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CHEBYSHEV WAVELET COLLOCATION METHOD FOR THE NUMERICAL SOLUTION OF ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. Wavelet analysis is relatively new developed mathematical tool for many problems. Wavelets permit the accurate representation of a variety of functions and operators. More over wavelets establish a connection with fast numerical algorithms. In this paper, Chebyshev wavelet collocation method is developed for the numerical solution of differential equations. Numerical examples are presented to verify the efficiency and accuracy of the proposed algorithm.

Keywords and phrases: Wavelets, Chebyshev wavelet collocation method, Differential equations.
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1. INTRODUCTION

In the past few years, the study of solution of differential equations has been fascinated the interest of many mathematicians and physicists. Many methods including numerical and perturbation have been used to solve differential equations. From the literature, it is observed that solving the differential equations representing physical phenomenon, is the most challenging task and needs huge efforts to handle the various problems. The preferred approach to solve them is to articulate the solution as a linear combination of so-called basis functions. These basis functions can for instance be plane waves, splines or finite elements. In the recent past, authors have worked with the finite elements and B-splines.
Wavelets have become a powerful tool for having applications in almost all the science and engineering field. The focus of wavelets has recently drawn a great deal of attention from mathematical scientists in various disciplines. Wavelets theory is a relatively new

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and an emerging area in mathematical research. Many families of wavelets have been proposed in the mathematical literature. In which, most simple one is the Haar wavelet. Haar wavelet has been used by many researchers because of its simplicity and better convergence. Some of the work regarding Haar wavelet can be found in [1-10]. The weaker side of using the Haar basis functions for approximating smooth functions is that they are lower in accuracy due to their non-smooth character. To cover this aspect, smooth Chebyshev wavelet [11, 12] is considered to get more accurate approximation. Chebyshev polynomials have many applications in numerical computations, interpolation, series truncation, etc. Since from many years, the connection between orthogonal polynomials and wavelet analysis has been explored. Chebyshev wavelet uses the Chebyshev polynomials as bases. Because of their improved smoothness and good interpolating properties, accuracy of Chebyshev wavelet is better than Haar wavelets.

The main objective of this paper is to develop Chebyshev wavelet collocation method (CWCM) to solve linear and nonlinear ordinary differential equations. The method consists of reducing the differential equation to a set of algebraic equations by first expanding the Chebyshev wavelet with unknown coefficients. By solving these coefficients, we get the required solution. Here we demonstrate the method by considering the some of the test problems.

The paper is organized as follows; Section 2 is devoted to the Preliminaries on wavelets. Method of solution is discussed in section 3. Numerical examples are presented in section 4. Section 5 deals with the concluding remarks of the paper.

2. PRELIMINARIES

2.1. Wavelets

Recently, wavelets have been applied extensively for signal processing in communications and physics research, and have proved to be a wonderful mathematical tool. Wavelets can be used for algebraic manipulations in the system of equations obtained which leads to better resulting system. Wavelets constitute a family of functions constructed from dilation and translation of a single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets;

$$\psi_{a,b}(t) = \frac{1}{\sqrt{|a|}}\psi\left(\frac{t-b}{a}\right), \quad a, b \in R, a \neq 0 \tag{1}$$

The best way to understand wavelets is through a multi-resolution analysis. Given a function $f \in L_2(R)$ a multi-resolution analysis (MRA) of $L_2(R)$ produces a sequence of subspaces $V_j, V_{j+1}, \ldots,$ such that the projections of f onto these spaces give finer and finer approximations of the function f as $j \to \infty$.
A multi-resolution analysis of $L_2(R)$ is defined as a sequence of closed subspaces $V_j \subset L_2(R), j \in Z$ with the following properties

(i) $\ldots \subset V_{-1} \subset V_0 \subset V_1 \subset \ldots$

(ii) The spaces V_j satisfy $\cup_{j\in Z} V_j$ is dense in $L_2(R)$ and $\cap_{j\in Z} V_j = 0$.

(iii) If $f(t) \in V_0$ then $f(2^j t) \in V_j$, i. e the spaces V_j are scaled versions of the central space V_0.

(iv) If $f(t) \in V_0$ then $f(2^j t - m) \in V_j$, i. e all the V_j are invarient under translation.

