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CRITICALITY AND THERMAL EXPLOSION IN THE FLOW OF REACTIVE VISCOUS THIRD GRADE FLUID FLOW IN A CYLINDRICAL PIPE WITH SURFACE COOLING

HAMMED OGUNSEYE AND S. S. OKOYA[1]

ABSTRACT. This work investigates the steady state momentum and heat transfer in a fully developed flow of viscous third-grade fluid through a cylindrical pipe, such that the general spatial dependence of viscosity is accounted for. Expression for the velocity profile is constructed analytically and displayed graphically. Comparison with previous documented result through reduction in emergent parameter demonstrates the accuracy of the solution approach. The energy equation with generalized Arrhenius kinetics is solved for thermal explosion using the Modified Adomian Decomposition semi analytical method. Comparisons are given for sets of results obtained via the Modified Adomian Decomposition Methods against those from other well known methods in the literature. The present study explores the effects of the Biot number Bi $\in \{0, \infty\}$ on critical values of emerging parameters (Frank-Kamenetskii parameter, δ_{cr} and the dimensionless maximum temperature, $\theta_{max\ cr}$). The present parametric examination of the effects of Biot number, inhomogeneity of the fluid, heat generating parameter, activation energy parameter (β) and the exponent of the generalized Arrhenius kinetic on the δ_{cr} and $\theta_{max\ cr}$ provides quantitative properties for the sensitized reaction kinetics.

Keywords and phrases: Third-grade fluid, heat generation, thermal transition, spatially varying viscosity, Modified Adomian Decomposition Method
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1. INTRODUCTION

It is imperative to note that the study of the motion of reactive non-Newtonian fluids with space dependent viscosity has applications

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in physics and engineering. A few examples of thermal explosion in liquid or gas flows with symmetric boundary conditions are in the impact testing of liquid explosives as recorded in [25], [27], [20] and the references contained there-in. Analytical and numerical studies have been undertaken which contribute to better understanding of the qualitative properties of steady reactive fluid flows. These properties include thermal explosion, extinction and transition amongst others.

The studies in the preceding paragraph have considered symmetric thermal boundary conditions. For asymmetric thermal boundary conditions obeying the cooling law, the following studies are relevant: Classical models for stationary thermal explosion with surface cooling have been investigated by Thomas [21], Boddington et al [8] and [9], Britz et al. [10], Okeremeta [19] among others. In a recent study, Makinde [16] investigated an incompressible viscous reactive third-grade fluid flow through a cylindrical pipe under Arrhenius kinetics with constant viscosity using a special type of the Hermite-Padé approximation technique. Furthermore, Chinyoka and Makinde [11] computationally investigated a reactive third grade fluid flow in a cylindrical pipe with convective cooling on its surface while Reynolds' model viscosity and generalized Arrhenius kinetics are considered. More recently, Lebelo and Makinde [13] numerically addressed the effects of various thermo-physical parameters on the temperature for a combustible reactive material in a cylindrical pipe with convective and radiative heat loses at the surface. They have considered the constant viscosity model in their analysis.

Various forms of the fluid viscosity model varying as a function of the spatial coordinates have been investigated in literature. Anand and Rajagopal [6] studied non-Newtonian fluids with shear dependent viscosity assuming a quadratic dependence on the spatial coordinate based on some experimental observation to understand the flow properties. Massoudi and Vaidya [17] found solutions for suggested linear space dependence and exponentially decaying space function in unsteady flows of inhomogeneous incompressible laminar fluids. Further studies by Massoudi and Vaidya [18] examined Stokes-type flow of an inhomogeneous incompressible laminar viscous fluid, where the viscosity depends spatially according to three models and obtained exact solutions in terms of special functions, the results were compared with the cases of constant viscosity as well as the slow flow regimes. Recently, analytical and numerical results were presented by Fiordilino et al. [12] for non-homogenous viscous fluid whose viscosity depends on spatial coordinate and temperature in a poiseuille device. Quite recently, Zhou et al. [28] investigated the influence of a shear-rate-dependent viscosity based on Krieger's viscosity - concentration correlation for unsteady one-dimensional flow of non-Newtonian fluid in an annulus.

The spatial variation of the effective viscosity of non-Newtonian fluids is not only important in the theoretical analysis of flow of such fluids, its physical implication is also shown in the experimentally motivated model for self expanding polyurethane foams [14].

Relatively few attempts have been made on the study of heat and mass transfer in pipe flow of non-Newtonian fluid with space dependent viscosity. In [14], Hayat et al. obtained analytical solutions for the velocity and temperature profiles via the homotopy analysis method (HAM) for the model governing the flow of the viscous fluid in a pipe for constant viscosity and variable space dependent viscosity with symmetric surface temperature. A further study on the flow of reactive viscous fluids with spatially distributed viscosity was presented by Ajadi [5] for the Arrhenius kinetics. He employed the HAM to obtain the velocity profile for the flow of a third-grade fluid in the pipe which was substituted into the viscous dissipation term in the nonlinear heat equation to obtain the temperature distribution in the domain. The resulting equation with symmetric boundary conditions was then tackled with variational technique to study the critical Frank-Kamenetskii parameter and the critical maximum temperature.

HAM is one of the analytical techniques employed in the literature to tackle fluid flow and heat transfer problems. Another widely used method is the Adomian Decomposition Method (ADM). Most of the semi - analytical solutions to the Newtonian equations involving Adomian Decomposition Method are given in a textbook [23] and in the review articles [24], [4] and Biyadi [7]. For some complex and non-linear fluids, the technique of Adomian Decomposition Method has been used for reactive fluid flows successfully by Adesanya et al. [2] and the references therein. The situation of employing the Modified Adomian Decomposition Method (MADM) are contained in [1], [3] and the references contained therein. So, a new discussion is required for a similar problem in the preceding paragraph with asymmetric thermal boundary condition using MADM.

