Journal of the Nigerian Mathematical Society
Vol. 36, Issue 2, pp. 435-457, 2017

ON THE RESPONSE OF A SIMPLY SUPPORTED NON-UNIFORM RAYLEIGH BEAM TO TRAVELLING DISTRIBUTED LOADS

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ABSTRACT. The problem of a simply supported non-uniform Rayleigh beam under travelling distributed loads is investigated in this paper. Both gravity and inertia effects of the distributed loads are taken into consideration. For the solution of the problem, the Galerkin's Weighted Residual (GWR) method and Struble's asymptotic technique are employed. From the analysis, it is seen that the displacement response of both moving distributed force and moving distributed mass problems decrease with increase in the values of foundation moduli. Furthermore, higher values of rotatory inertia correction factor reduce the transverse deflection of the beam structure when both force and inertia effects are taken into consideration. Analytical and Numerical solutions show that for the same natural frequency, the critical speed for the system consisting of a non-uniform simply supported Rayleigh beam traversed by a moving distributed force at constant velocity is higher than that of the moving distributed mass problem. Thus, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

Keywords and phrases: Travelling distributed loads, non-uniform Rayleigh beam, moving distributed force, moving distributed mass
2010 Mathematical Subject Classification: 70F99

1. INTRODUCTION

The transverse motion of beam structures with elastic foundation supports which are under the influence of moving loads are widely used in many areas, such as mechanical and aerospace Engineering. Long historical research work in this area can be seen in literature of Engineering mechanics [1,2]. When structural members are under the passage of moving loads, the interaction between the passing load and the structure makes the dynamic response analysis very complex [3]. By virtue of the relevance in the analysis and design

Received by the editors May 19, 2014; Revised: July 10, 2017; Accepted: July 13, 2017
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of railway trucks, bridges, elevated roadways, decking slabs etc, this area of research has continued to motivate both experimental and numerical studies. However, in most of the studies [4-11] the scope has been limited to structural members having uniform cross-section whether the inertia of the moving load is considered or not, in addition, the moving loads have been idealized as moving concentrated loads which acts at a certain point on the structure and along a single line in space as they move [12]. In reality, the cross-sections of structural members such as bridges, girders, hull of ships, concrete slabs etc are of variable cross-sections and in practice, it is known that loads are actually distributed over a small segment or over the entire length of the structural member as they traverse the structure.

Until recently, the literature on one-dimensional structures such as non-uniform beams subjected to dynamic loads is very meagre. The problem of flexural vibrations of non-uniform beams under moving loads was first tackled by Kolousek[13]. The problem was solved using normal-mode analysis. Zheng et al[14] studied the vibration of multispan non-uniform beams under moving loads. The authors solved the problem using modified beam vibration functions. Oni[15] investigated the problem of a non-uniform beam carrying moving concentrated masses under tensile stress and resting on elastic foundation. For the close form solution of the problem, he used the versatile technique of Galerkin and modified asymptotic method of Struble. All the pertinent components of inertia term were considered and results showed that resonance is reached earlier in the moving mass problem than in the moving force problem. He concluded that as the ratio of the mass of the load to the mass of the beam increases, the response amplitude of the beam increases. Oni and Awodola[16] proposed an elegant method based on the generalized Galerkin's method and Struble's asymptotic technique to assess the vibration under a moving concentrated load of a non-uniform Rayleigh beam on variable elastic foundation for some illustrative examples of classical boundary conditions. They concluded that higher values of rotatory inertia correction factor are required for a more noticeable effect in the case of clamped-clamped end conditions than those of simply supported end conditions for both moving concentrated force and moving concentrated mass problems. Ajibola[17] investigated the transverse displacement of clamped-clamped non-uniform Rayleigh beams under moving concentrated masses resting on a constant elastic foundation.

It is pointed out, that, at this juncture, all the above authors have modelled their moving loads as concentrated loads which do not well represent the distribution of the load on the elastic structures they traverse. Studies that considered moving loads as distributed loads can be found in [18-23], though only structural members with uniform cross-section were investigated. Authors who have carried out studies on elastic systems with variable cross-section include Huang and Li [24] who presented a new approach for free vibration of axially functionally graded beams with non-uniform cross-section. Their governing equation with varying coefficients was transformed using Fredholm integral equation and natural frequencies were determined by requiring that the resulting Fredholm integral equation has a non-trivial solution. Shahba et al [25] who investigated the effects of taper ratio, elastic constraint, attached mass and the material non-homogeneity on the natural frequencies and critical buckling load using finite element approach. Shahba and Rajasekaram [26] studied free vibration and stability of tapered Euler-Bernoulli beams made of axially functionally graded materials. Huang et al [27] introduced an auxiliary function to change the coupled governing equations with variable coefficients for the deflection and rotation to a single governing equation. Power series for unknown functions were used to transform the single equation to a system of linear algebraic equations to obtain a characteristic equation in natural frequencies for different boundary conditions. Banerjee and Jackson [28] studied the free vibration of a rotating tapered Rayleigh beam employing a dynamic stiffness method of solution. Tang et al [29] obtained exact frequency equations of free vibration of exponentially non-uniform functionally graded Timoshenko beams. It was observed that the gradient index has a strong influence on the natural frequencies. Shafiei et al [30] investigated the impact of shear deformation on natural frequencies of a rotating non-uniform functionally graded (FG) Timoshenko and Euler-Bernoulli microbeam considering different values of the material length scale parameter, angular velocity and rate of cross-section change. These works however, are on free vibration of graded beams. This paper therefore presents the problem of the flexural motions of a non-uniform Rayleigh beam resting on an elastic foundation and under the actions of travelling distributed loads. Both gravity and inertia effects of the distributed loads are taken into consideration. The influence of the pertinent structural parameters in the dynamical system shall be investigated. Conditions under which resonance is reached is obtained for both the moving distributed force problem and moving distributed mass problem.

