle Bol loop relative to that of right Bol loop obtained from the Lemma 3, we have

$$\begin{aligned} x \oplus (x \oplus y) &= x \circ (x \oplus y)\backslash\backslash x \\ &= x \circ [x \circ y\backslash\backslash x]\backslash\backslash x \\ &= yM_x^{\circ} L_x^{\circ} M_x^{\circ} L_x^{\circ} \\ &= yi \qquad \text{using Lemma 4(iii)} \end{aligned}$$

(ii) Using similar approach to (i) above, we have

$$\begin{aligned} x \oplus (y \oplus z) &= x \circ (y \oplus z)\backslash\backslash x \\ &= x \circ [(y \circ z\backslash\backslash y)]\backslash\backslash x \\ &= zM_y^{\circ} L_y^{\circ} M_x^{\circ} L_x^{\circ} \end{aligned} \tag{6}$$

$$\begin{aligned} (x \oplus y) \oplus (x \oplus z) &= (x \circ y\backslash\backslash x) \circ (x \circ z\backslash\backslash x)\backslash\backslash (x \circ y\backslash\backslash x) \\ &= zM_x^{\circ} L_x^{\circ} M_{(x\circ y\backslash\backslash x)}^{\circ} L_{(x\circ y\backslash\backslash x)}^{\circ} \\ &= zM_x^{\circ} L_x^{\circ} M_{(yM_x^{\circ} L_x^{\circ})}^{\circ} L_{(yM_x^{\circ} L_x^{\circ})}^{\circ} \\ &= zM_x^{\circ} L_x^{\circ} (M_x^{\circ} L_x^{\circ})^{-1} M_y^{\circ} M_x^{\circ} L_x^{\circ} (M_x^{\circ} L_x^{\circ})^{-1} L_y^{\circ} M_x^{\circ} L_x^{\circ} \\ \because \theta = M_x^{\circ} L_x^{\circ} \in AUM(Q) &= zM_y^{\circ} L_y^{\circ} M_x^{\circ} L_x^{\circ} \end{aligned} \tag{7}$$

From equations (6) and (7) the result follows. □

Theorem 2: Let $(Q,\cdot)$ be a right Bol loop and let $(Q,\circ)$ be its corresponding middle Bol loop. Let the isotope $(Q,*)$ of middle Bol loop be defined relative to that of right Bol loop such that, for all $x,y \in Q$, $x * y = (xR_f^{\circ} I \circ yI)IR_f^{\circ -1}$. Let T be a permutation of Q and let $(Q,\oplus)$ and $(Q,\otimes)$ be the cores of $(Q,\circ)$ and $(Q,*)$ respectively. Then

$$xT \otimes yT = (x \oplus y)T \tag{8}$$

for all $x, y \in Q$, if and only if

$$[(x \circ y \backslash\backslash x)T \circ f]I = (xT \circ f)I \circ (yT \circ f)I \backslash\backslash (xT \circ f)I \tag{9}$$

for all $x, y \in Q$.

Proof: Using equation (5) obtained from the Lemma 7 for the core of middle Bol loop $(Q, \circ)$ relative to that of the right Bol loop $(Q, \cdot)$, then the core $(Q, \otimes)$ of middle Bol loop $(Q, *)$ is given by

$$x \otimes y = x * y \backslash\backslash^* x \tag{10}$$

Equation (10) implies that

$$x \otimes y = x * p = (xR_f^\circ I \circ pI)IR_f^{\circ -1} \tag{11}$$

where $p = y \backslash\backslash^* x \Rightarrow x = y * p = (yR_f^\circ I \circ pI)IR_f^{\circ -1} \Rightarrow p = (yR_f^\circ I \backslash\backslash xR_f^\circ I)I$. This expression in place of p in equation (11) gives

$$x \otimes y = x * p = (xR_f^\circ I \circ yR_f^\circ I \backslash\backslash xR_f^\circ I)IR_f^{\circ -1} \tag{12}$$

Equation (12) is true if and if

$$[(x \circ y \backslash\backslash x)T \circ f]I = (xT \circ f)I \circ (yT \circ f)I \backslash\backslash (xT \circ f)I \tag{13}$$

□

Corollary 1: The core is an isotopic invariant for middle Bol loop. Furthermore the isotopic middle Bol loops have isomorphic cores.