Consequently, this study extends the work of Zaturska [27], Hayat et al. [14] and Ajadi [5] to accommodate the generalized Arrhenius kinetics as well as the convective cooling and seek the transitional values for a third grade fluid flow with spatially varying viscosity through a pipe of circular cross section. The governing equations for the conservation of momentum and energy are non-dimensionalized. Using the MADM as a solution technique, the flow velocity in the pipe is obtained analytically and with graphical illustrations, we present the results. Subsequently, the velocity gradient is substituted into the energy equation which is also resolved analytically via the MADM. We therefore investigate the phenomenon of criticality for convective cooling i.e. all Biot numbers, Bi, from the Semenov limit (Bi $\to 0$) to the Frank - Kamenetskii limit (Bi $\to \infty$) as well as the influence of the viscous heating parameter and exponent of the pre-exponential factor. Graphical representations of computed values for both criticality and transition (disappearance of criticality) of the Frank-Kamenetskii parameter and the maximum temperature are given. To validate the model, the computed results are compared with earlier results in the literature and the two sets show good agreement.

2. PROBLEM SET UP

We investigate the fully developed laminar flow of an incompressible non-Newtonian fluid with spatially varying viscosity through a circular cylinder of radius a and adopt the cylindrical coordinate system $(\bar{r}, 0, \bar{z})$, where the $\bar{z}$-axis lies along the centerline of the pipe and $\bar{r}$ is transverse to it. Furthermore, we assume one-dimensional laminar flow with constant thermal conductivity, constant pressure gradient along the pipe but space dependent viscosity. Figure 1 illustrates the geometry of the problem. In particular, we shall investigate the Poiseuile flow in a pipe of circular cross section and following Hayat et al. [14], Ajadi [5], Okoya [20] the momentum and heat balance equations can be written as:

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}(2\alpha_1+\alpha_2)\left[\frac{d\bar{w}}{d\bar{r}}\right]^2\right)=\frac{\partial\bar{p}}{d\bar{r}}, \tag{1}$$

$$0=\frac{\partial\bar{p}}{d\bar{\phi}}, \tag{2}$$

$$\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\bar{\mu}\frac{d\bar{w}}{d\bar{r}}\right)+\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(2\bar{r}\beta_3\left[\frac{d\bar{w}}{d\bar{r}}\right]^3\right)=\frac{\partial\bar{p}}{\partial\bar{z}}, \tag{3}$$

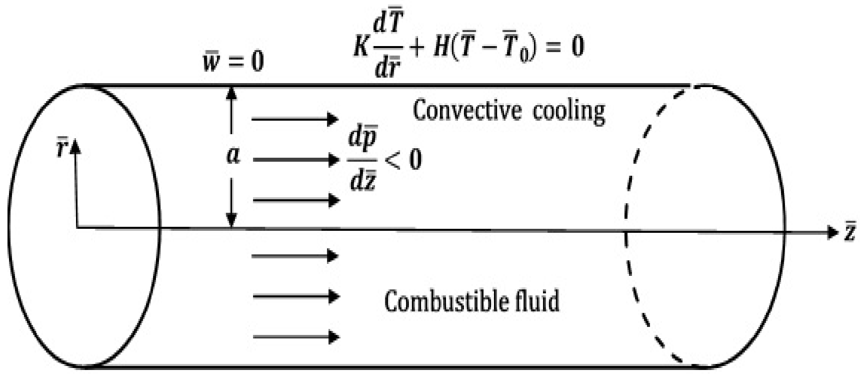

Fig. 1. Geometry of the physical model and coordinate system.

$$K\left(\frac{1}{\bar{r}}\frac{d}{d\bar{r}}\left(\bar{r}\frac{d\bar{T}}{d\bar{r}}\right)\right)+\bar{\mu}\left(\frac{d\bar{w}}{d\bar{r}}\right)^2+2\beta_3\left(\frac{d\bar{w}}{d\bar{r}}\right)^4+QC_0A_0\left(\frac{k\bar{T}}{\nu\hbar}\right)^m\exp\left(\frac{-E}{R\bar{T}}\right)=0, \quad (4)$$

subject to the following boundary conditions

$$\frac{d\bar{w}}{d\bar{r}} = \frac{d\bar{T}}{d\bar{r}} = 0 \text{ at } \bar{r} = 0; \ \bar{w} = K\frac{d\bar{T}}{d\bar{r}} + H(\bar{T} - \bar{T}_0) = 0 \text{ at } \bar{r} = a. \quad (5)$$

where Q is the heat of reaction, C_0 is the initial concentration of the reactant species, H is the surface heat transfer coefficient, K is the thermal conductivity, A is the rate constant, $\hbar$ is the Planck's number, k is the Boltzmann's constant, ν is the vibration frequency and m is a numerical exponent, E is the activation energy, R is the universal gas constant and $\bar{T}_0$ is a suitable reference temperature and $d\bar{p}/d\bar{z}$ is the emerging pressure gradient in the axial directions. In deriving equation (3) we have neglected the body force but the viscosity μ is defined as

$$\bar{\mu}(\bar{r}) = (\bar{r}/a)^q\bar{\mu}_0 \quad (6)$$

where $\bar{\mu}_0$ is a reference viscosity. Typically, $q \in \{0, 1\}$ (e.g. see, [6], [14], [5], [18] and the references contained therein). Most importantly, we proposed for very general circumstance where q is any natural number. The number q has a physical interpretation and can be thought of as a measure of the inhomogeneity of the fluid. Here equation (3) is to be integrated for a given $\frac{\partial\bar{p}}{\partial\bar{z}}$ and once the flow field is determined, the actual pressure field can be obtained from equations (1) and (3).

Denoting

$$r = \frac{\bar{r}}{a}, \ w = \frac{\bar{w}}{\bar{w}_0}, \ C = \frac{\bar{R}^2}{\bar{\mu}_0\bar{w}_0}\frac{d\bar{p}}{d\bar{z}}, \ \theta = \frac{(\bar{T} - \bar{T}_0)E}{R\bar{T}_0^2}, \ \mu = \frac{\bar{\mu}}{\bar{\mu}_0},$$

we have from equations (3) - (5) the dimensionless governing equations

$$\frac{d\mu}{dr}\frac{dw}{dr}+\frac{\mu}{r}\left(\frac{dw}{dr}+r\frac{d^2w}{dr^2}\right)+\frac{\Lambda}{r}\left(\frac{dw}{dr}\right)^2\left(\frac{dw}{dr}+3r\frac{d^2w}{dr^2}\right)=C, \quad (7)$$

