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BLOCK UNIFICATION ALGORITHM FOR 2D AND 3D ELLIPTIC PDEs

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ABSTRACT. A continuous linear multistep method (LMM) is constructed and used to obtain a block linear multistep method (BLMM) of order 2. The BLMM is then extended on the entire interval of interest and combined as a block unification method to solve elliptic partial differential equations (PDEs)in two and three dimensions via the method of lines. In particular, the method is used to solve elliptic PDE by converting the PDE into a system of ordinary differential equations (ODEs) by replacing one of the spatial derivatives with the central difference method. The stability and convergence properties of the method are discussed. We have tested the accuracy of the BLMM on several numerical examples.

KeyWords: Elliptic PDEs; Method of lines; 2D and 3D equations
2010 Mathematical Subject Classification: 65N06, 65N40

1. INTRODUCTION

In this paper, we consider the following equation in the rectangular region Υ, where $\Upsilon = [a, b] \times [c, d]$ and a, b, c, d are real numbers.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + p(x, y)\frac{\partial u}{\partial x} + q(x, y)\frac{\partial u}{\partial y} + r(x, y)u = g(x, y), \qquad (1)$$

subject to suitable boundary conditions prescribed on its boundary $\partial\Upsilon$. We note that $u(x, y)$ denotes the dependent variable, x and y are spatial variables, $g(x, y)$ is a distributed source, $p(x, y), q(x, y)$, $r(x, y)$ are continuous functions, and when $r(x, y) = 0$, (1) becomes the two-dimensional convection diffusion equation given in Sun and Zhang [14]. The performance of the method is not restricted to (1) as the method can also be applied to 3D problems and problems involving Neumann boundary conditions. The method of lines approach is commonly used for solving partial differential equations

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(PDEs), whereby the PDE is converted into a system of ODEs replacing the appropriate derivatives by finite difference approximations (see Lambert [11], Ramos and Vigo-Aguiar [15], and Brugnano and Trigiante [1]). Our objective is to convert the elliptic PDE into a system of ordinary differential equations (ODE) by replacing one of the spatial derivatives using the central difference method. The resulting system of ODEs is then solved using a BLMM. Specifically, we discretize the x variable such with mesh spacings $\Delta x = (b-a)/M$, $x_m = a + m\Delta x$, $m = 0, 1, \ldots, M$. We then define $\mathbf{u} = [u_1(y), \ldots, u_{M-1}(y)]^T$ and $\mathbf{g} = [g_1(y), \ldots, g_m(y)]^T$, where $u_m(y) \approx u(x_m,\ y)$ and $g_m(y) \approx g(x_m,\ y)$; furthermore we replace the partial derivatives $\frac{\partial^2 u(x,\ y)}{\partial x^2}$ and $\frac{\partial u(x,\ y)}{\partial x}$ occurring in (1) by central difference approximations $\frac{\partial^2 u(x_m,\ y)}{\partial x^2} = \frac{u(x_{m+1},y)-2u(x_m,y)+u(x_{m-1},y)}{(\Delta x)^2}$ and $\frac{\partial u(x_m,\ y)}{\partial x} = \frac{u(x_{m+1},y)-u(x_{m-1},y)}{(2\Delta x)}$, $m = 1, \ldots, M-1$.
The problem (1) then leads to the resulting semi-discrete problem

$$\frac{d^2 u_m}{dy^2} = -\frac{(u_{m+1} - 2u_m + u_{m-1})}{(\Delta x)^2} - \frac{p(x_m, y)(u_{m+1} - u_{m-1})}{(2\Delta x)} - q(x_m, y)\frac{du_m}{dy} + g_m, \tag{2}$$

which can be written in the form

$$\mathbf{u}'' = \mathbf{f}(y,\ \mathbf{u},\ \mathbf{u}'), \tag{3}$$

subject to the boundary conditions $\mathbf{u}(c) = \mathbf{u}_0,\ \mathbf{u}(d) = \mathbf{u}_M$ or $\mathbf{u}'(c) = \mathbf{u}'_0,\ \mathbf{u}'(d) = \mathbf{u}'_M$, where for the special case $\mathbf{f}(y, \mathbf{u}) = \mathbf{A}\mathbf{u} + \mathbf{g}$, $\mathbf{A}$ is a matix of dimension $M-1 \times M-1$, or for the general case $\mathbf{f}(y, \mathbf{u}, \mathbf{u}') = \mathbf{A}\mathbf{U} + \mathbf{g}$, $U = (u, u')^T$, T is the transpose, $\mathbf{A}$ is a matrix of dimension $2M-1 \times 2M-1$, both arising from the semi-discretized system (2) which is expressed in the form (3) and solved by the BLMM.

The paper is organized as follows. In section two, we derive a continuous LMM which is used to formulate the BLMM. The computational aspects of the method is given in section three. Numerical examples are given in section four to show the accuracy of the method. Finally, the conclusion of the paper is discussed in section five.