(v) There exists $\phi \in V_0$ such that $\phi(t-m); m \in Z$ is a Riesz basis in V_0

The space V_j is used to approximate general functions by defining appropriate projection of these functions onto these spaces. Since the union of all the V_j is dense in $L_2(R)$, so it guarantees that any function in $L_2(R)$ can be approximated arbitrarily close by such projections. As an example the space $\{V_j, j \in Z\}$ can be defined like

$$V_j = W_j \oplus V_{j-1} = W_{j-1} \oplus W_{j-2} \oplus V_{j-2} = \ldots = \oplus_{j=1}^{J+1} W_j \oplus V_0$$

For each j the space W_j serves as the orthogonal complement of V_j in V_{j+1}. The space W_j include all the functions in V_{j+1} that are orthogonal to all those in V_j under some chosen inner product. The set of functions which form basis for the space W_j are called wavelets.

2.2. Chebyshev wavelets and operational matrix of integration

Here, we presented a family of wavelets, called Chebyshev wavelets, which are derived from Chebyshev polynomials. For any positive integer k, the Chebyshev wavelets family is defined on the interval

[0, 1)[11] as follows;

$$C_{n,m}(t) = \begin{cases} \frac{\alpha_m 2^{k/2}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & for\ \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ \\ 0, & otherwise \end{cases} \tag{2}$$

where $n = 1, 2, \ldots, 2^k$ and $m = 0, 1, \ldots, M-1$, M is the maximum order of the Chebyshev polynomial and $\alpha_m = \begin{cases} \sqrt{2}, & m = 0 \\ \\ 2, & otherwise. \end{cases}$

Here $T_m(t)$ are the well known Chebyshev polynomials of order m. Chebyshev polynomials can be calculated recursively with the help of the following equations;

$$T_0(t) = 1, T_1(t) = t,\ T_{m+1}(t) = 2tT_m(t) - T_{m-1}(t), m = 1, 2, 3, \ldots$$

Equivalently, for any positive integer k, the Chebyshev wavelets family is defined as follows;

$$C = C_i(t) = \begin{cases} \frac{\alpha_m 2^{k/2}}{\sqrt{\pi}} T_m(2^{k+1}t - 2n + 1), & for\ \frac{n-1}{2^k} \le t < \frac{n}{2^k} \\ \\ 0, & otherwise \end{cases} \tag{3}$$

where $i = n + 2^k m$. By varying the values of i with respect to the collocation points $t_j = \frac{j-0.5}{N}, j = 1, 2, \ldots, N$, we get the Chebyshev matrix of order $N \times N$, where $N = 2^k M$ and Chebyshev polynomials used in the approximation are of degree less than M.
The integration of Chebyshev wavelet is given as

$$\int_0^t C(t)dt = PC(t) = P_1 \tag{4}$$

$$\int_0^t PC(t)dt = P^2C(t) = P_2 \tag{5}$$

and in general

$$\int_0^t P^{n-1}C(t)dt = P^nC(t) = P_n,\ n > 0 \tag{6}$$

where P is the $N \times N$ operational matrix for integration and is given as

$$P = \begin{pmatrix} D & U & U & \dots & U \\ 0 & D & U & \dots & U \\ 0 & 0 & \ddots & \ddots & U \\ \vdots & \vdots & \ddots & D & U \\ 0 & 0 & \dots & 0 & D \end{pmatrix}$$

where U and D are $M \times M$ matrices given by

$$U = \frac{\sqrt{2}}{2^k} \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{3} & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ -\frac{1}{15} & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ -\frac{1}{M(M-2)} & 0 & 0 & \dots & 0 \end{pmatrix}$$

and

$$D = \frac{1}{2^k} \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{2}} & 0 & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{4\sqrt{2}} & 0 & \frac{1}{8} & 0 & \dots & 0 & 0 & 0 \\ -\frac{1}{3\sqrt{2}} & -\frac{1}{4} & 0 & \frac{1}{12} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ -\frac{1}{2\sqrt{2}(M-1)(M-3)} & 0 & 0 & 0 & \dots & -\frac{1}{4(M-3)} & 0 & \frac{1}{4(M-1)} \\ -\frac{1}{2\sqrt{2}(M)(M-2)} & 0 & 0 & 0 & \dots & 0 & -\frac{1}{4(M-3)} & 0 \end{pmatrix}$$

3. CHEBYSHEV WAVELET COLLOCATION METHOD OF SOLUTION

Consider the second order differential equation of the form

$$y'' + ay' + by = f(t) \tag{7}$$

with the initial conditions $y(0) = \alpha, y'(0) = \beta$
where a & b are dependent or independent variables or constants, $f(t)$ is a non homogeneous function and α & β are real constants.
Now the method is as follows,
Step 1: Let us assume that

$$y'' = \sum_{i=1}^{N} a_i C_i(t) \tag{8}$$

where $a_i, i = 1, 2, \dots, N$ are the Chebyshev wavelet coefficients to be determined.