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\left(\frac{dw}{dr}\right)^2\left(\mu+\Lambda\left(\frac{dw}{dr}\right)^2\right)+\delta(1+\beta\theta)^m\exp\left(\frac{\theta}{1+\beta\theta}\right)=0, \quad (8)$$

while

$$\mu(r)=r^q, \quad (9)$$

and the boundary conditions

$$\frac{dw}{dr}=\frac{d\theta}{dr}=0 \text{ at } r=0;\ w=\frac{d\theta}{dr}+\mathrm{Bi}\theta=0 \text{ at } r=1, \quad (10)$$

where the emerging dimensionless parameters are

$$\beta=\frac{R\bar{T}_0}{E},\ \Gamma=\frac{\bar{\mu}_0\bar{w}_0^2a^q}{K\bar{T}_0\beta},\ \delta=\frac{QEA_0a^2C_0k^m\bar{T}_0^{m-2}}{\nu^m\hbar^mRK}\exp\left(-\frac{E}{R\bar{T}_0}\right),$$

$$\mathrm{Bi}=\frac{aH}{K},\ \Lambda=\frac{\beta_3\bar{w}_0^2}{\bar{\mu}_0}.$$

Here, β is the activation energy, Γ is the viscous heating parameter which is related to Prandtl and Eckert numbers, Λ may also depend on temperature, they are taken as constants for simplicity in this study, Bi is the Biot number and δ is the usual Frank-Kamenetskii parameter. Due to symmetry, θ_{max} is the maximum value of θ at the centre of the cylinder.

It is worthy of note that the viscosity in equation (9) is monotonic and positive as depicted in Figure 2. Concerning the viscosity at the origin, we observe that as one approaches the origin the effects of the non-Newtonian part of the fluid's constitutive relation dominates over the Newtonian part. One may also attribute this behaviour to the possibility of spatial variation of the specie concentration which was not solved for in this case. However, the main idea of this study is to investigate the capability of the proposed methodology.

It is worth noting that in the classical sense, Semenov assumed a uniform temperature in the material ($K \to \infty$) and arbitrary surface cooling resulting in the limit case of Bi $\to 0$ while Frank-Kamenetskii assumed an infinite heat transfer coefficient at the surface with the limit case of Bi $\to \infty$

3. LIMITING CASES

1. When Bi $\to \infty$ and $q = 1$ equations (7) - (10) corresponds to the case of a viscosity function with linear dependence on r, which has

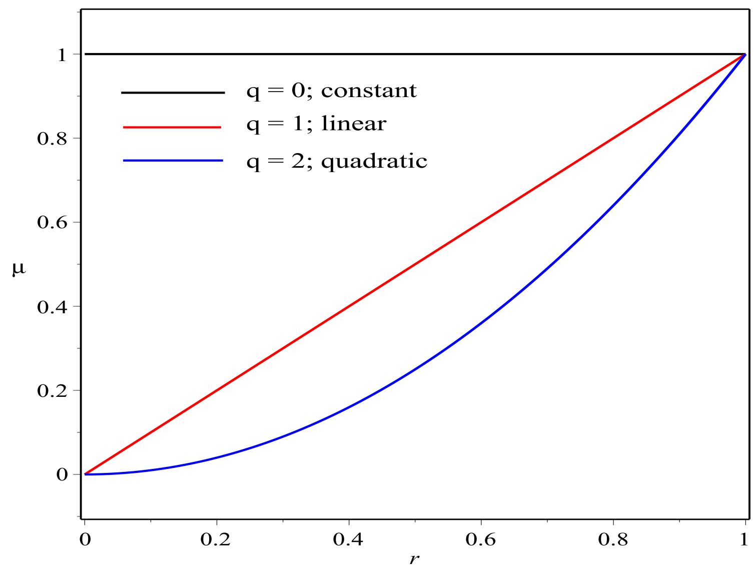

Fig. 2. The plots show the three viscosity models studied in this article as a function of r.

been considered in the flow of third grade fluid for the heat transfer process in a non-reactive system [14] ($\delta = 0$) and for reactive system [5] ($m = 0$).
2. The case of Bi $\to \infty$ and $q = \beta = \Lambda = 0$ corresponds to the classical constant viscosity case has been studied in [27] (reactive Newtonian fluid flows and heat transfer),
3. The scenarios for stationary thermal explosion contained in [8], [9], [21] are similar to ours.

4. REMARKS ON SOLUTION OF THE VELOCITY

In view of the choice of equation (9), the nonlinear velocity and temperature equations are decoupled and offers the possibility to attempt to investigate whether or not there exists velocity profile in closed - form. Two points about equation (7) are that it is non-autonomous and independent of w and can hence be integrated once to yield

$$\Lambda\left(\frac{dw}{dr}\right)^3 + \mu(r)\frac{dw}{dr} - \frac{Cr}{2} = 0,\ w(1) = 0 \qquad (11)$$

In the special case of $\Lambda = 0$ (Newtonian fluid) the nonlinear equation (11) reduces to linear form and the can be integrated once for any viscosity and the viscous dissipation term can be eliminated from equation (8) to obtain

$$\frac{d^2\theta}{dr^2}+\frac{1}{r}\frac{d\theta}{dr}+\Gamma\frac{C^2r^2}{4\mu(r)}+\delta(1+\beta\theta)^m\exp\left(\frac{\theta}{1+\beta\theta}\right)=0, \tag{12}$$

$$\frac{d\theta}{dr}=0 \text{ at } r=0;\ \frac{d\theta}{dr}+\mathrm{Bi}\theta=0 \text{ at } r=1. \tag{13}$$

The third term on the left hand side of equation (12) reveals that the singularity at the origin is removable when $q \leq 2$. If $q > 2$, the system blows up.

When $\Lambda \neq 0$, equation (11) entails a unique positive real root, dw/dr, which is the Cardan cubic equation emanating from equation (11) with the constraint that the discriminant $D(r;q,C,)>0$. It is evident that the end result of dw/dr is without confronting cube roots of complex number but it displays the solution in a somewhat doubtful matter. But it is much easier to develop an iterative-based alternative to Cardan method. Hence, the need to employ the Modified Adomian Decomposition method.