Proof: From Theorem 2, only the isotopes of $(Q, \cdot)$ where $x \otimes y = (xR_f^\circ I \circ yI)IR_f^{\circ -1}$ is considered, for all $x, y \in Q$. Let $(Q, \oplus)$ and $(Q, \otimes)$ be the cores of $(Q, \circ)$ and $(Q, *)$ respectively. Since $(Q, \circ)$ is a middle Bol loop, then the expression (9) written in terms of translations, gives

$$(x \circ y \backslash\backslash x)TR_f^\circ I = xTR_f^\circ I \circ yTR_f^\circ I \backslash\backslash xTR_f^\circ I. \tag{14}$$

Hence, (9) holds for all $x, y \in Q$ with $\theta = TR_f^\circ I$. So, by Theorem 2 $(Q, \oplus)$ and $(Q, \otimes)$ are isomorphic. □

Remark 1: The cores of middle Bol loops, exhibits some level of semi-automorphism with respect to $\theta = TR_f^\circ I$, as shown in Corollary 1 above.

Corollary 2: A middle Bol loop $(Q, \circ)$ satisfies the identity

$$(x \circ f^{-1} \backslash\backslash x) \circ f = (x \circ f) \circ (x \circ f), \forall\, f, x \in Q \tag{15}$$

if and only if for each principal isotope $(Q, *)$, where $x * y = (xR_f^\circ I \circ yI)R_f^{\circ -1}I$, $x \otimes y = x \oplus y, \forall\, x, y \in Q$. $(Q, \otimes)$ and $(Q, \oplus)$ being cores of

$(Q,*)$ and $(Q,\circ)$ respectively. In particular, A middle Bol loop of order 2 satisfies the Cross Inverse Property.

Proof: If $(Q,\circ)$ satisfies the identity (15), then for all $x,y \in Q, x\otimes y = x\oplus y$ for all principal isotope $(Q,*)$ of $(Q,\circ)$.

Conversely, suppose that $x\otimes y = x\oplus y, \forall\, x,y \in Q$ and all principal isotope $(Q,*)$ of the type as stated above. Then set $T = i$ in (9) to obtain:

$$[(x\circ y\backslash\backslash x)\circ f]I = (x\circ f)I\circ(y\circ f)I\backslash\backslash(x\circ f)I \tag{16}$$

for all $f,x,y \in Q$. In particular, for $y = f^{-1}$ in (16) gives

$$(x\circ f^{-1}\backslash\backslash x)\circ f = (x\circ f)\circ(x\circ f)$$

which is the required identity. □

Theorem 3: Let $(Q,\cdot)$ be a right Bol loop and $(Q,\circ)$ be its corresponding middle Bol loop with the core $(Q,\oplus)$. If

i $(x\oplus y)\circ z = (x\circ z)\oplus(y\circ z)$ and
ii $x\circ(y\oplus z) = (x\circ y)\oplus(x\circ z)$

for all $x,y,z \in Q$, then $(Q,\circ)$ is Moufang.

Proof:

i

$$\begin{aligned} (x\oplus y)\circ z &= (x\circ y\backslash\backslash x)\circ z \quad (17)\\ (x\circ z)\oplus(y\circ z) &= (x\circ z)\circ(y\circ z)\backslash\backslash(x\circ z) \quad (18)\end{aligned}$$

combining (17) and (18) to obtain

$$(x\circ y\backslash\backslash x)\circ z = (x\circ z)\circ(y\circ z)\backslash\backslash(x\circ z) \tag{19}$$

Now set $x = 1$ in (19) to obtain

$$\begin{aligned} yI\circ z &= z\circ(y\circ z)\backslash\backslash z\\ yI &= yR_zM_zL_zR_z^{-1}\\ yI &= yM_zL_z^{-1}L_zR_z^{-1} \because R_zM_z = M_zL_z^{-1} \text{ from Lemma 3(iv)}\\ yI &= yM_zR_z^{-1}\\ yI &= (y\backslash\backslash z)//z\\ y\circ(yI\circ z) &= z\end{aligned}$$

which implies that Q is an inverse property loop, and hence Moufang. Thus, the result follows.

ii Observe that,

$$\begin{aligned} x\circ(y\oplus z) &= (x\circ y)\oplus(x\circ z)\\ \Leftrightarrow\ x\circ(y\circ z\backslash\backslash y) &= (x\circ y)\circ(x\circ z)\backslash\backslash(x\circ y)\end{aligned}$$

for all $x,y,z\in Q$. setting $y=1$ in the last expression gives $(x\circ z)\circ z^{-1}=x$. Which similarly implies Q has inverse property and hence Moufang.

□

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