Step 2: Integrating (8) twice with respect to the given condition we get,

$$y' = \beta + \sum_{i=1}^{N} a_i P_{1i}(t) \tag{9}$$

and

$$y = \alpha + \beta t + \sum_{i=1}^{N} a_i P_{2i}(t) \tag{10}$$

Step 3: Substituting the values of (8)-(10) in (7) then we have,

$$\sum_{i=1}^{N} a_i C_i(t) + a\left(\beta + \sum_{i=1}^{N} a_i P_{1i}(t)\right) + b\left(\alpha + \beta t + \sum_{i=1}^{N} a_i P_{2i}(t)\right) = f(t) \tag{11}$$

Step 4: Solving (11), we get Chebyshev wavelet coefficients a_i, substituting these a_i in (10), we obtain the solution of the problem (7). The error will be calculated by using $E = |y_e - y_a|$ and $E_{max} = max|y_e - y_a|$, where y_e & y_a are exact and approximate solution respectively.

The convergence analysis of the Chebyshev wavelet is given through the following Lemma,

Lemma: Assume that the $y(t) \in L_2(R)$ with the bounded first derivative on (0, 1), then the error norm at k^{th} level satisfies the following inequality $\|e_k(t)\| \leq A2^{-(3/2)(N/2)}$, where $A = \sqrt{\frac{K}{7}}C$ is some real constant.

Proof: The error at k^{th} level may be defined as,

$$|e_k(t)| = |y(t) - y_k(t)| = |\sum_{i=N+1}^{\infty} a_i C_i(t)|$$

where

$$y_k(t) = \sum_{i=1}^{N=2^{k+1}} a_i C_i(t)$$

$$\|e_k(t)\|^2 = \int_{-\infty}^{\infty} \left\langle \sum_{i=N+1}^{\infty} a_i C_i(t), \sum_{l=N+1}^{\infty} a_l C_l(t) \right\rangle dt$$

$$= \sum_{i=N+1}^{\infty} \sum_{l=N+1}^{\infty} a_i a_l \int_{-\infty}^{\infty} C_i(t) C_l(t) dt$$

$$\|e_k(t)\|^2 \leq \sum_{i=N+1}^{\infty} |a_i|^2$$

But

$$|a_i| \leq C2^{-\frac{3i}{2}} max|y'(t)|$$

where

$$C = \int_0^1 |tC_2(t)|dt,\ t \in \left(\frac{n-1}{2^k}, \frac{n}{2^k}\right)$$

. Then

$$\|e_k(t)\|^2 \leq \sum_{i=N+1}^{\infty} KC^2 2^{-3i}$$

where $|y'(t)| \leq K$, $\forall\ t \in (0,1)$, where K is positive constant.

$$\|e_k(t)\|^2 \leq KC^2 \frac{1}{7} 2^{-3(N/2)}$$

$$\|e_k(t)\| \leq \sqrt{\frac{K}{7}} C2^{-(3/2)(N/2)}$$

$\|e_k(t)\| \leq A2^{-(3/2)(N/2)}$, where $A = \sqrt{\frac{K}{7}}C$ is some real constant.

From the above lemma, the error bound is inversely proportional to the level of the resolution of the Chebyshev wavelets. This ensures that the convergence of the Chebyshev wavelet approximation by increasing the level of resolution.

Rate of convergence $R_c(N)$:
The rate of convergence is defined as $R_c(N) = \frac{log(E_{max}(N/2)/(E_{max}(N))}{log2}$.

4. NUMERICAL EXPERIMENTS

In this section, we consider some of the examples to demonstrate the applicability of the proposed method.