5. MAD SOLUTION FOR VELOCITY DISTRIBUTION

We are interested in the solutions of equations (7) - (10) for positive natural number q using MADM and in particular we first seek solution analytically for the velocity. In this case, equation (7) can now be rewritten in a non symmetric differential operator form as

$$L_1w=C-\frac{\Lambda}{r}\left(\frac{dw}{dr}\right)^3-3\Lambda\left(\frac{dw}{dr}\right)^2\frac{d^2w}{dr^2} \tag{14}$$

where $L_1=r^{-1}\frac{d}{dr}\left(r^{q+1}\frac{d}{dr}\right)$. To overcome the singularity at $r=0$, the inverse operator L_1^{-1} is therefore considered as a two-fold integral operator defined as

$$L_1^{-1}=\int_0^r y^{-(q+1)}\left(\int_0^y \tau(\cdot)d\tau\right)dy \tag{15}$$

Applying L_1^{-1} to equation (14), it then follows;

$$w(r)=a_0+\frac{1}{2(q+2)}Cr^2-\Lambda L_1^{-1}\frac{1}{r}N_1(w)-3\Lambda L_1^{-1}N_2(w) \tag{16}$$

where $w(0)=a_0$. $N_1(w)=(w')^3$ and $N_2(w)=(w')^2(w'')$ are the nonlinear terms in equation (16).

In Adomian decomposition method, $w(r), N_1(w)$ and $N_2(w)$ are decomposed as series

$$w(r) = \sum_{k=0}^{\infty} w_k(r), \qquad N_1(w) = \sum_{k=0}^{\infty} A_k, \quad N_2(w) = \sum_{k=0}^{\infty} B_k. \quad (17)$$

The nonlinear terms $N_1(w)$ and $N_2(w)$ are decomposed into Adomian polynomials, the few terms of the Adomian polynomials are given by

$$\begin{aligned} A_0 &= (w_0')^3 \\ A_1 &= (w_0')^3 \\ B_0 &= (w_0')^2 (w_0'\prime) \\ B_1 &= (w_0')^2 (w_1'\prime) + 2 (w_0') (w_1') (w_0'\prime) \end{aligned} \quad (18)$$

Now substituting equation (17) into equation (16) yields

$$\sum_{k=0}^{\infty} w_k(r) = a_0 + \frac{1}{6} C r^2 - \Lambda L_1^{-1} \frac{1}{r} \sum_{k=0}^{\infty} A_k - 3\Lambda L_1^{-1} \sum_{k=0}^{\infty} B_k. \quad (19)$$

The recursive relation for the approximate analytical solution for the velocity is given as

$$\begin{aligned} w_0(r) &= a_0 + \frac{1}{6} C r^2 \\ w_{k+1}(r) &= -\Lambda L_1^{-1} \left[A_k + 3B_k \right], \qquad k \geq 1 \end{aligned} \quad (20)$$

The recursive relations (20) with the prescribed values of the parameters involved are coded in Maple to generate the first 6 terms of w_k as the series solution. The undetermined values of a_0 and b_0 are calculated from the boundary condition at $r = 1$, by taking the diagonal Padé approximants $[N/N]$ that approximate $w(r)$ using $w(r) = 0$ for selected N. The five term solution for the velocity at padé [15/15] is obtained via a symbolic computing platform (MAPLE) as

$$w(r) = \frac{C(r^2-1)}{2(2+q)} - \frac{\Lambda C^3 (r^4 - 1)}{(2+q)^3(4+q)} + \frac{4\Lambda^2 C^5 (r^6-1)}{(2+q)^5(4+q)(6+q)}$$

$$- \frac{472\Lambda^3 C^7 (r^8-1)}{(2+q)^7(4+q)^3(6+q)(8+q)(10+q)} + \frac{27968\Lambda^4 C^9 (r^{10}-1)}{(2+q)^9(4+q)^2(6+q)(8+q)}. \quad (21)$$

It is evident from equation (21) that when $q = 0$ and 1, we recover the first four terms solution obtain in [5] with the non-zero auxiliary parameter as unity. Hence solution (21) is an extension of an earlier work and since the effects of C and Λ on the velocity profiles were reported earlier, we will study only the effect of q. To accomplish this we plot in Figure 3 the expression for velocity distribution $w(r)$, given by equation (21) for various values of q. The graphical presentation reveals that when the inhomogeneity

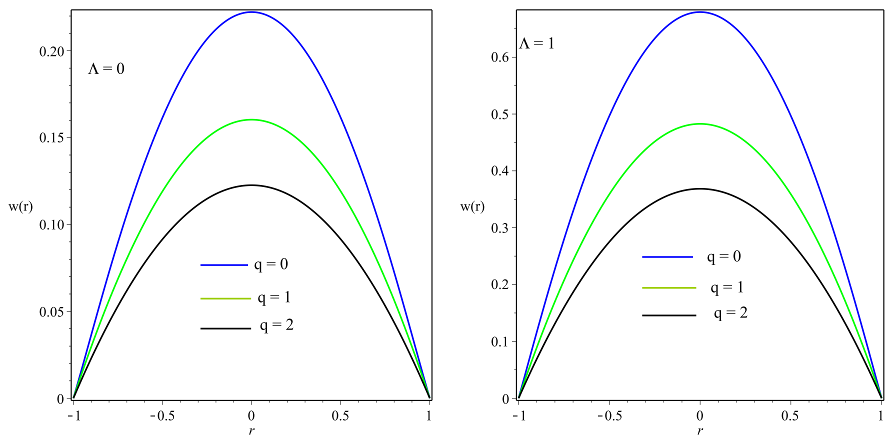

Fig. 3. Velocity profile for various values of q when $\Lambda = 1, C = -1$.

parameter increases then the maximum velocity magnitude profile decreases.

6. THERMAL EXPLOSION CALCULATION AND VALIDATION

Now, we turn our attention to the heat equation (8). The velocity gradient is obtained from equation (21) and is substituted into equation (8). We then generate the corresponding symmetric differential operator form of the resulting equation. As is usual in the theory of thermal explosion, the condition for criticality is that $d\delta/d\theta_{max} = 0$. By insertion of the criticality criterion into the resulting equation a new set of equations are derived. For the sake of brevity, no relevant equations will be presented here. But it suffices to say that the numerical experiments were conducted using MAPLE in accordance with the MADM discussed in the preceding paragraph. In Figure 4, the diagrams of θ_{max} against δ for reactive Newtonian and third grade fluid flows are illustrated, where $\beta = 0.05$, $m = -2$, $\Gamma = q = 1$ and Bi = 10,000. In practice, it is necessary to know the relative error in the comparison of MADM with numerical calculation. This is achieved for special cases in Tables 2 - 4.