2. CONTINUOUS LMM and BLMM

We propose a BLMM for (3) in which on the partition $\prod_N$, $h > 0, y_n = y_0 + nh$, $n = 0, 1, \ldots, N$, the two-step $[y_n, u_n, u'_n] \mapsto [y_{n+2} =$

$y_n + 2h, u_{n+2}, u'_{n+2}]$ is given by the equations

$$\begin{cases} \alpha_{0,1}u_n + \alpha_{1,1}u_{n+1} + \alpha_{2,1}u_{n+2} = h^2(\beta_{0,1}f_n + \beta_{1,1}f_{n+1} + \beta_{2,1}f_{n+2}), \\ h\alpha'_{0,2}u'_n + \alpha_{1,2}u_{n+1} = h^2(\beta_{0,2}f_n + \beta_{1,2}f_{n+1} + \beta_{2,2}f_{n+2}), \\ h\alpha'_{1,3}u'_{n+1} + \alpha_{1,3}u_{n+1} + \alpha_{0,3}u_n = h^2(\beta_{0,3}f_n + \beta_{1,3}f_{n+1} + \beta_{2,3}f_{n+2}), \\ h\alpha'_{1,4}u'_{n+2} + \alpha_{1,4}u_{n+1} + \alpha_{0,4}u_n = h^2(\beta_{0,4}f_n + \beta_{1,4}f_{n+1} + \beta_{2,4}f_{n+2}), \end{cases} \tag{4}$$

where $\prod_N : c = y_0 < y_1 < y_2 < \ldots < y_N = d$, and $\alpha'_{0,2}$, $\alpha'_{1,3}$, $\alpha'_{1,4}$, $\alpha_{i,j}$, $\beta_{i,j}$, $i = 0, 1, 2,\ j = 1, \ldots, 4$ are coefficients that are uniquely determined. We note that u_{n+i} denote the numerical approximation to the analytical solution $u(y_{n+i})$, $f_{n+i} = f(y_{n+i}, u_{n+i}, u'_{n+i})$, $i = 0, 1, 2$. In order to determine the coefficients of (4), we derive a continuous method based on the two-step method of Richtmyer and Morton (LMM) [12], since it was shown by Dahlquist [4] to be the most accurate unconditionally stable LMM. Thus, on the interval $[y_n, y_n+2h]$, we approximate the exact solution by the interpolating function $u(y)$ of the form

$$u(y) = \sum_{j=0}^{3} \ell_j y^j, \tag{5}$$

where ℓ_j are parameters to be uniquely determined. We impose that the interpolating function (5) coincides with the analytical solution at the points y_{n+i}, $i = 0, 1$ and satisfies the scalar form of the differential equation (3) at the points $y_{n+i}, i = 0, 1, 2$ to obtain the following of five equations:

$$\begin{cases} \ell_0 + \ell_1 y_n + \ell_2 y_n^2 + \ell_3 y_0^3 = u_n, \\ \ell_0 + \ell_1 y_{n+1} + \ell_2 y_{n+1}^2 + \ell_3 y_{n+1}^3 = u_{n+1}, \\ 2\ell_2 + 6\ell_3 y_n = f_n + \eta \top_2^*(y_n), \\ 2\ell_2 + 6\ell_3 y_{n+1} = f_{n+1} + \eta \top_2^*(y_{n+1}), \\ 2\ell_2 + 6\ell_3 y_{n+2} = f_{n+2} + \eta \top_2^*(y_{n+2}) \end{cases} \tag{6}$$

where the perturbation term involves η as a parameter with $\top_m^*(y_{n+j})$ obtained from the shifted Chebychev's polynomial, $\top_m^*(y)$ of degree $m = 2$. We note that the perturbation term is included to ensure that the continuous method produces the RMM as a by-product since it is unconditionally stable (see Dahlquist [4]). The shifted the Chebyshev's polynomial $\top_m^*(y)$ of degree $m = 2$ is obtained from the Chebychev's polynomial of degree m given by

$\top_m(\xi) = \cos\{m\ arc\ \cos\xi\},\ \xi \,\epsilon\, [-1, 1], m = 0, 1, 2, \ldots,$

which can also be defined by the recurrence relation $T_0(\xi) = 1$, $T_1(\xi) = \xi$, $T_{m+1}(\xi) = 2\xi T_m(\xi) - T_{m-1}(\xi)$. The shifted Chebyshev's polynomials $\top^*_m(y)$ is obtained by transforming the Chebychev's polynomials $\top_m(\xi)$ defined on $[-1, 1]$ to the interval $[y_n, y_{n+2}]$ (see Johnson and Riess [9]). It is easily shown that after a simple algebraic computation $\xi = \frac{2(y-y_n)}{y_{n+2}-y_n} - 1$. Thus, we define the shifted Chebyshev's polynomials of degree m on the interval $[y_n, y_{n+2}]$ as

$\top^*_m(y) = \top_m(\xi) = \cos[m \ \cos^{-1}(\frac{2(y-y_n)}{y_{n+2}-y_n} - 1)]$.