2. MATHEMATICAL FORMULATION

Consider the flexural motion of a non-prismatic Rayleigh beam resting on an elastic foundation and carrying a mass M. The mass M is assumed to touch the beam at time $t = 0$ and travel across it with constant velocity c. The governing equation of motion with damping neglected is given by the fourth order partial differential equation [1].

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V}{\partial x^2}\right] + \mu(x)\frac{\partial^2 V(x,t)}{\partial t^2} - \mu(x)R_0\frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} + KV(x,t) = P(x,t) \tag{1}$$

where x is the spatial coordinate, t is the time, $V(x,t)$ is the transverse displacement, E is Young's modulus, R_0 is the measure of rotatory inertia correction factor, K is the elastic foundation constant and the beam properties such as the moment of inertia $I(x)$ and mass per unit length $\mu(x)$ of the beam are considered as varying along the length L of the beam.For this problem, the distributed load moving on the beam under consideration has mass commensurable with the mass of the beam. Consequently, the load inertia is not negligible but significantly affects the behaviour of the dynamical system. Thus, the distributed load $P(x,t)$ takes the form,

$$P(x,t) = P_f(x,t)\left[1 - \frac{1}{g}\frac{d^2V(x,t)}{dt^2}\right] \tag{2}$$

where $P_f(x,t)$ is the continuous moving force acting on the beam model given by

$$P_f(x,t) = MgH(x - ct) \tag{3}$$

where c is the velocity of the distributed mass, the time t is assumed to be limited to that interval of time within which the mass μ is on the beam, that is

$$0 \le ct \le L \tag{4}$$

g is the acceleration due to gravity, and $H(x - ct)$ is the Heaviside function defined as

$$H(x-ct) = \begin{cases} 0, & x < ct \\ 1, & x > ct \end{cases} \tag{5}$$

$\frac{d^2}{dt^2}$ is the convective acceleration operator defined as

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2c\frac{\partial^2}{\partial x \partial t} + c^2\frac{\partial^2}{\partial x^2} \tag{6}$$

The Rayleigh beam under consideration is simply supported. Thus the deflection and the bending moments vanish identically, that is

$$V(0,t) = 0 = V(L,t) \tag{7}$$

$$\frac{\partial^2}{\partial x^2}V\,(0,t) = 0 = \frac{\partial^2}{\partial x^2}V\,(0,t) \tag{8}$$

For simplicity the initial conditions are

$$V(x,0) = V_t(x-0) = 0 \tag{9}$$

Adopting examples in [31], $I(x)$ and $\mu(x)$ are taken to be of the form

$$I(x) = I_0\Big(1 + Sin\frac{\pi x}{L}\Big)^3 \tag{10}$$

$$\mu(x) = \mu_0\Big(1 + Sin\frac{\pi x}{L}\Big) \tag{11}$$

3. OPERATIONAL SIMPLIFICATION

In order to solve equation (1), equations (2), (3), (6), (10) and (11) are substituted into equation (1), after some simplifications and rearrangements (1) becomes

$$\begin{gathered} EI_0\left[\frac{\partial^4 V(x,t)}{\partial x^4}\left\{\frac{5}{2} + \frac{15}{4}Sin\frac{\pi x}{L} - \frac{1}{4}Sin\frac{3\pi x}{L} - \frac{3}{2}Cos\frac{2\pi x}{L}\right\} + \right. \\ \left. \frac{\partial^2 V(x,t)}{\partial x^2}\left\{\frac{9\pi^2}{4L^2}Sin\frac{3\pi x}{L} - \frac{15\pi^2}{4L^2}Sin\frac{\pi x}{L} + \frac{6\pi^2}{L^2}Cos\frac{2\pi x}{L}\right\} + \right. \\ \mu_0\left(1 + Sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial t^2} - \mu_0 R_0\left[\left(1 + Sin\frac{\pi x}{L}\right)\frac{\partial^4 V(x,t)}{\partial x^2 \partial t^2} + \right. \\ \left. \frac{\pi}{L}Cos\frac{\pi x}{L}.\,\frac{\partial^3 V(x,t)}{\partial x \partial t^2}\right] + KV(x,t) + MH(x-ct) \\ \left[\frac{\partial^2 V\,(x,t)}{\partial t^2} + 2c\frac{\partial^2 V\,(x,t)}{\partial x \partial t} + c^2\,\frac{\partial^2 V\,(x,t)}{\partial x^2}\right] = MgH(x-ct) \end{gathered} \tag{12}$$