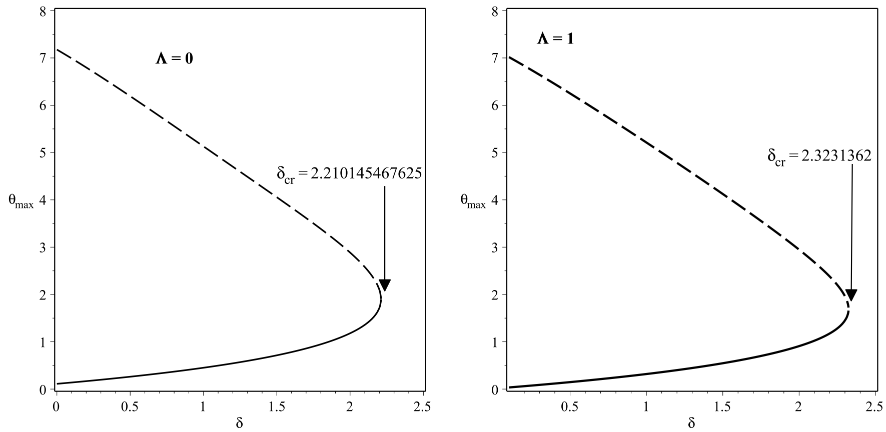

Fig. 4. Visualization of the bifurcation diagram from MADM when $\beta = 0.05$, $m = -2$, $\Gamma = q = 1$ and $\mathrm{Bi} = 10{,}000$.

TABLE 2: Comparison of exact solutions of δ_{cr} and $\theta_{max\ cr}$ obtained in Thomas (1958) with results obtained by HAM when $C = -2$, $\beta = \Lambda = \Gamma = 0$.

	δ_{cr}			$\theta_{max\ cr}$		
Bi	Exact [21]	Present	Error	Exact [21]	Present	Error
0.01	0.007339	0.007329	0.14 %	1.002494	0.977403	2.50 %
0.1	0.071760	0.071640	0.17 %	1.024370	0.991425	3.22 %
1	0.575799	0.573715	0.36 %	1.187803	1.149668	3.21 %
1 x 10^1	1.653837	1.653536	0.02 %	1.377654	1.356159	1.56 %
1 x 10^2	1.960593	1.966384	-0.29 %	1.386196	1.375987	0.74 %
1 x 10^3	1.996006	2.001641	-0.28 %	1.386293	1.358498	2.00 %
1 x 10^4	1.999600	2.005244	-0.28 %	1.386294	1.358414	2.01 %
1 x 10^5	1.999960	2.005244	-0.26 %	1.386294	1.348879	2.70 %
1 x 10^6	1.999996	2.005244	-0.26 %	1.386294	1.348053	2.76 %
1 x 10^7	2.000000	2.005244	-0.26 %	1.386294	1.347971	2.76 %

TABLE 3: Series solutions of δ_{cr} and $\theta_{max\ cr}$ obtained in Zaturska (1981) compared with results obtained by MADM when $C = -2$, $\beta = \Lambda = q = 0$ and $\mathrm{Bi} \to \infty$.

	δ_{cr}			θ_{max}		
Γ	[27]	Present	Error	[27]	Present	Error
0	1	1	0.0 %	1	1	0.0 %
0.05	0.997161	0.997140	2.11x10^{-3}%	1.002183	1.006533	-1.29 %
0.1	0.994330	0.994279	5.13x10^{-3}%	1.004366	1.015117	-1.07 %
0.15	0.991506	0.991196	3.13x10^{-2}%	1.006549	1.003818	0.27 %
0.2	0.988689	0.988461	2.41x10^{-2}%	1.008732	1.019899	-1.11 %
0.25	0.985880	0.985385	5.02x10^{-2}%	1.010915	1.007326	0.35 %

TABLE 4: Numerical solutions of δ_{cr} and $\theta_{max\ cr}$ obtained in Boddington et al. (1984) compared with results obtained by MADM when $C = -2$, $\Gamma = \Lambda = m = 0$ and $\beta = 0.05$.

	δ_{cr}			$\theta_{max\ cr}$		
Bi	Num. [9]	Present	Error	Num. [9]	Present	Error
1	0.6075005	0.6055478	0.32 %	1.3261112	1.2514834	5.63 %
1 x 10^1	1.7512941	1.7518748	-0.033 %	1.5502305	1.5179320	2.08 %
1 x 10^2	2.0768671	2.0823412	-0.27 %	1.5607587	1.5388387	1.40 %
1 x 10^3	2.1143881	2.1197928	-0.26 %	1.5608794	1.5287421	2.06 %
1 x 10^4	2.1219191	2.1236584	-0.082 %	1.5608511	1.5303134	1.96 %

TABLE 5: Confirmation of MADM with numerical computation for δ_{cr} and $\theta_{max\ cr}$ when $\beta = 0.01$, $\Gamma = 10$, $\Lambda = 0$, $C = m = -2$ and $q = 1$.

	δ_{cr}			$\theta_{max\ cr}$		
Bi	Num.	Present	Error	Num.	Present	Error
1	0.401817	0.396128	1.42 %	1.693054	1.587689	6.22 %
1 x 10^1	1.515574	1.508845	0.44 %	1.577054	1.531033	2.92 %
1 x 10^2	1.842715	1.841194	0.08 %	1.555438	1.526034	1.89 %
1 x 10^3	1.880695	1.879318	0.07 %	1.552530	1.520972	2.03 %
1 x 10^4	1.884552	1.882993	0.08 %	1.552231	1.513727	2.48 %
1 x 10^5	1.884939	1.883605	0.07 %	1.552200	1.522114	1.94 %
1 x 10^6	1.884977	1.883605	0.07 %	1.552197	1.520462	2.04 %
1 x 10^7	1.884981	1.883605	0.07 %	1.552198	1.520303	2.05 %

In order to illustrate that the MADM for equations (8) - (10) for the case of $q = 1$ is consistent with the non-special cases, we adopted the modified shooting method in [20] to obtain numerical solutions of equation (7) and (8) subject to the boundary conditions (10). Table 5 accordingly depicted the error margin between the numerical solution and the semi-analytical method of MADM. The relative error for small Bi is larger than those for large values of Bi.