Thus, the system of (6) is solved with the aid of Mathematica to obtain ℓ_j and the perturbation parameter η. The continuous LMM and its first derivative are constructed by substituting the values of ℓ_j into equation (5) to give

$$\begin{cases} \alpha_0(y)u_n + \alpha_1(y)u_{n+1} + h^2(\beta_0(y)f_n + \beta_1(y)f_{n+1} + \beta_2(y)f_{n+2}), \\ u'(y) = \frac{d}{dy}(u(y)), \end{cases} \tag{7}$$

where $\alpha_0(y)$, $\alpha_1(y)$, $\beta_j(y)$, $j = 0, 1, 2$ are continuous coefficients.

2.1 BLMM and its block extension

The coefficients given in (4) are specified by evaluating (7) at $y = y_{n+2}$ and $y = \{y_n, y_{n+1}, y_{n+2}\}$ to give

$$\begin{cases} u_{n+1} = u_n - hu'_n - \frac{h^2}{24}(-7f_n - 6f_{n+1} + f_{n+2}), \\ u_{n+2} - 2u_{n+1} = u_n + \frac{h^2}{4}(f_n + 2f_{n+1} + f_{n+2}), \\ hu'_{n+1} - u_{n+1} = -u_n + \frac{h^2}{24}(5f_n + 6f_{n+1} + f_{n+2}), \\ hu'_{n+2} - u_{n+1} = -u_n + \frac{h^2}{24}(5f_n + 18f_{n+1} + 13f_{n+2}). \end{cases} \tag{8}$$

Remark 1: We note that the method (8) is locally obtained on $[y_n, y_{n+2}]$ and can be applied to solve time-dependent problems in a block-by-block fashion. However, the semi-discrete problem (3) must be solved simultaneously over the whole interval $[c, d]$; in which case, for $n = 0, 2, \ldots, N-2$, the BLMM (8) is used to generate an extended block global method which can then solve problem (3).

2.2 Convergence analysis

The order of each method in (8) is given by the vector $p = (2, 3, 2, 2)^T$

and local truncation errors associated with (8) are given by

$$\begin{cases} \tau_{i+1} = \frac{-1}{45}h^5u^{(5)}(y_i+\theta_i)+O(h^6), \\ \tau_{i+2} = \frac{-1}{6}h^4u^{(4)}(y_i+\theta_i)+O(h^5), \\ h\tau'_{i+1} = -\frac{1}{12}h^4u^{(4)}(y_i+\theta_i)+O(h^5), \\ h\tau'_{i+2} = \frac{-1}{6}h^4u^{(4)}(y_i+\theta_i)+O(h^5),\ i=0,2,\ldots,N-2,\ |\theta_i|\leq 1. \end{cases} \tag{9}$$

The method (8) can be expressed in block form as

$$A_0\mathbf{V}_\mu = A_1\mathbf{V}_{\mu-1} + h^2B_1\mathbf{F}_{\mu-1} + h^2B_0\mathbf{F}_\mu,\ \mu=1,\ldots\ \Gamma,\ n=0,2,\ldots\ N-2, \tag{10}$$

where the positive integer $\Gamma = N/k$ is the number of blocks, $k = 2$ is the step number, $\mathbf{V}_\mu = (u_{n+1}, u_{n+2}, hu'_{n+1}, hu'_{n+2})^T$, $\mathbf{F}_\mu = (f_{n+1}, f_{n+2}, hf'_{n+1}, hf'_{n+2})^T$, $\mathbf{V}_{\mu-1} = (u_{n-1}, u_n, hu'_{n-1}, hu'_n)^T$, $\mathbf{F}_{\mu-1} = (f_{n-1}, f_n, hf'_{n-1}, hf'_n)^T$, and A_0, A_1, B_0, and B_1 are matrices each of dimension 4 whose entries are given by the coefficients of (8).

Let the local truncation error be defined by $Ł(h) = (\tau_{i+2}, h\tau'_i, h\tau'_{i+1}, h\tau'_{i+2})^T, i = 0, 2, \ldots, N-2$,
and let the exact form of the system is given by (10) be defined as

$$A_0\overline{\mathbf{V}}_\mu = A_1\overline{\mathbf{V}}_{\mu-1} + h^2B_1\overline{\mathbf{F}}_{\mu-1} + h^2B_0\overline{\mathbf{F}}_\mu + Ł(h), \tag{11}$$
$$\mu = 1,\ldots\ \Gamma,\ n = 0,2,\ldots\ N-2,$$