It is evident that exact closed form solution to this equation is impossible. As a result of this, an approximate solution is sought. One of the approximate methods for solving this class of dynamical problems is the Galerkin's Weighted Residual method as employed in Ref [32]. This method requires that the solution of equation (12) takes the form

$$V_n(x,t) = \sum_{m=1}^{n} W_m\,(t)\,U_m(x) \tag{13}$$

where $W_m(t)$ are coordinates in modal space and $U_m(x)$ are the normal modes of free vibration of the beam which is often chosen to satisfy the pertinent boundary conditions (7) and (8). Since our elastic system has simple supports at the edges $x = 0$ and $x = L$, evidently, $U_m(x)$ can be chosen as

$$U_m(x) = Sin\frac{m\pi x}{L} \tag{14}$$

Consequently,

$$V_n(x,t) = \sum_{m=1}^{n} W_m\,(t)\,Sin\frac{m\pi x}{L} \tag{15}$$

Equation (15) when substituted into equation (12) yields

$$\sum_{m=1}^{n}\Bigg\{\frac{EI_0}{\mu_0}\Bigg[\left(\frac{5}{2}+\frac{15}{4}Sin\frac{\pi x}{L}-\frac{1}{4}Sin\frac{3\pi x}{L}-\frac{3}{2}Cos\frac{2\pi x}{L}\right)\left(\left(\frac{m\pi}{L}\right)^4 Sin\frac{m\pi x}{L}\right)+\left(\frac{9\pi^2}{4L^2}Sin\frac{3\pi x}{L}-\frac{15\pi^2}{4L^2}Sin\frac{\pi x}{L}+\frac{6\pi^2}{L^2}\,Cos\frac{2\pi x}{L}\right)\left(-\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}\right)\Bigg]W_m\,(t)+\left(1+Sin\frac{\pi x}{L}\right)Sin\frac{m\pi x}{L}\ddot{W}_m(t)$$
$$-R_0\Bigg[\left(1+Sin\frac{\pi x}{L}\right)\left(-\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}\right)+\left(\frac{\pi}{L}Cos\frac{\pi x}{L}\right)\left(\left(\frac{m\pi}{L}\right)Cos\frac{m\pi x}{L}\right)\Bigg]\ddot{W}_m\,(t)+\frac{K}{\mu_0}W_m\,(t)\,Sin\frac{m\pi x}{L}+$$
$$\frac{MH\,(x-ct)}{\mu_0}\Bigg[Sin\frac{m\pi x}{L}\ddot{W}_m(t)+2c\left(\left(\frac{m\pi}{L}\right)Cos\frac{m\pi x}{L}\right)\dot{W}_m\,(t)+c^2\left(-\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}\right)W_m\,(t)\Bigg]\Bigg\}-\frac{Mg}{\mu_0}H\,(x-ct)=0 \tag{16}$$

In order to determine $W_m(t)$, it is required that the expression on the left hand side of equation (16) be orthogonal to the function $U_k(x)$.
Hence,

$$\int_0^L\left(\sum_{m=1}^{n}\Bigg\{\frac{EI_0}{\mu_0}\Bigg[\left(\frac{5}{2}+\frac{15}{4}Sin\frac{\pi x}{L}-\frac{1}{4}Sin\frac{3\pi x}{L}-\frac{3}{2}Cos\frac{2\pi x}{L}\right)\right.$$

$$\left(\left(\frac{m\pi}{L}\right)^4 Sin\frac{m\pi x}{L}\right)+\left(\frac{9\pi^2}{4L^2}Sin\frac{3\pi x}{L}-\frac{15\pi^2}{4L^2}Sin\frac{\pi x}{L}+\right.$$

$$\left.\frac{6\pi^2}{L^2}Cos\frac{2\pi x}{L}\right)\left(-\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}\right)\Bigg]W_m(t)+\left(1+Sin\frac{\pi x}{L}\right)$$

$$Sin\frac{m\pi x}{L}\ddot{W}_m(t)-R_0\left[\left(1+Sin\frac{\pi x}{L}\right)\left(-\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}\right)\right.$$

$$\left.+\left(\frac{\pi}{L}Cos\frac{\pi x}{L}\right)\left(\left(\frac{m\pi}{L}\right)Cos\frac{m\pi x}{L}\right)\right]\ddot{W}_m(t)+\frac{K}{\mu_0}Sin\frac{m\pi x}{L}$$

$$W_m(t)+\frac{MH(x-ct)}{\mu_0}\left[Sin\frac{m\pi x}{L}\ddot{W}_m(t)+2c\left(\left(\frac{m\pi}{L}\right)Cos\right.\right.$$

$$\left.\left.\frac{m\pi \dot{x}}{L}\right)W_{\ m}(t)+c^2\left(-\left(\frac{m\pi}{L}\right)^2 Sin\frac{m\pi x}{L}\right)W_m(t)\right]\Bigg\}$$

$$\left.-\frac{Mg}{\mu_0}H(x-ct)\right)Sin\frac{k\pi x}{L}dx=0 \tag{17}$$

A rearrangement of the above equation yields

$$\sum_{m=1}^{n}\{[(Q_1(m,k)+Q_3(m,k))-R_0(Q_4(m,k)+Q_2(m,k)+$$

$$Q_5(m,k))]\ddot{W}_{\ m}(t)+\left[P_1(T_0+T_1)+\frac{K}{\mu_0}Q_1(m,k)\right]W_m(t)+$$