Test problem 1. First consider the equation [13],

$$ty'' + y' + ty = 0 \tag{12}$$

with the initial conditions $y(0) = 1, y'(0) = 0$
Let us assume that

$$y'' = \sum_{i=1}^{N} a_i C_i(t) \tag{13}$$

where $a_i, i = 1, 2, \ldots, N$ are the Chebyshev wavelet coefficients to be determined.
Integrating (13) twice with respect to the given condition we get,

$$y' = y'(0) + \sum_{i=1}^{N} a_i P_{1i}(t) \tag{14}$$

and

$$y = 1 + \sum_{i=1}^{N} a_i P_{2i}(t) \tag{15}$$

Substituting the values of (13)-(15) in (12) then we have,

$$t\left(\sum_{i=1}^{N} a_i C_i(t)\right) + \left(\sum_{i=1}^{N} a_i P_{1i}(t)\right) + t\left(1 + \sum_{i=1}^{N} a_i P_{2i}(t)\right) = 0 \tag{16}$$

By solving (16), we get Chebyshev wavelet coefficients a_i=[-4.2773e-01, 1.4478e-02, 3.5516e-03, -2.3540e-05, -8.5639e-06, 3.3414e-06, -1.2491e-06, 7.1445e-07, -3.4923e-01, 4.0442e-02, 2.8330e-03, -8.7498e-05, -3.0734e-06, 6.7520e-08, -3.4026e-11, 2.4824e-10] for N=16. Substituting these a_i in (15), we obtain the solution (CWCM) of the problem (12) and is presented in comparison with exact solution which is known as the Bessel function of the zero order [14], $y(t) = J_0(t) = \sum_{q=0}^{N} \frac{(-1)^q}{(q!)^2} \left(\frac{t}{2}\right)^{2q}$ and Haar wavelet collocation method (HWCM) (as the method explained in [9, 10]) solution in Table 1 for N=8 (M=4 & k=1), Table 2 for N=16 (M=8 & k=1) & Fig. 1 for N=32 (M=8 & k=2). The error analysis for higher values of N is given in Table 3.

Table 1. Comparison of numerical solutions with exact solution for N=8 of Test problem 1.

t	RHFM [13]	HWCM	CWCM	Exact
0.0	1.0	1.0	1.0	1.0
0.1	0.99747	0.99750	0.99749	0.99750
0.2	0.99000	0.99003	0.99001	0.99003
0.3	0.97762	0.97763	0.97761	0.97760
0.4	0.96038	0.96040	0.96039	0.96040
0.5	0.93845	0.93847	0.93847	0.93847
0.6	0.91200	0.91200	0.91200	0.91201
0.7	0.88118	0.88120	0.88119	0.88120
0.8	0.84627	0.84629	0.84628	0.84629
0.9	0.80750	0.80752	0.80752	0.80752
1.0	0.76518	0.76519	0.76518	0.76520

Table 2. Comparison of numerical solutions with exact solution for N=16 of Test problem 1.

$t(= 1/32)$	HWCM	CWCM	Exact
1	0.99975591	0.99974113	0.99975587
3	0.99780398	0.99778925	0.99780394
5	0.99390583	0.99389116	0.99390579
7	0.98807288	0.98805831	0.98807283
9	0.98032219	0.98030775	0.98032216
11	0.97067646	0.97066217	0.97067643
13	0.95916389	0.95914978	0.95916388
15	0.94581813	0.94580424	0.94581814
17	0.93067816	0.93066773	0.93067820
19	0.91378816	0.91377839	0.91378825
21	0.89519738	0.89518825	0.89519754
23	0.87495996	0.87495148	0.87496020
25	0.85313477	0.85312694	0.85313512
27	0.82978521	0.82977805	0.82978569
29	0.80497901	0.80497254	0.80497966
31	0.77878800	0.77878225	0.77878885

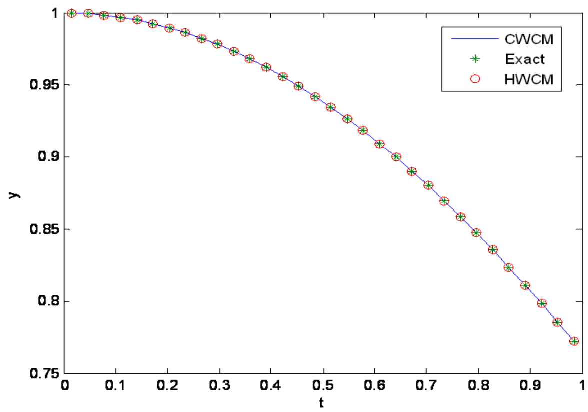

Fig. 1. Comparison of numerical solutions with exact solution for N=32 of Test problem 1.