where
$\overline{\mathbf{V}}_\mu = ((u(y_{n+1}), u(y_{n+2}), hu'(y_{n+1}), hu'(y_{n+2}))^T$,
$\overline{\mathbf{F}}_\mu = (f(y_{n+1}, u(y_{n+1}), u'(y_{n+1})), f(y_{n+2}, u(y_{n+2}), u'(y_{n+2}))$,
$hf'(y_{n+1}, u(y_{n+1}), u'(y_{n+1})), hf'(y_{n+2}, u(y_{n+2}), u'(y_{n+2}))^T$,
$\overline{\mathbf{V}}_{\mu-1} = (u(y_{n-1}), u(y_n), hu'(t_{n-1}), hu'(y_n))^T$,
$\overline{\mathbf{F}}_\mu = (f(y_{n-1}, u(y_{n-1}), u'(y_{n-1})), f(y_n, u(y_n), u'(y_n))$,
$hf'(y_{n-1}, u(y_{n-1}), u'(y_{n-1})), hf'(y_n, u(y_n), u'(y_n))^T$.

Theorem 1: Let $\mathbf{V}_\mu$ be an approximation of the solution vector $\overline{\mathbf{V}}_\mu$ for the system obtained on the partition $\aleph_N$ from the method (10). If $e_i = |u(y_i) - u_i|$, $he'_i = |hu'(y_i) - hu'_i|$, where the exact solution $u(y)$ is several times differentiable on $[c, d]$ and if $\|E\| = \|\mathbf{V}_\mu - \overline{\mathbf{V}}_\mu\|$, then, the BLMM is convergent of order 2, which implies that $\|E\| = O(h^2)$.

Proof. See Jator [8].

□

2.3 Stability of the BLMM

The linear-stability of the BLMM is discussed by applying the method to the test equation $u'' = \lambda u$, where λ is expected to run through the (negative) eigenvalues of the Jacobian matrix $\frac{\partial f}{\partial u}$ (see

Sommeijer [13]). Letting $q = \lambda h^2$, it is easily shown that the application of (10) to the test equation yields

$$\mathbf{V}_\mu = M(q)\mathbf{V}_{\mu-1}\ , M(q) := (A_0 - qB_0)^{-1}(A_1 + qB_1), \qquad (12)$$

where the matrix $M(q)$ is the amplification matrix which determines the stability of the method.

Definition 1: The interval $[-q_0, 0]$ is the stability interval, if in this interval $\rho(q) \leq 1$, where $\rho(q)$ is the spectral radius of $M(q)$ and q_0 is the stability boundary (see [13]).

Remark 2: We found that $\rho(q) \leq 1$ if $q\ \epsilon\ [-\infty, -4] \bigcup [-3, 0]$

3. COMPUTATIONAL ASPECTS

We begin by converting (1) into (3) by discretizing π_M, given by

$$\pi_M := \{a = x_0 < x_1 < \ldots < x_M = b,\ x_m = x_{m-1} + \Delta x\},$$

where $\Delta x = \frac{b-a}{M}$ is a constant step-size of the partition of π_M, $m = 1, 2, \ldots, M$, M is a positive integer and m the grid index. The resulting system of ODEs (3) is then solved on the partition $\prod_N$. We emphasize the block unification of (10) lead to a single matrix of finite difference equations, which is solved to provide all the solutions of (3) on the entire grid given by the rectangle $[a, b] \times [c, d]$.

Step 1: Use the block unification of (10) for $\mu = 1, n = 0$ to obtain $\mathbf{V}_1$ on the rectangle $[y_0, y_2] \times [a, b]$, for $\mu = 2, n = 2$, $\mathbf{V}_2$ is obtained on the rectangle$[y_2, y_4] \times [a, b]$, and on the rectangles $[y_4, y_6] \times [c, d]$, $\ldots, [y_{N-2}, y_N] \times [c, d]$, for $\mu = 3, \ldots, \Gamma$, $n = 4, 8 \ldots, N-2$, we obtain $\mathbf{V}_3 \ldots, \mathbf{V}_\Gamma$.

Step 2: Solve unified block given by the system $\mathbf{V}_1 \bigcup \mathbf{V}_2 \bigcup \ldots \bigcup \mathbf{V}_{\Gamma-1} \bigcup \mathbf{V}_\Gamma$ obtained in step 1, noting that $u_m(y_n) \approx u(x_m,\ y_n)$, $m = 1, 2, \ldots, M, n = 1, 2, \ldots, N$.

Step 3: The solution of (1) is approximated by the solutions in step 2 as $\mathbf{u}(y_n) = [u(x_1, y_n), \ldots, u(x_M, y_n)]^T, n = 1, 2, \ldots, N$.

BLMM is implemented in a block unification fashion using a Mathematica 10.0 code, enhanced by the feature $NSolve[\ ]$ for linear problems, while nonlinear problems were solved using the Newton's method enhanced by the feature $FindRoot[\ ]$ (see Keiper and Gear [10]). The implementation is summarized in Algorithm 1.