$$\frac{M}{\mu_0}\left[Q_1(t)\ddot{W}_m(t)+2cQ_2(t)\dot{W}_m(t)+\right.$$

$$\left.c^2Q_3(t)W_m(t)\right]\}-\frac{Mg}{\mu_0}Q_4(t)=0 \tag{18}$$

where

$$T_0=Q_9(m,k)+Q_{10}(m,k)-(Q_{11}(m,k)+Q_{12}(m,k)) \tag{19}$$

$$T_1=Q_6(m,k)+Q_7(m,k)-Q_8(m,k),\qquad P_1=\frac{EI_0}{\mu_0} \tag{20}$$

$$Q_1(t)=\int_0^L H(x-ct)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{21}$$

$$Q_2(t)=\frac{m\pi}{L}\int_0^L H(x-ct)Cos\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{22}$$

$$Q_3(t)=-\left(\frac{m\pi}{L}\right)^2\int_0^L H(x-ct)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{23}$$

$$Q_4(t)=\int_0^L H(x-ct)Sin\frac{k\pi x}{L}dx \tag{24}$$

$$Q_1(m,k) = \int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{25}$$

$$Q_2(m,k) = -\left(\frac{m\pi}{L}\right)^2 \int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{26}$$

$$Q_3(m,k) = \int_0^L Sin\frac{\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{27}$$

$$Q_4(m,k) = \frac{m\pi^2}{L^2} \int_0^L Cos\frac{\pi x}{L} Cos\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{28}$$

$$Q_5(m,k) = -\left(\frac{m\pi}{L}\right)^2 \int_0^L Sin\frac{\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{29}$$

$$Q_6(m,k) = -\frac{9m^2\pi^4}{4L^4} \int_0^L Sin\frac{3\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{30}$$

$$Q_7(m,k) = -\frac{6m^2\pi^4}{L^4} \int_0^L Cos\frac{2\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{31}$$

$$Q_8(m,k) = -\frac{15m^2\pi^4}{4L^4} \int_0^L Sin\frac{\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{32}$$

$$Q_9(m,k) = \frac{5m^4\pi^4}{2L^4} \int_0^L Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{33}$$

$$Q_{10}(m,k) = \frac{15m^4\pi^4}{4L^4} \int_0^L Sin\frac{\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{34}$$

$$Q_{11}(m,k) = \frac{m^4\pi^4}{4L^4} \int_0^L Sin\frac{3\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{35}$$

$$Q_{12}(m,k) = \frac{3m^4\pi^4}{2L^4} \int_0^L Cos\frac{2\pi x}{L} Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{36}$$

In order to evaluate the integrals in (21), (22) and (23), use is made of the Fourier series representation of the Heaviside unit step function, (see Ref [33]);

$$H(x-ct) = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty} \frac{Sin((2n+1)\pi(x-ct))}{2n+1}, \qquad 0 < x < L \tag{37}$$

Substituting (37) into (18) and simplifying yields

$$\sum_{m=1}^{n} \Big\{ \Delta_1(m,k)\ddot{W}_m(t) + \triangle_2(m,k)W_m(t) + \in_0 L\left[(Q_{1A}(m,k)\ +\right.$$

$$\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}Q_{1B}\left(n,m,k\right)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin\left(2n+1\right)\pi ct}{2n+1}$$

$$Q_{1C}\left(n,m,k\right)\right)\ddot{W}_m\left(t\right)+2c\left(Q_{2A}\left(m,k\right)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos(2n+1)\pi ct}{2n+1}\right.$$

$$\left.Q_{2B}\left(n,m,k\right)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin\left(2n+1\right)\pi ct}{2n+1}Q_{2C}\left(n,m,k\right)\right)\dot{W}_m\left(t\right)c^2$$

$$\left(Q_{3A}\left(m,k\right)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Cos\left(2n+1\right)\pi ct}{2n+1}\;Q_{3B}\left(n,m,k\right)-\right.$$

$$\left.\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{Sin\left(2n+1\right)\pi ct}{2n+1}Q_{3C}\left(n,m,k\right)\right)W_m(t)\Bigg]\Bigg\}$$

$$=\frac{\in_0 gL^2}{\lambda_k}\left[-(-1)^k+Cos\frac{k\pi ct}{L}\right] \tag{38}$$

where

$$\Delta_1\left(m,k\right)=Q_1\left(m,k\right)+Q_3\left(m,k\right)-R_0\left(Q_4\left(m,k\right)+Q_2\left(m,k\right)+Q_5(m,k)\right) \tag{39}$$

$$\Delta_2\left(m,k\right)=P_1(T_0+T_1)+\frac{K}{\mu_0}Q_1(m,k)\qquad \in_0=\frac{M}{\mu_0 L} \tag{40}$$

$$Q_{1A}(m,k)=\frac{1}{4}\int_0^L Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{41}$$

$$[Q_{1B}(n,m,k)=\int_0^L Sin(2n+1)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}\pi xdx \tag{42}$$

$$Q_{1C}(n,m,k)=\int_0^L Cos(2n+1)Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}\pi xdx \tag{43}$$

$$Q_{2A}(m,k)=\frac{m\pi}{4L}\int_0^L Cos\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{44}$$

$$Q_{2B}(n,m,k)=\frac{m\pi}{L}\int_0^L Sin(2n+1)\pi xCos\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{45}$$

$$Q_{2C}(n,m,k)=\frac{m\pi}{L}\int_0^L Cos(2n+1)\pi xCos\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{46}$$

$$Q_{3A}(m,k)=-\frac{m^2\pi^2}{4L^2}\int_0^L Sin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{47}$$

$$Q_{3B}(n,m,k)=-\frac{m^2\pi^2}{L^2}\int_0^L Sin(2n+1)\pi xSin\frac{m\pi x}{L}Sin\frac{k\pi x}{L}dx \tag{48}$$

$$Q_{3C}(n,m,k) = -\frac{m^2\pi^2}{L^2}\int_0^L Cos(2n+1)\pi x Sin\frac{m\pi x}{L} Sin\frac{k\pi x}{L}dx \tag{49}$$