Table 3. Error analysis of Test problem 1.

M	k	N	E_{max}(HWCM)	E_{max}(CWCM)	Rate of convergence $R_c(N)$ HWCM	Rate of convergence $R_c(N)$ CWCM
8	1	16	8.5105e-07	1.4741e-05	-	-
8	2	32	2.3152e-07	9.4554e-07	1.8781	3.9626
8	3	64	5.9981e-08	5.9477e-08	1.9486	3.9907
8	4	128	1.5243e-08	3.7233e-09	1.9764	3.9977
8	5	256	3.8408e-09	2.3280e-10	1.9887	3.9994
8	6	512	9.6392e-10	1.4551e-11	1.9944	3.9999

Test problem 2. Now consider another linear equation [13],

$$y'' + 2ty' = 0 \tag{17}$$

with the initial conditions $y(0) = 0, y'(0) = \frac{2}{\sqrt{\pi}}$
Using the method explained in section 3, we get the Chebyshev wavelet coefficients a_i=[-4.2891e-01, -2.7902e-01, 2.8381e-02, 3.7904e-03, -3.5667e-04, -2.5188e-05, 2.5283e-06, 1.0918e-07, -8.0593e-01, 1.7658e-02, 3.4446e-02, -2.4630e-03, -2.4303e-04, 2.8453e-05, 6.3348e-07, -1.6602e-07] for N=16. Using these coefficients, we obtained the CWCM solution and is presented in comparison with HWCM (as the method explained in [9, 10]) solution and exact solution $y(t) = \frac{2}{\sqrt{\pi}} \int_0^t e^{-x^2} dx$ in Table 4 for N=8 (M=4 & k=1), Table 5 for N=16 (M=8 & k=1) & Fig. 2 for N=32 (M=8 & k=2). The error analysis for higher values of N is given in Table 6.

Table 4. Comparison of numerical solutions with exact solution for N=8 of Test problem 2.

t	RHFM [13]	HWCM	CWCM	Exact
0.0	0	0	0	0
0.1	0.11244	0.11217	0.11238	0.11246
0.2	0.22268	0.22214	0.22248	0.22270
0.3	0.32861	0.32783	0.32836	0.32863
0.4	0.42837	0.42739	0.42817	0.42839
0.5	0.52047	0.51935	0.52031	0.52050
0.6	0.60384	0.60260	0.60363	0.60386
0.7	0.67779	0.67648	0.67752	0.67780
0.8	0.74208	0.74074	0.74186	0.74210
0.9	0.79689	0.79554	0.79679	0.79691
1.0	0.84269	0.84132	0.84245	0.84270

Table 5. Comparison of numerical solutions with exact solution for N=16 of Test problem 2.

$t(= 1/32)$	HWCM	CWCM	Exact
1	0.03522748	0.03511946	0.03525037
3	0.10540829	0.10532205	0.10547644
5	0.17477305	0.17470729	0.17488488
7	0.24279871	0.24275136	0.24295170
9	0.30899290	0.30896124	0.30918372
11	0.37290452	0.37288533	0.37312919
13	0.43413281	0.43412257	0.43438691
15	0.49233460	0.49232967	0.49261347
17	0.54722945	0.54722058	0.54752844
19	0.59860272	0.59862871	0.59891738
21	0.64630646	0.64636185	0.64663270
23	0.69025821	0.69033803	0.69059246
25	0.73043797	0.73053786	0.73077729
27	0.76688353	0.76699984	0.76722566
29	0.79968453	0.79981432	0.80002789
31	0.82897545	0.82911651	0.82931915

Table 6. Error analysis of Test problem 2.