Algorithm 1 Block Unification Algorithm

procedure ENTER PARTITIONS($\pi_M, \prod_N, variables$)
    For $\mu = 1, \ldots, \Gamma$, $n = 0, 2, \ldots, N-2$, generate $\mathbf{V}_1$, $\mathbf{V}_2$, $\ldots$, $\mathbf{V}_\Gamma$.
    System $= \mathbf{V}_1 \bigcup \mathbf{V}_2 \bigcup \ldots \bigcup \mathbf{V}_{\Gamma-1} \bigcup \mathbf{V}_\Gamma$.
    $NSolve[System, variables]$ ▷ If the system is linear
    $FindRoot[System, variables]$ ▷ If the system is nonlinear
    $u_m(y_n) \approx u(x_m,\ y_n)$, $m = 1, 2, \ldots, M, n = 1, 2, \ldots, N$.
    $\mathbf{u}(y_n) = [u(x_1, y_n), \ldots, u(x_M, y_n)]^T, n = 1, 2, \ldots, N$.
end procedure

4. NUMERICAL EXAMPLES

In this section, the performance of the BLMM is tested on selected problems from the literature. The results given by the BLMM is compared to the well known finite difference method (FDM) and other methods given in the literature. In each example, it is demonstrated numerically or graphically that the BLMM is superior in terms of accuracy. The global error is given by $Error = Max|u(x_m, y_n) - u_m(y_n)|)$, $m = 1, 2, \ldots, M, n = 1, 2, \ldots, N$.

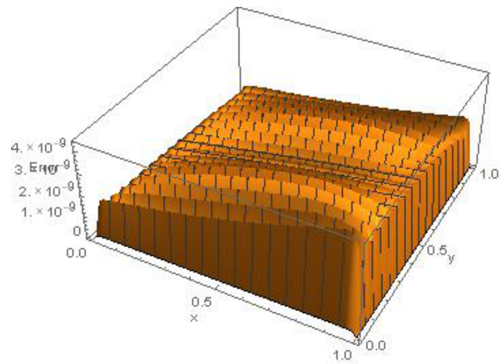
Example 4.1: We solve the given Laplace equation (see Xu and Wang [17])

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 \le x \le 1,\ 0 \le y \le 1,$$

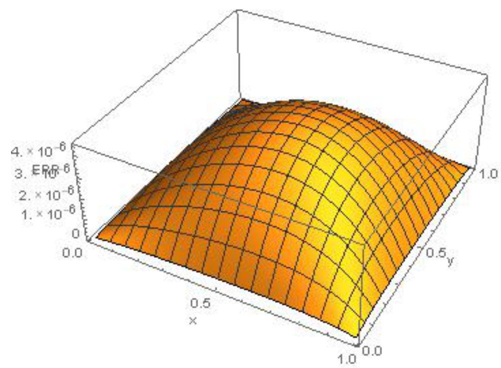
subject to boundary conditions $u(x, y) = e^x \cos(y)$ on the boundary of the domain.
The exact solution is given by $u(x, y) = e^x \cos(y)$.

TABLE 1. Errors for Example 4.1

	Xu and Wang [17]	BLMM
$M = N = 64$	Error	Error
16	3.90×10^{-5}	1.28×10^{-6}
24	1.74×10^{-5}	2.68×10^{-7}
32	9.77×10^{-6}	8.76×10^{-8}
40	6.26×10^{-6}	3.67×10^{-8}
48	4.35×10^{-6}	1.80×10^{-8}



(a) BLMM



(b) FDM

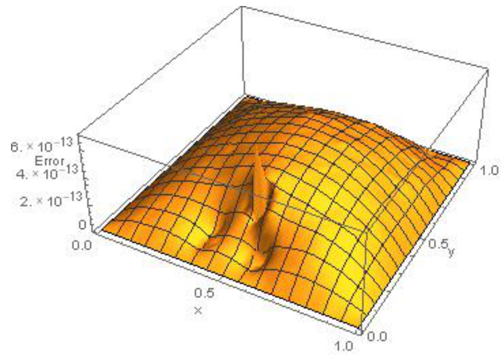
FIGURE 1. Errors for Example 4.1, $h = 1/64, \Delta x = 1/64$.

Example 4.2: We solve the given Poisson equation (see Xu and Wang [17])

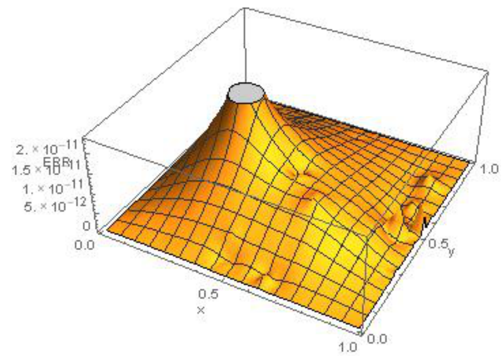
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2(3x + x^2 + y^2),\ 0 \le x \le 1,\ 0 \le y \le 1,$$

subject to boundary conditions $u(x, y) = x^2(x + y^2) + 2$

The exact solution is given by $u(x, y) = x^2(x + y^2) + 2$.