Simple computations show that Bi $\to \infty$ can be truncated since extensive computation see Table 2: So that Bi $\in [0, \infty)$ can be carefully replaced by a finite domain Bi $\in [0, 1 \text{ x } 10^5)$. With this choice, numerical calculation shows that the system converges to a fixed value for Bi from 1 x 10^5 and above. In fact, the relative error in the table reveals that the error values are bounded above in the form $\delta_{cr} \leq 0.36\%$ and $\theta_{max\ cr} \leq 3.22\%$. Tables 3 and 4 give a comparison of MADM with series and numerical solutions, respectively, for values of $\Gamma \leq 0.25$ and Bi ≥ 1. In the tables, the values for the MADM were calculated from equations (9) and (10).

It is seen that the results from MADM give very good agreement with the series and numerical solutions with both having a relative error of less than 1.3 % and 5.7 %, respectively.

The apparent large relative error in the computed values for criticality when Bi ≤ 1 is due to the difficulties with computing in the neighbourhood of small values of Bi. Consequently, in that region, our MADM solutions are better than the early computed ones.

Now, we turn our attention to discussing the critical results for the possible range of values for Bi as depicted in the literature for the sensitized reaction. We note that as Bi $\to \infty$ and $\beta \to 0$ we have the classical values of $\delta_{cr} = 2$, 2.01 and $\theta_{max\ cr} = 1.3863$, 1.47 for viscous heating parameter $\Gamma = 0$ [21] and 1 [5], respectively. The response of δ_{cr} and $\theta_{max\ cr}$ to variation of q and Γ from Figure 2 (a) are displayed in Table 6 for $\Gamma \in \{0, 1\}$. It is evident that that the special case of $q = 0$, δ_{cr} and $\theta_{max\ cr}$ are lower bounds of $q \neq 0$.

TABLE 6: Thermal critical values as functions of q and Γ as Bi $\to \infty$ and $\beta \to 0$ when $m = -2$.

	$\Gamma = 0$		$\Gamma = 1$	
q	δ_{cr}	$\theta_{max\ cr}$	δ_{cr}	$\theta_{max\ cr}$
0	2	1.3863	2.01	1.47
1	2.3863	1.7687	2.2101	1.8601

7. RESULTS

The default parameter values in subsequent analysis are as follows: $m = -2$, $\Gamma = 1$, $q = 1$ $C = -2$, $\Lambda = 1$ and $\beta = 0.05$. Therefore, in any graph where any of these parameter values is not explicitly mentioned, it will be understood that such parameter takes default values.

With the reaction defined by the sensitized reaction kinetics with $m =$ -2, we investigate the dependence of critical parameters on the viscous heating parameter, Γ, Biot number, Bi, numerical exponent of the pre-exponential factor, m, pressure gradient, C and the activation energy parameter, β.

The results displayed in Figures 5 - 12, show comparison of the MADM approximate critical values (thermal explosion) for the reactive Newtonian and third grade fluid flows for sensitized reaction for various parameter values.

Figure 5 shows the critical maximum temperature against the activation energy parameter for different Biot number Bi. It should be noted that Bi decreases the critical maximum temperature in the case of reactive Newtonian model while for the reactive third grade fluid flow, there is a reverse effect. We also observed from the plots that the non-Newtonian parameter reduces the critical maximum temperature. The complimentary Frank-Kamenetskii parameter in Figure 6 increases with increasing parameter β for both reactive Newtonian and non-Newtonian fluid flows. This is consistent with the fact that the activation energy yields an augment in the critical value of δ (see [22, Page 152]). An increase in the non-Newtonian parameter increases the value of δ_{cr}. This implies that the critical value of δ is enhanced with increasing the work due to deformation.

Figures 7 and 8 show the duo of critical maximum temperature and critical Frank-Kamenetskii parameter against pressure gradient for different values of Biot number. We observed that the reaction parameter $\theta_{max\ cr}$ (or δ_{cr} in Figure 8) is a convex (or concave) function of hydrodynamic parameter C and the minimum (maximum) occur at the $C = 0$ for both scenarios. It is depicted in Figure 6 that δ_{cr} has an opposite behavior as compared to the $\theta_{max\ cr}$.

In Figures 9 and 10, $\theta_{max\ cr}$ and δ_{cr} are plotted as functions of the viscous dissipation parameter Γ for different values of Biot number Bi. As can be seen, $\theta_{max\ cr}$ increases (decreases) as Γ (or Bi) increases for the two cases. This is physically true since viscous dissipation is an additional heat source within the pipe while the Biot number is a cooling effect at the walls of the vessel. Furthermore, $\theta_{max\ cr}$ for the Newtonian case serves as upper bound for all Γ. It is again observed that δ_{cr} rise with increasing values of Bi for all Γ.

As can be seen, Figures 11 and 12 illustrate the combustion process at the transitional values of β, δ and θ_{max}. The δ_{cr} value rises with increasing values of Bi while the reverse is the case for $\theta_{max\ cr}$ in the flow of a reactive Newtonian as well as that of a third grade fluid. Figure 9 (or Figure 10) shows that $\theta_{max\ cr}$ (or δ_{cr}) increases with increase in the activation energy parameter. The transition values in the case of reactive Newtonian fluid flow are greater when compared with the flow of reactive third grade fluid for the two reactive parameters due to it's non-Newtonian nature. It is remarkable how well the trends fit even with the classical case where there is no fluid flow (see Boddington et al. [9]).