M	k	N	E_{max}(HWCM)	E_{max}(CWCM)	Rate of convergence $R_c(N)$ HWCM	CWCM
8	1	16	3.4369e-04	3.0786e-04	-	-
8	2	32	8.5965e-05	2.0952e-05	1.9993	3.8771
8	3	64	2.1493e-05	1.3367e-06	1.9999	3.9703
8	4	128	5.3735e-06	8.3873e-08	1.9999	3.9943
8	5	256	1.3434e-06	5.2434e-09	2.0000	3.9996
8	6	512	3.3585e-07	3.2761e-10	2.0000	4.0005

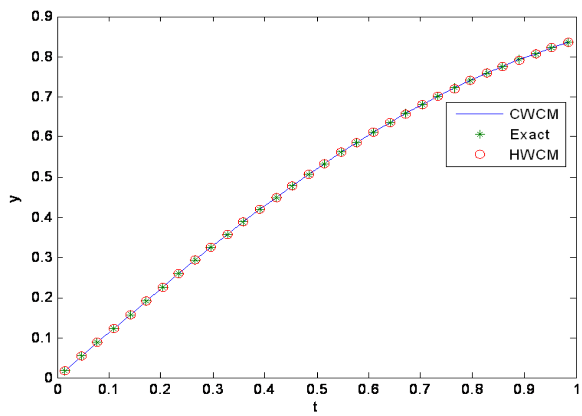

Fig. 2. Comparison of numerical solutions with exact solution for N=32 of Test problem 2.

Test problem 3. Next consider the non homogeneuos equation [4],

$$y'' + y' = sint + tcost \tag{18}$$

with the initial conditions $y(0) = 1, y'(0) = 1$
As in the previous examples, we get Chebyshev wavelet coefficients a_i=[-6.5204e-01, 1.6522e-01, -2.3848e-03, -1.9848e-03, 2.1932e-05, 3.3819e-06, -2.4529e-08, -2.1695e-09, -3.0793e-01, 6.1689e-02, -2.2022e-02, -1.1444e-03, 7.9077e-05, 2.1288e-06, -7.6517e-08, -1.4250e-09] for N=16. Using these coefficients, we obtained the CWCM solution and is presented in comparison with exact solution $y(t) = cost + \frac{5}{4}sint + \frac{1}{4}(t^2 sint - tcost)$ and HWCM (as the method explained in [9, 10]) solution in Table 7 for N=16 (M=8 & k=1) & Fig. 3 for N=32 (M=8 & k=2). Absolute error calculated and compared with [3, 4] and HWCM is shown in Table 8. Excellent agreement is found in the results so obtained. The error analysis for higher values of N is given in Table 9.

Table 7. Comparison of numerical solutions with exact solution for N=16 of Test problem 3.

$t(=1/32)$	HWCM	CWCM	Exact
1	1.03077719	1.03078300	1.03076684
3	1.08952687	1.08952417	1.08949571
5	1.14475187	1.14474034	1.14469969
7	1.19671638	1.19669592	1.19664324
9	1.24568785	1.24565861	1.24559410
11	1.29193225	1.29189460	1.29181852
13	1.33570941	1.33566394	1.33557658
15	1.37726844	1.37721595	1.37711766
17	1.41684338	1.41686402	1.41667605
19	1.45464900	1.45465929	1.45446673
21	1.49087685	1.49087787	1.49068148
23	1.52569157	1.52568458	1.52548513
25	1.55922754	1.55921395	1.55901221
27	1.59158576	1.59156713	1.59136390
29	1.62283125	1.62280924	1.62260531
31	1.65299065	1.65296703	1.65276319

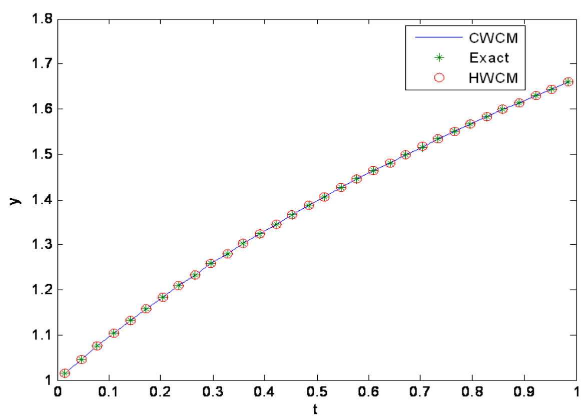

Fig. 3. Comparison of numerical solutions with exact solution for N=32 of Test problem 3.

Table 8. Calculated absolute error for Test problem 3.

t	1/32	7/32	13/32	19/32	25/32	31/32
Ref. [3]	0.000010	0.000201	0.001559	0.010914	0.001132	0.010545
Ref. [4]	0.000010	0.00020	0.001443	0.010902	0.001021	0.010442
HWCM	0.000010	0.000073	0.000132	0.000182	0.000215	0.000227
CWCM	0.000016	0.000052	0.000087	0.000192	0.000201	0.000203

Table 9. Error analysis of Test problem 3.