(a) BLMM



(b) FDM

FIGURE 2. Approximate and exact solutions for Example 4.2, $h = 1/128, \Delta x = 1/128$.

Example 4.3: We solve the given PDE to Dirichlet boundary conditions (see Volkov et al. [6])

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2\pi(2\pi y^2 - 2\pi y - 1)e^{\pi(1-y)} \sin(\pi x),$$

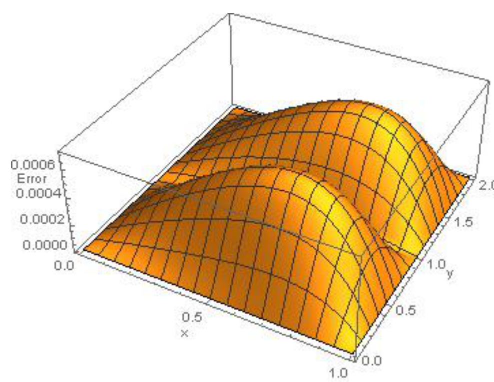
$u(x, 0) = u(x, 1),\ 0 \leq x \leq 1,$

The exact solution is given by

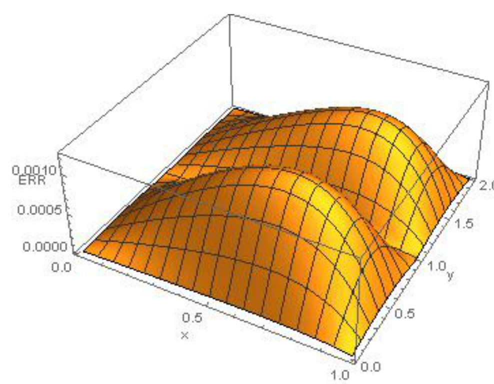
$$u(x, y) = e^{\pi x} \sin(\pi y) + e^{\pi(1-y)y} \sin(\pi x).$$

TABLE 2. Errors for Example 4.3

	BLMM	Volkov et al. [6]
$M = N$	Err	Err
16	1.286×10^{-3}	3.266×10^{-2}
32	3.153×10^{-4}	8.210×10^{-3}
64	7.913×10^{-5}	2.053×10^{-3}
128	1.975×10^{-5}	5.128×10^{-4}



(a) BLMM



(b) FDM

FIGURE 3. Errors for Example 4.3, $h = 1/128, \Delta x = 1/128$.

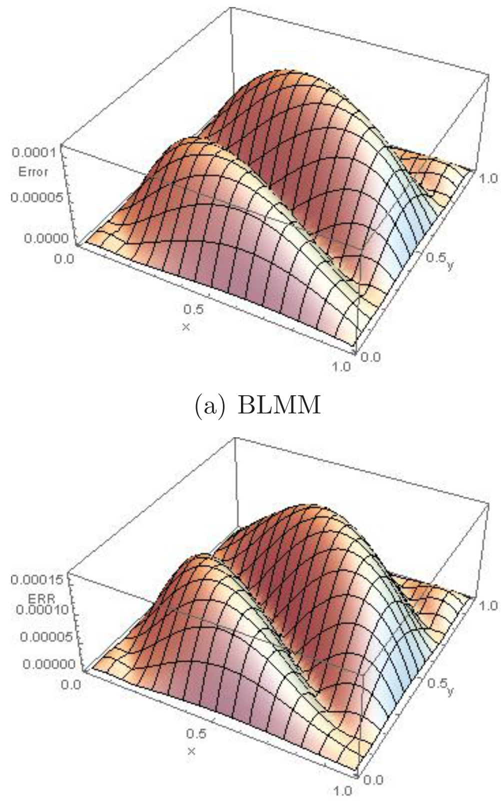
Example 4.4: We consider the given two-dimensional convection diffusion equation (see Sun and Zhang [14]).

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+p(x,y)\frac{\partial u}{\partial x}+q(x,y)\frac{\partial u}{\partial y}=g(x,y)\ 0<x<1,\ 0<y<1,$$

The exact solution is given by $u(x, y) = \cos[4x + 6y]$ and the convection coefficients are $p(x, y) = 10x(x - 1)(1 - 2y)$ and $q(x, y) = -10y(y-1)(1-2x)$. The Dirichlet boundary conditions and $g(x, y)$ are chosen accordingly.