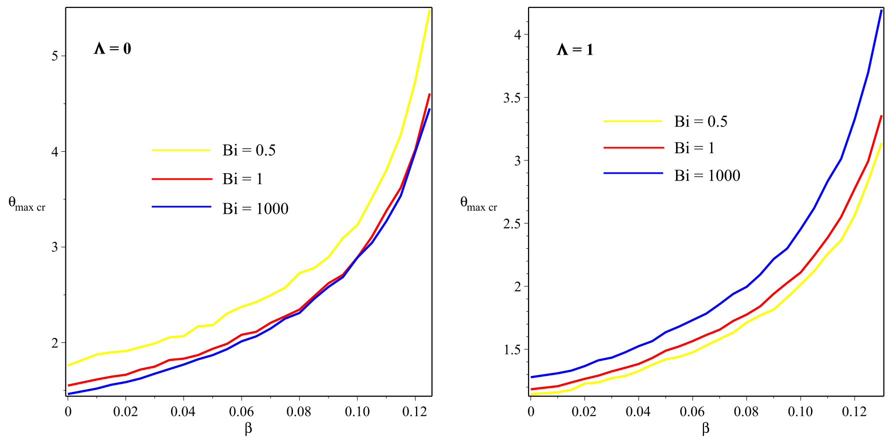

Fig. 5. Plot of $\theta_{max\ cr}$ over β for different values of Bi.

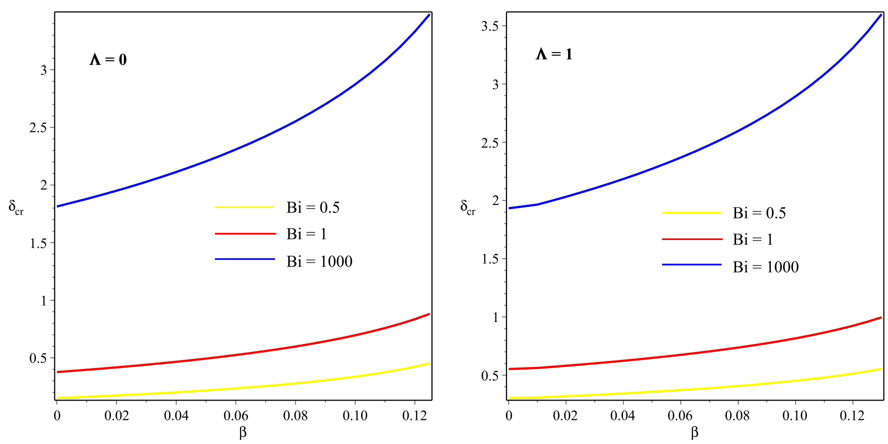

Fig. 6. Graphs of δ_{cr} versus β for different values of Bi.

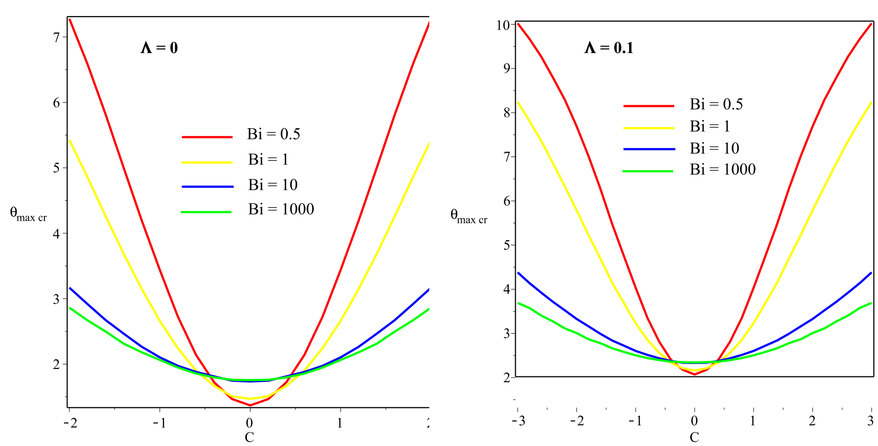

Fig. 7. Variations of θ_{cr} over C for different values of Bi.

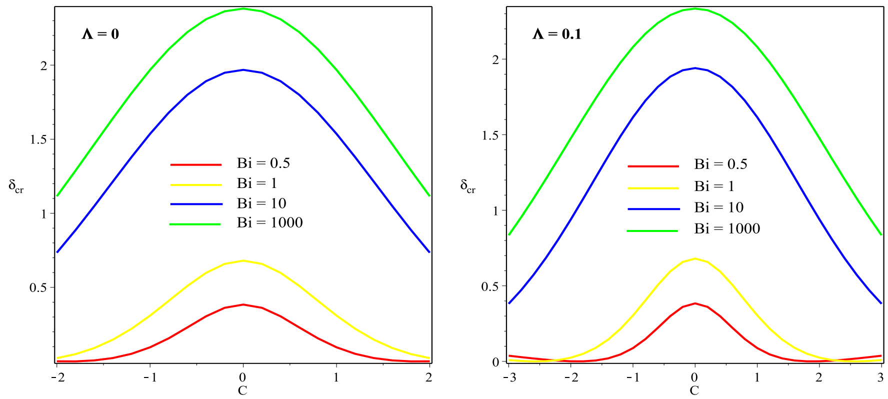

Fig. 8. δ_{cr} variations over C for different values of Bi.

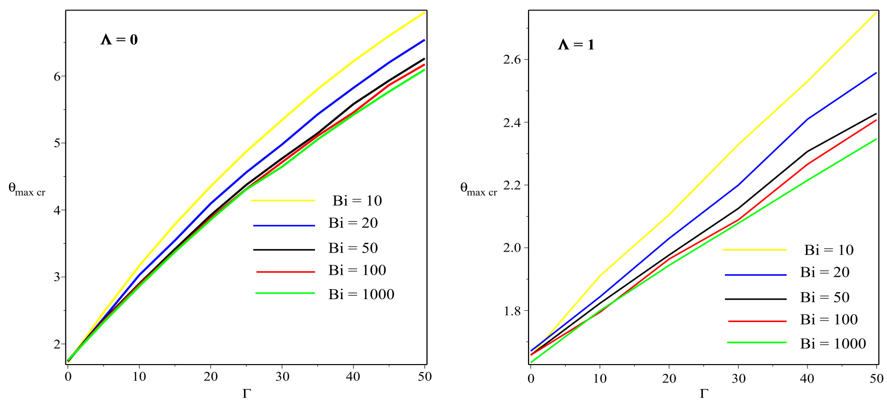

Fig. 9. $\theta_{max\ cr}$ distributions over Γ for different values of Bi.