M	k	N	E_{max}(HWCM)	E_{max}(CWCM)	Rate of convergence $R_c(N)$ HWCM	CWCM
8	1	16	2.2746e-04	2.0393e-04	-	-
8	2	32	5.6881e-05	1.3882e-05	1.9996	3.8768
8	3	64	1.4223e-05	8.8981e-07	1.9997	3.9636
8	4	128	3.5560e-06	5.5525e-08	1.9999	4.0023
8	5	256	8.8903e-07	3.4592e-09	2.0000	4.0046
8	6	512	2.2225e-07	2.1582e-10	2.0000	4.0025

Test problem 4. Finally, consider the nonlinear non homogeneuos equation (Van Der Pol equation),

$$y'' + (1 + y^2)y' + y = -sint - sintcos^2t \tag{19}$$

with the initial conditions $y(0) = 1, y'(0) = 0$
Let us assume that

$$y'' = \sum_{i=1}^{N} a_i C_i(t) \tag{20}$$

where $a_i, i = 1, 2, \dots, N$ are the Chebyshev wavelet coefficients to be determined.
Integrating (20) twice with respect to the given condition we get,

$$y' = y'(0) + \sum_{i=1}^{N} a_i P_{1i}(t) \tag{21}$$

and

$$y = 1 + \sum_{i=1}^{N} a_i P_{2i}(t) \tag{22}$$

Substituting the values of (20)-(22) in (19) then we have,

$$\left(\sum_{i=1}^{N} a_i C_i(t)\right) + \left(1 + \left(1 + \sum_{i=1}^{N} a_i P_{2i}(t)\right)^2\right)\left(\sum_{i=1}^{N} a_i P_{1i}(t)\right)$$

$$+\left(1+\sum_{i=1}^{N} a_i P_{2i}(t)\right) = -sint - sintcos^2t \quad (23)$$

By solving (23), we get the Chebyshev wavelet coefficients a_i=[-8.2001e-01, 6.4380e-02, 1.8632e-02, 6.9939e-04, -2.4365e-04, -2.5380e-06, 1.0502e-06, 1.0538e-08, -4.5197e-01, 1.8563e-01, 6.2173e-03, -2.3145e-03, -6.8639e-05, 1.6865e-05, 2.6586e-07, -5.5992e-08] for N=16. Substituting these a_i in (22), we obtain the CWCM solution of the problem (19) and is presented in comparison with HWCM (as the method explained in [9, 10]) solution and exact solution $y(t) = cost$ in Table 10 for N=16 (M=8 & k=1) & Fig. 4 for N=32 (M=8 & k=2). The error analysis for higher values of N is given in Table 11.

Table 10. Comparison of numerical solutions with exact solution for N=16 of Test problem 4.

$t(=1/32)$	HWCM	CWCM	Exact
1	0.999497	0.999431	0.999511
3	0.995451	0.995525	0.995608
5	0.987193	0.987735	0.987817
7	0.974569	0.976103	0.976169
9	0.957474	0.960697	0.960709
11	0.935849	0.941614	0.941497
13	0.909679	0.918982	0.918609
15	0.878998	0.892961	0.892133
17	0.843882	0.863803	0.862174
19	0.804456	0.831622	0.828848
21	0.760886	0.796706	0.792285
23	0.713387	0.759325	0.752629
25	0.662212	0.719764	0.710033
27	0.607655	0.678324	0.664665
29	0.550044	0.635311	0.616702
31	0.489736	0.591034	0.566330

Table 11. Error analysis of Test problem 4.

M	k	N	E_{max}(HWCM)	E_{max}(CWCM)	Rate of convergence $R_c(N)$ HWCM	Rate of convergence $R_c(N)$ CWCM
8	1	16	7.6593e-02	2.4704e-02	-	-
8	2	32	7.9091e-02	2.6372e-02	-0.0463	-0.0943
8	3	64	8.0345e-02	2.7253e-02	-0.0227	-0.0474
8	4	128	8.0973e-02	2.7704e-02	-0.0112	-0.0237
8	5	256	8.1287e-02	2.7931e-02	-0.0056	-0.0118
8	6	512	8.1444e-02	2.8045e-02	-0.0028	-0.0059