TABLE 3. Errors for Example 4.4

	FDM ($p = 2$)	BLMM ($p = 2$)
$M = N$	Err	Err
4	1.85×10^{-1}	2.47×10^{-1}
8	4.36×10^{-2}	3.69×10^{-2}
16	1.07×10^{-2}	7.64×10^{-3}
32	2.65×10^{-3}	1.79×10^{-3}
64	6.62×10^{-4}	4.41×10^{-4}
128	1.66×10^{-4}	1.10×10^{-4}



(a) BLMM

(b) FDM

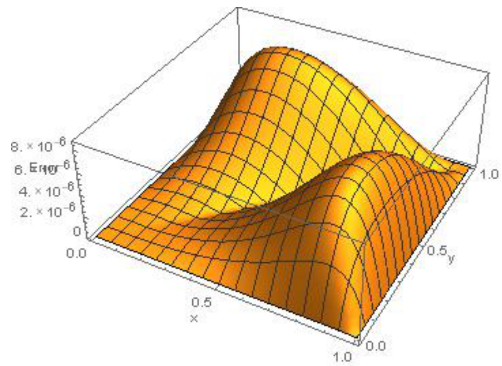
FIGURE 4. Error for Example 4.4, $h = 1/128, \Delta x = 1/128$.

Example 4.5: We consider the given two-dimensional Helmoltz equation (see Sun and Chenney [14]).

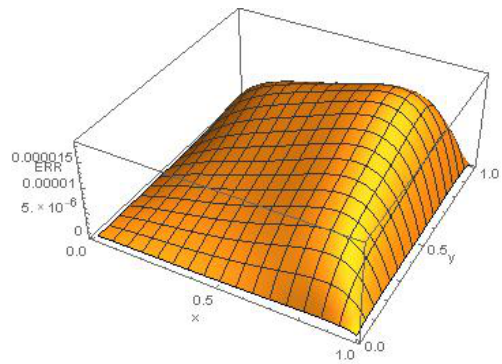
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + 25u = 0,\ 0 < x < 1,\ 0 < y < 1,$$

The exact solution is given by $u(x, y) = \frac{1}{2\cosh[5]}(\cosh[5x]+\cosh[5y])$.

The Dirichlet boundary conditions are chosen accordingly. This example was chosen to demonstrate that the EBNUM can be used to solve the Helmoltz equation. The results produced by the EBNUM are accurate as shown by the graphical evidence given in Figure 4.



(a) BLMM



(b) FDM

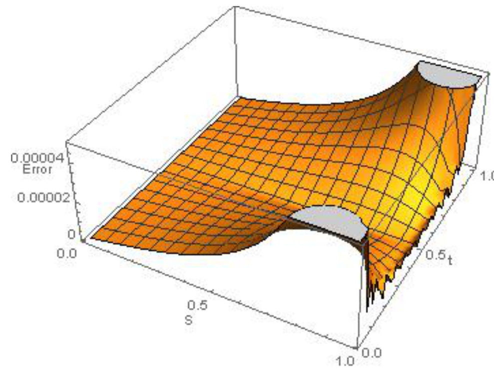
FIGURE 5. Errors for Example 4.5, $h = 1/128, \Delta x = 1/128$.

Example 4.6: We solve the given PDE (1) to Neumann boundary conditions (see Zill and Cullen [16])

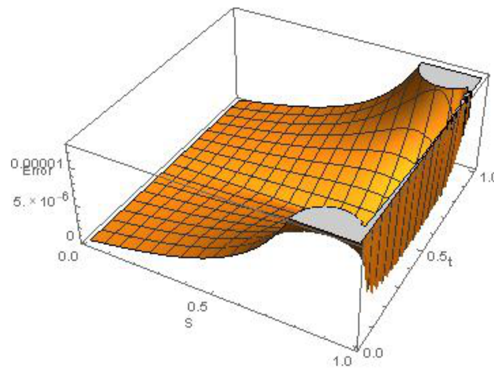
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,\ 0 < x < 1,\ 0 < y < 1,$$

$$u(0, y) = 0,\ u(1, y) = 1 - y,\ \frac{\partial u}{\partial y}|_{y=0} = 0,\ \frac{\partial u}{\partial y}|_{y=1} = 0.$$

The exact solution is given by $u(x, y) = \frac{x}{2} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{1-(-1)^n}{n^2 \sinh(n\pi)} \sinh(n\pi x) \cos(n\pi y)$.



(a) BLMM: M=N=64



(b) BLMM: M=N=128

FIGURE 6. Errors for Example 4.6.

Example 4.7: We consider the given 3D poisson equation (see Zhang [18]).

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=\Theta(x,y,z),\ 0\le x\le 1,\ 0\le y\le 1,\ 0\le z\le 1,$$

where $\Theta(x,y,z)=3\pi^2\sin(\pi x)\sin(\pi y)\sin(\pi z)$. The exact solution is given by $u(x,y,z)=\sin(\pi x)\sin(\pi y)\sin(\pi z)$ and the initial and boundary conditions are chosen accordingly.