Substituting results of the integrals in (25)-(36) and (41)-(49) into (38), after some simplification and rearrangement, one obtains

$$\sum_{m=1}^{n}\Bigg\{\triangle_{11}\ddot{W}_m(t) + \triangle_{22}W_m(t) + \in_0 L\Bigg(\Bigg[\frac{L}{8} + \frac{L^2}{2\pi^2}$$

$$\sum_{n=0}^{\infty}(2n+1)\Bigg(\Bigg[\frac{(-1)^{m+k}Cos(2n+1)\pi L - 1}{((2n+1)L)^2 - (m+k)^2} -$$

$$\frac{(-1)^{m-k}Cos(2n+1)\pi L - 1}{((2n+1)L)^2 - (m-k)^2}\Bigg]\frac{Cos(2n+1)\pi ct}{2n+1}\Bigg)\Bigg]\ddot{W}_m(t) +$$

$$+\frac{2cm\pi}{L}\Bigg[\frac{2Lk\pi}{k^2\pi^2 - m^2\pi^2} + \frac{L}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\Bigg(\frac{Sin(2n+1)\pi ct}{2n+1}$$

$$\Bigg[\frac{(m-k)\left\{(-1)^{m-k}Cos(2n+1)\pi L - 1\right\}}{((2n+1)L)^2 - (m-k)^2} -$$

$$\frac{(m+k)\left\{(-1)^{m+k}Cos(2n+1)\pi L - 1\right\}}{((2n+1)L)^2 - (m+k)^2}\Bigg]\Bigg)\Bigg]\dot{W}_m(t) - \frac{c^2m^2\pi^2}{L^2}$$

$$\Bigg[\frac{L}{8} + \frac{L}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\Bigg(\Bigg[\frac{(-1)^{m+k}Cos(2n+1)\pi L - 1}{((2n+1)L)^2 - (m+k)^2} -$$

$$\frac{(-1)^{m-k}Cos(2n+1)\pi L - 1}{((2n+1)L)^2 - (m-k)^2}\Bigg]\frac{Cos(2n+\ddot{1})\pi ct}{2n+1}\Bigg)\Bigg]W_m(t)\Bigg)$$

$$= \frac{MgL}{\mu_0 k\pi}\left[-(-1)^k + Cos\frac{k\pi ct}{L}\right] \tag{50}$$

where

$$\Delta_{11} = \left[\frac{L}{2} + \frac{L}{4\pi}AA1 - R_0\left(\frac{m\pi}{4L}AA2 - \frac{m^2\pi^2}{2L} - \frac{m^2\pi}{4L}AA1\right)\right] \tag{51}$$

$$\Delta_{22} = \Bigg[\frac{EI_0}{\mu_0}\left(\frac{5m^4\pi^4}{4L^3} + \frac{15m^2\pi^3}{16L^3}\left(1+m^2\right)AA1\right.$$

$$\left. - \frac{m^2\pi^4}{4L^4}\left(9+m^2\right)AA3\right) + \frac{KL}{2\mu_0}\Bigg] \tag{52}$$

and

$$AA1 = 2\begin{cases} 2, & 1 \pm 2m \; is \; even \\ \frac{-2}{1-4m^2}, & 1 \pm 2m \; is \; odd \end{cases} \tag{53}$$

$$AA2 = \begin{cases} 0, & 1 \pm 2m \; is \; even \\ \frac{-8m}{1-4m^2}, & 1 \pm 2m \; is \; odd \end{cases} \tag{54}$$

$$AA3 = \frac{L}{3\pi} - \frac{3L}{\pi}\begin{cases} 0, & 3 \pm 2m \; is \; even \\ \frac{1}{9-4m^2}, & 3 \pm 2m \; is \; odd \end{cases} \tag{55}$$

Equation (50) is now the fundamental equation of our problem when the non-uniform Rayleigh beam has simple supports at all edges. In what follows, two special cases of equation (50) are discussed.