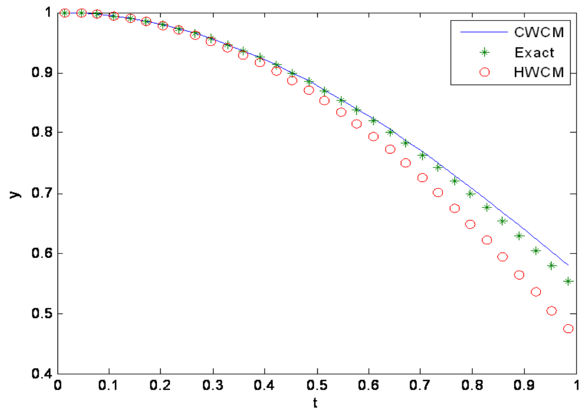

Fig. 4. Comparison of numerical solutions with exact solution for N=32 of Test problem 4.

5. CONCLUSIONS

In this paper, we proposed a simple and straightforward numerical method based on Chebyshev wavelet for solving differential equations. By observing the error analysis, the performance of Chebyshev wavelet is superior to the other (Haar wavelet solution) which is justified through the illustrative examples. The main advantage of this method is its simplicity and small computation costs with faster convergence. Hence, the present method is a very reliable, simple, fast, flexible, and convenient method.

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NOMENCLATURE

E	Absolute error
E_{max}	Maximum of absolute error
$R_c(N)$	Rate of convergence

REFERENCES

[1] U. Lepik, *Application of the Haar wavelet transform to solving integral and differential Equations*, Proc. Estonian Acad Sci. Phys. Math. **56** (1) 28-46, 2007.

[2] S. Islam, I. Aziz, B. Sarler, *The numerical solution of second-order boundary-value problems by collocation method with the Haar wavelets*, Math. Comput. Model. **52** 1577-1590, 2010.

[3] P. Chang, P. Piau, *Haar wavelet matrices designation in numerical solution of ordinary differential equations*, Int. J. Appl. Math. **38** (3) 11, 2008.

[4] S. Dhawan, S. Arora, S. Kumar, *Wavelet based numerical scheme for differential equations*, Int. J. of Differential Equations and Applications **12** (2) 85-94, 2013.

[5] N. M. Bujurke, C. S. Salimath, S. C. Shiralashetti, *Numerical solution of stiff systems from nonlinear dynamics using single-term Haar wavelet series*, Nonlinear Dyn. **51** 595-605, 2008.

[6] N. M. Bujurke, C. S. Salimath, S. C. Shiralashetti, *Computation of eigenvalues and solutions of regular SturmLiouville problems using Haar wavelets*, J. Comput. Appl. Math. **219** 90-101, 2008.

[7] N. M. Bujurke, S. C. Shiralashetti, C. S. Salimath, *An application of single-term Haar wavelet series in the solution of nonlinear oscillator equations*, J. Comput. Appl. Math. **227** 234-244, 2009.

[8] S. C. Shiralashetti, A. B. Deshi, *An efficient Haar wavelet collocation method for the numerical solution of multi-term fractional differential equations*, Nonlinear Dyn. **83** 293-303, 2016.

[9] S. C. Shiralashetti, A. B. Deshi, P. B. Mutalik Desai, *Haar wavelet collocation method for the numerical solution of singular initial value problems*, Ain Shams Engg. J. **7** 663-670, 2016.

[10] S. C. Shiralashetti, P. B. Mutalik Desai, A. B. Deshi, *Comparison of Haar Wavelet Collocation and Finite Element Methods for Solving the Typical Ordinary Differential Equations*, Int. J. Basic Sciences and Applied Computing **1** (3) 1-11, 2015.

[11] M. T. Kajani, A. H. Vencheh, M. Ghasemi, *The Chebyshev wavelets operational matrix of integration and product operation matrix*, Int. J. Comp. Math. **86** (7) 1118-1125, 2009.

[12] M. Arsalani, M. A. Vali, *Numerical solution of nonlinear variational problems with moving boundary conditions by using Chebyshev wavelets*, Appl. Math. Sci. **5** (20) 947-964, 2011.

[13] M. Razzaghi, Y. Ordokhani, *Solution of differential equations via rationalized Haar Functions*, Math. Comput. Simulat. **56** 235-246, 2001.

[14] P. V. O'Neil, *Advanced Engineering Mathematics*, Wadsworth Publishing Company, 1987.

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