TABLE 4. Errors for Example4.7

	BLMM	Zhang [18]
$M = N$	Err	Err
4	2.21×10^{-2}	5.30×10^{-2}
8	4.56×10^{-3}	1.30×10^{-2}
16	1.09×10^{-3}	3.22×10^{-3}
32	2.69×10^{-4}	8.04×10^{-4}

Example 4.8: We consider the following one-dimensional nonlinear Sine-Gordon equation given in Dehghan and Shokri [5])

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} - \sin(u), \ -3 < x < 3, \ 0 < t < 1,$$

$$u(x,0) = 4\arctan(e^{\frac{x}{\sqrt{1-C^2}}}), \ u_t(x,0) = -\frac{4Ce^{\frac{x}{\sqrt{1-C^2}}}}{\sqrt{1-C^2}(1+e^{2\frac{x}{\sqrt{1-C^2}}})}.$$

The exact solution is given by $u(x,t) = 4\arctan(sech(x)t)$, C is the velocity of the solitary wave, and the boundary conditions are given according. The problem was solved for $C = 0.5$, $\Delta t = 0.125$, and $\Delta x = 0.04$. In order to solve this PDE using the BLMM, we carry out the semi-discretization of the spatial variable x using the second order finite difference method to obtain the following second order system in the second variable t.

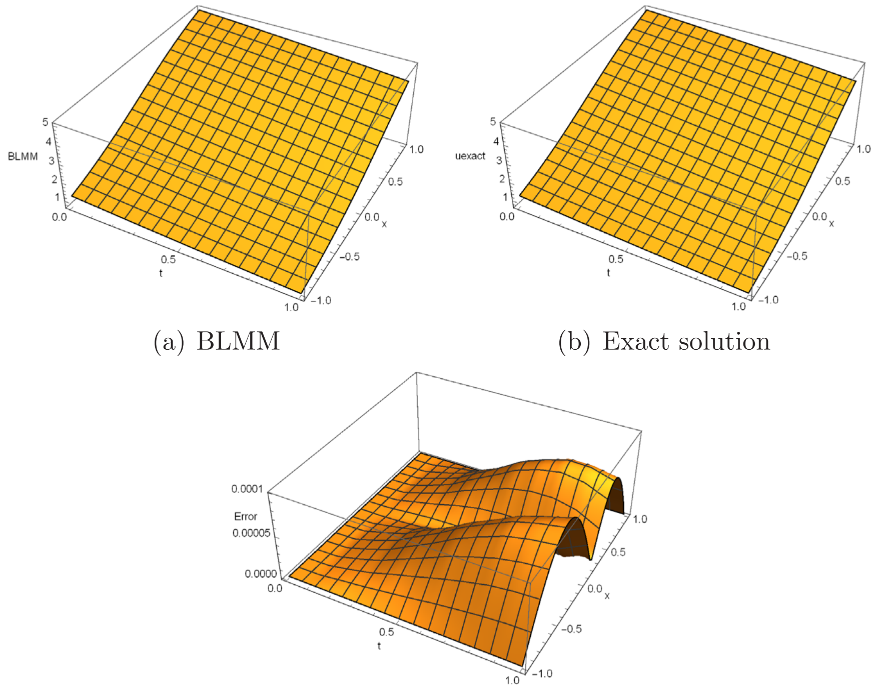
$$\begin{cases} \frac{\partial^2 u_m}{\partial t^2} - \frac{(u_{m+1} - 2u_m + u_{m-1})}{(\Delta x)^2} = g_m, \ 0 < t < 1, =, \ m = 1, \dots, M-1, \\ u(x_m, 0) = u_m, \ u_t(x_m, 0) = u'_m, \end{cases} \tag{13}$$

where $\Delta x = (b-a)/M$, $x_m = a + m\Delta x$, $m = 0, 1, \dots, M$, $\mathbf{u} = [u_1(t), \dots, u_M(t)]^T$, $\mathbf{g} = [g_1(t), \dots, g_m(t)]^T$, $u_m(t) \approx u(x_m, \ t)$ and $g_m(t) \approx g(x_m, \ t) = \sin(u_m)$, which can be written in the form

$$\mathbf{u}'' = \mathbf{f}(t, \ \mathbf{u}), \tag{14}$$

subject to the boundary conditions $\mathbf{u}(t_0) = \mathbf{u}_0$, $\mathbf{u}'(t_0) = \mathbf{u}'_0$ where $\mathbf{f}(t, \ \mathbf{u}) = \mathbf{Au} + \mathbf{g}$, and $\mathbf{A}$ is an $M - 1 \times M - 1$, matrix arising from the semi-discretized system and $\mathbf{g}$ is a vector of constants.

The results produced by the BLMM are presented in Figure 7 and show that the method can also cope with nonlinear PDEs.



(a) BLMM (b) Exact solution

(c) Errors

FIGURE 7. Graphical evidence for Example 4.7

5. CONCLUSION

In this paper, we have developed and implemented a BLMM based on a block unification strategy which is used to solve elliptic PDEs in 2D and 3d via the method of lines. The results given in the Section 4 show that the approach can be competitive with existing methods in the literature. Our future research will be to search for higher order LMMs for elliptic PDEs including a study of the conditioning of the matrices arising from the semi-discretization of the PDEs.

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