3.1. Non-uniform Rayleigh Beam traversed by moving distributed force

An approximate model of the system when the inertia effect of the moving distributed mass M is neglected is obtained when ϵ_0 is set to zero in equation (50). We then have the moving distributed force problem associated with the system given by,

$$\ddot{W}_m(t) + \beta_f^2 W_m(t) = \frac{MgL}{\mu_0 k\pi\Delta_{11}}\left[-(-1)^k + Cos\frac{k\pi ct}{L}\right]. \tag{56}$$

where

$$\beta_f^2 = \frac{\Delta_{22}}{\Delta_{11}} \tag{57}$$

Solving equation (56) using the method of Laplace transforms and Convolution theory in conjunction with the initial conditions (9), one obtains

$$W_m(t) = \frac{PL}{\mu_0 k\pi\Delta_{11}}\left[\frac{Cos\alpha_c t - Cos\beta_f t}{\beta_f^2 - \alpha_c^2} + \frac{(-1)^k Cos\beta_f t}{\beta_f} - \frac{(-1)^k}{\beta_f}\right] \tag{58}$$

where

$$\alpha_c = \frac{k\pi c}{L} \qquad P = Mg \tag{59}$$

substituting (58) into (15), we have

$$V_n(x,t) = \frac{2}{L}\sum_{m=1}^{\infty}\left\{\frac{PL}{\mu_0 k\pi\Delta_{11}}\left[\frac{Cos\alpha_c t - Cos\beta_f t}{\beta_f^2 - \alpha_c^2} + \frac{(-1)^k Cos\beta_f t}{\beta_f} - \frac{(-1)^k}{\beta_f}\right] Sin\frac{m\pi x}{L}\right\} \tag{60}$$

Equation (60) represents the transverse displacement response to a distributed force moving at constant velocity of a simply supported non-uniform Rayleigh beam resting on elastic foundation.

3.2. Non-uniform Rayleigh Beam traversed by moving distributed mass

If the moving load has mass commensurable with that of the elastic beam, the inertia effect of the moving mass is not negligible and must be taken into consideration. Thus, $\in_0 \neq 0$ and we are required to solve the entire equation (50). This is termed the moving distributed mass problem. Evidently, an exact closed form solution of this equation is not possible. Thus we resort to the approximate analytical solution technique which is a modification of the asymptotic method of Struble discussed extensively in [32]. To this end, equation (50) is simplified and rearranged to take the form,

$$\ddot{W}_m(t) + \frac{2cm\pi\in_0 R_{II}(m,k,t)}{\Delta_{11}\left[1+\frac{\in_0 L}{\Delta_{11}}R_I(m,k,t)\right]}\dot{W}_m(t) -$$

$$-\left(\frac{\frac{c^2m^2\pi^2\in_0 L R_I(m,k,t)}{L^2\Delta_{11}} - \beta_f^2}{1+\frac{\in_0 L}{\Delta_{11}}R_I(m,k,t)}\right)W_m(t)$$

$$= \frac{PL\left[-(-1)^k + Cos\frac{k\pi ct}{L}\right]}{\mu_0.k\pi.\Delta_{11}\left[1+\frac{\in_0 L}{\Delta_{11}}R_I(m,k,t)\right]} \tag{61}$$

where

$$R_I(m,k,t) = \frac{L}{8} + \frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\,N(m,k)\frac{Cos(2n+1)\pi ct}{2n+1} \tag{62}$$

$$R_{II}(m,k,t) = \frac{2Lk\pi}{k^2\pi^2 - m^2\pi^2} + \frac{L}{2\pi^2}$$

$$\sum_{n=0}^{\infty}(2n+1)F(m,k)\,\frac{Sin\,(2n+1)\,\pi ct}{2n+1} \tag{63}$$

and

$$N(m,k) = \left[\frac{(-1)^{m+k}Cos\,(2n+1)\,\pi L - 1}{((2n+1)\,L)^2 - (m+k)^2} - \right.$$

$$\left.\frac{(-1)^{m-k}Cos\,(2n+1)\,\pi L - 1}{((2n+1)\,L)^2 - (m-k)^2}\right] \tag{64}$$

$$F(m,k) = \left[\frac{(m-k)\left\{(-1)^{m-k} Cos\,(2n+1)\,\pi L - 1\right\}}{((2n+1)\,L)^2 - (m-k)^2} - \frac{(m+k)\left\{(-1)^{m+k} Cos\,(2n+1)\,\pi L - 1\right\}}{((2n+1)\,L)^2 - (m+k)^2} \right] \quad (65)$$

By means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the distributed moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (61). Thus, the right hand of equation (61) is set to zero and a parameter $\in_1 < 1$ is considered for any arbitrary mass ratio $\in_0$, defined as

$$\in_1 = \frac{\in_0}{1+\in_0} \quad (66)$$

which implies

$$\in_0 = \in_1 + o\left(\in_1^2\right) \quad (67)$$

and

$$\frac{1}{1+\in_0\left(\frac{L}{\Delta_{11}} R_I(m,k,t)\right)} = 1 - \in_0\left(\frac{L}{\Delta_{11}} R_I\,(m,k,t)\right) + o\left(\in_0^2\right) \quad (68)$$

whenever

$$\left|\frac{L}{\Delta_{11}} R_I\,(m,k,t)\right| < 1 \quad (69)$$

When $\in_1 = 0$, a case corresponding to the case when the inertia effect of the mass of the system is neglected is obtained, then the solution of (61) can be written in the form,

$$W_m\,(t) = \psi_m Cos(\beta_f t - \theta_m) \quad (70)$$

where ψ_m and θ_m are constants.

Since $\in_1 < 1$, Struble's technique requires that the asymptotic solution of the homogeneous part of equation (61) be of the form (see Ref [34]),

$$W_m(t) = \psi(m,t) Cos(\beta_f t - \theta(m,t)) + \in_1 W_1(m,t) + o(\in_1^2) \quad (71)$$

where $\psi(m,t)$ and $\theta(m,t)$ are slowly varying functions of time.