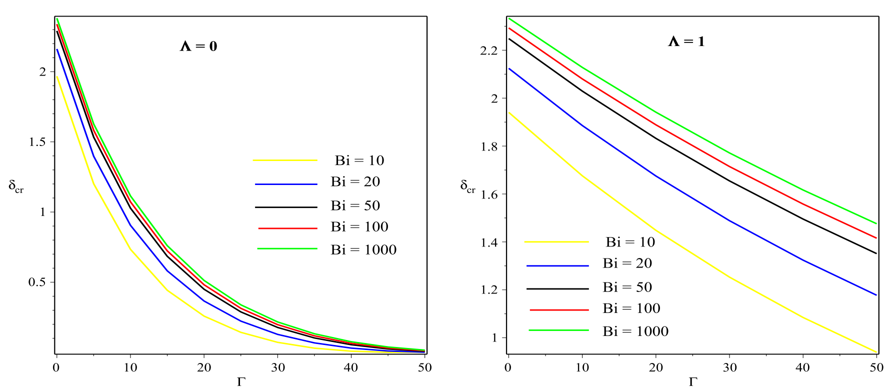

Fig. 10. δ_{cr} distributions with Γ for different values of Bi.

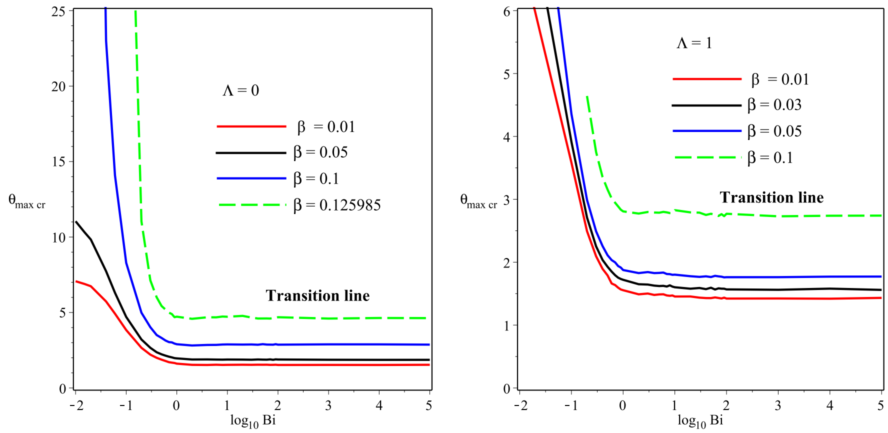

Fig. 11. Distributions of $\theta_{max\ cr}$ against log10 Bi at five different β demonstrating the procedure to determine transition.

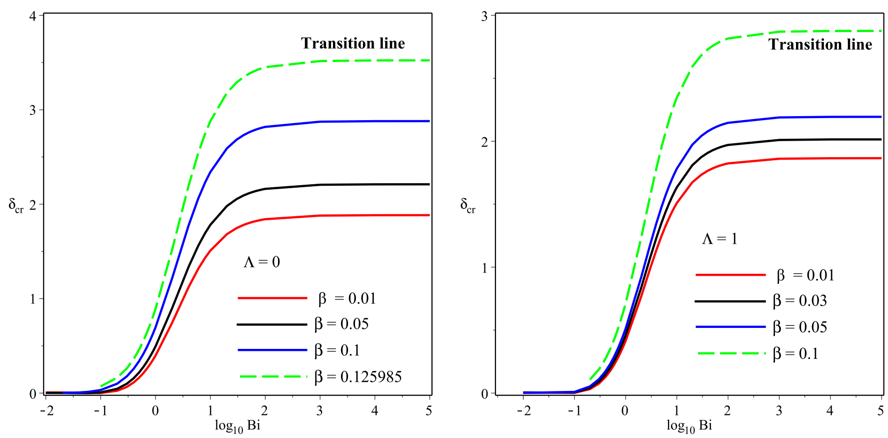

Fig. 12. Distributions of δ_{cr} against log10 Bi at five different β demonstrating the procedure to determine transition.

8. CONCLUSIONS

In this work, we utilized the Modified Adomian Decomposition Method to develop analytical solutions to the equations governing the flow of a reactive viscous third grade fluid with spatially varying viscosity through a cylindrical pipe with surface cooling. The resulting solutions were derived via the diagonal Padé approximation techniques. We further verified the validity and efficiency of the method for various values of q, Γ and Bi. With graphical illustrations the axial flow velocity, critical and transitional values of the maximum temperature as well as the Frank-kamenetskii parameters as functions of emerging variables and parameters were presented. The main findings can be summarizes as follows:

1. The MADM supplies reliable result in the form of analytical approximation converging with relative error less than 5%.
2. The present analysis reveals that the velocity profile diminishes, due to the influence of the inhomogeneity parameter.
3. Also this paper investigated the problem of thermal explosion of reactive viscous fluid with spatial dependent viscosity and irreversible exothermic reactions. The proposed MADM for thermal explosion, is shown to be very accurate for all values of Bi except when $\mathrm{Bi} = 1$ and it is in excellent agreement with numerical results.
4. Furthermore, it has been shown that spatial dependence of the fluid viscosity strongly influences the associated state parameters in the flow of Newtonian and third grade explosive liquid. This conclusion further justifies the necessity to account for the spatial dependence of viscosity in some theoretical consideration of reactive fluid flow.
5. The flow of a reactive non-Newtonian fluid flow with spatial viscosity behaviour for all parameter variations followed the trends on well established Newtonian reactive fluid flow, but the dependence on non-Newtonian parameter was even more pronounced.
6. It was also shown in general that the spatial variation of the fluid viscosity does not distort the nature of the transition diagram for both the Newtonian and third-grade fluid flows. Thus, confirming the usefulness of this model in describing the state behaviour.
7. The present study can also be reduced to Newtonian fluid by taking non-Newtonian parameter $\Lambda = 0$, as limiting case.

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Department of Mathematics, University of Lagos, Akoka, Yaba, Lagos, Nigeria.
E-mail address: `ogunseyehammed@gmail.com`

Department of Mathematics, Obafemi Awolowo University, Ile - Ife, 220005, NIGERIA.
E-mail addresses: `ssokoya@gmail.com & sokoya@oauife.edu.ng`