Substituting equation (71) and its derivatives into the homogeneous part of equation (61) while taking into account (67) and retaining terms to $o(\in_1)$, one obtains

$$-2\dot{\psi}\,(m,t)\,\beta_f Sin\,(\beta_f t - \theta\,(m,t)) + 2\psi\,(m,t)\,\dot{\theta}(m,t)\beta_f$$

$$Cos\left(\beta_f t-\theta(m,t)\right)-\frac{2cm\pi\in_1}{\Delta_{11}}\left[\frac{2Lk\pi}{k^2\pi^2-m^2\pi^2}+\frac{L}{2\pi^2}\right.$$

$$\sum_{n=0}^{\infty}(2n+1)\left(\left[\frac{(m-k)\left\{(-1)^{m-k}Cos\,(2n+1)\,\pi L-1\right\}}{((2n+1)\,L)^2-(m-k)^2}-\right.\right.$$

$$\left.\left.\left.\frac{(m+k)\left\{(-1)^{m+k}Cos\,(2n+1)\,\pi L-1\right\}}{((2n+1)\,L)^2-(m+k)^2}\right]\frac{Sin\,(2n+1)\,\pi ct}{2n+1}\right)\right]$$

$$\psi\,(m,t)\,\beta_f Sin\left(\beta_f t-\theta\,(m,t)\right)-\frac{c^2m^2\pi^2\in_1 L}{\Delta_{11}.L^2}\left[\frac{L}{8}+\frac{L^2}{2\pi^2}\right.$$

$$\sum_{n=0}^{\infty}(2n+1)\left(\left[\frac{(-1)^{m+k}Cos\,(2n+1)\,\pi L-1}{((2n+1)\,L)^2-(m+k)^2}-\right.\right.$$

$$\left.\left.\left.\frac{(-1)^{m-k}Cos\,(2n+1)\,\pi L-1}{((2n+1)\,L)^2-(m-k)^2}\right]\frac{Cos(2n+1)\pi ct}{2n+1}\right)\right]\ \psi\,(m,t)$$

$$Cos\left(\beta_f t-\theta(m,t)\right)-\frac{\beta_f^2\in_1 L\psi\,(m,t)}{\Delta_{11}}\left[\frac{L}{8}+\right.$$

$$\frac{L^2}{2\pi^2}\sum_{n=0}^{\infty}(2n+1)\left(\left[\frac{(-1)^{m+k}Cos\,(2n+1)\,\pi L-1}{((2n+1)\,L)^2-(m+k)^2}-\right.\right.$$

$$\left.\left.\left.\frac{(-1)^{m-k}Cos\,(2n+1)\,\pi L-1}{((2n+1)\,L)^2-(m-k)^2}\right]\frac{Cos\,(2n+1)\,\pi ct}{2n+1}\right)\right]$$

$$Cos\left(\beta_f t-\theta(m,t)\right)=0 \tag{72}$$

To obtain variational equations, we equate the coefficients of $Sin\left(\beta_f t-\theta(m,t)\right)$ and $Cos\left(\beta_f t-\theta(m,t)\right)$ on both sides of equation (72). To do this, we note the following trigonometric identities,

$$\begin{aligned}&\frac{Cos(2n+1)\pi ct}{2n+1}Sin\left(\beta_f t-\theta(m,t)\right)=\\&\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}+\beta_f t-\theta(m,t)\right]\\&-\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}-\beta_f t+\theta(m,t)\right]\\&\frac{Sin(2n+1)\pi ct}{2n+1}Sin\left(\beta_f t-\theta(m,t)\right)=\\&\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}-\beta_f t+\theta(m,t)\right]\end{aligned} \tag{73}$$

$$-\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}+\beta_f t-\theta(m,t)\right] \tag{74}$$

$$\frac{Cos(2n+1)\pi ct}{2n+1}Cos\left(\beta_f t-\theta(m,t)\right)=$$
$$\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}+\beta_f t-\theta(m,t)\right]$$
$$+\frac{1}{2}Cos\left[\frac{(2n+1)\pi ct}{2n+1}-\beta_f t+\theta(m,t)\right] \tag{75}$$

$$\frac{Sin(2n+1)\pi ct}{2n+1}Cos\left(\beta_f t-\theta(m,t)\right)=$$
$$\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}+\beta_f t-\theta(m,t)\right]$$
$$+\frac{1}{2}Sin\left[\frac{(2n+1)\pi ct}{2n+1}-\beta_f t+\theta(m,t)\right] \tag{76}$$

Neglecting terms that do not contribute to the variational equation, equation (72) reduces to

$$-2\dot{\psi}\left(m,t\right)\beta_f Sin\left(\beta_f t-\theta\left(m,t\right)\right)+2\psi\left(m,t\right)\dot{\theta}\left(m,t\right)\beta_f$$
$$Cos\left(\beta_f t-\theta\left(m,t\right)\right)-\frac{2cm\pi\in_1}{\Delta_{11}}\left[\frac{2Lk\pi}{k^2\pi^2-m^2\pi^2}\right]\psi\left(m,t\right)$$
$$\beta_f Sin\left(\beta_f t-\theta\left(m,t\right)\right)-\frac{c^2m^2\pi^2\in_1 L^2}{8\Delta_{11}.L^2}\ \psi\left(m,t\right)Cos$$
$$\left(\beta_f t-\theta(m,t)\right)-\frac{\beta_f^2\in_1 L^2}{8\Delta_{11}}\psi\left(m,t\right)\ Cos\left(\beta_f t-\theta(m,t)\right)=0 \tag{77}$$

Then, the variational equations are respectively

$$-2\dot{\psi}(m,t)-\frac{4cmkL\in_1}{\Delta_{11}\left(k^2-m^2\right)}\psi(m,t)=0 \tag{78}$$

and

$$2\dot{\theta}(m,t)\beta_f-\frac{c^2m^2\pi^2\in_1}{8\Delta_{11}}-\frac{\beta_f^2\in_1 L^2}{8\Delta_{11}}=0 \tag{79}$$

solving equations (78) and (79) respectively, one obtains

$$\psi(m,t)=\psi_m \tag{80}$$

and

$$\theta(m,t)=\frac{\beta_f^2L^2+c^2m^2\pi^2}{16\Delta_{11}\beta_f}\in_1 t+\theta_m \tag{81}$$

where

$$\psi_m=Ae^{-\chi^t},\qquad \chi=\frac{2cmkL\in_1}{\Delta_{11}\left(k^2-m^2\right)} \tag{82}$$

A and θ_m are constants.
Therefore, when the mass effect of the particle is considered, the first approximation to the homogeneous system is given by

$$W_m(t) = \psi_m Cos\left(\Omega_f t - \theta_m\right) \tag{83}$$

where,

$$\Omega_f = \frac{16\triangle_{11}\beta_f^2 L^2 - \in_1\left\{\beta_f^2 L^2 + c^2 m^2 \pi^2\right\}}{16\Delta_{11}\beta_f L^2} \tag{84}$$

is called the modified natural frequency of the free system due to the presence of the moving distributed mass. Thus, to solve the non-homogeneous equation (61), the differential operator which acts on $W_m\left(t\right)$ is replaced by the equivalent free system operator defined by the modified frequency Ω_f, i.e

$$\ddot{W}_m(t) + \Omega_f^2 W_m(t) = \frac{\in_1 L^2 g}{k\pi.\Delta_{11}}\left[-(-1)^k + Cos\frac{k\pi ct}{L}\right] \tag{85}$$

Solving equation (85) in conjunction with the initial condition, one obtains expression for $W_m(t)$. Thus, in view of (15)

$$V_n\left(x,t\right) = \frac{2}{L}\sum_{m=1}^{n}\frac{\in_1 L^2 g}{k\pi.\Delta_{11}}$$

$$\left[\frac{Cos\alpha_c t - Cos\Omega_f t}{\Omega_f^2 - \alpha_c^2} + \frac{(-1)^k Cos\Omega_f t}{\Omega_f} - \frac{(-1)^k}{\Omega_f}\right] Sin\frac{m\pi x}{L} \tag{86}$$

Equation (86) represents the transverse displacement response to a distributed mass moving with constant velocity of a simply supported non-uniform Rayleigh beam resting on elastic foundation.

4. DISCUSSION OF THE SOLUTIONS

4.1. Analytical Solutions

For analytical results, we will establish conditions under which resonance occurs for an undamped system such as this. Resonance takes place when the motion of the vibrating structure becomes unbounded. In actual practice, when this happens, the structure would collapse as the intensive vibrations cause cracks or permanent deformation in the vibrating systems. Equation (3.49) clearly shows that the non-uniform simply supported Rayleigh beam traversed by a moving distributed force reaches a state of resonance whenever

$$\beta_f = \frac{k\pi c}{L} \tag{87}$$

while equation (86) indicates that the same non-uniform beam under the action of a moving distributed mass experiences resonance effect when

$$\Omega_f = \frac{k\pi c}{L} \tag{88}$$

Evidently,

$$\Omega_f = \beta_f \left[1 - \frac{\eta_1}{16\Delta_{11}} \left\{1 + \frac{c^2 m^2 \pi^2}{\beta_f^2 L^2}\right\}\right] = \frac{k\pi c}{L} \tag{89}$$

Equations (87) and (89) show that for the same natural frequency, the critical speed for the system consisting of a non-uniform simply supported Rayleigh beam traversed by a moving distributed force at constant velocity is greater than that of the moving distributed mass problem. Thus, for the same natural frequency, resonance is reached earlier in the moving distributed mass system than in the moving distributed force system.

4.2. Numerical Solutions

Numerical results for the analysis presented above are presented in plotted curves. An elastic non-uniform Rayleigh beam of length L=12.192m is considered. Other values used are c=8.128ms^{-1}, $\in_0 = 0.2$ and E=2109$\times 10^9$kg/m. The values of the rotatory inertia correction factor R_0 are varied between 0.5 and 9.5, while the values of the foundation modulli K are varied between 0 and 4000000Nm^2. Figure 1 displays the deflection profile of a finite non-uniform simply supported Rayleigh beam under the action of distributed forces for various values of foundation modulli K and fixed rotatory inertia correction factor R_0=5. The graph shows that the response amplitude decreases as the value of K increases. In Figure 2, the deflection profile of the same beam under the action of distributed forces for various values of rotatory inertia correction factor R_0 and fixed foundation modulli K=4000 is displayed. It is evident that the response amplitude decreases as the value of R_0 increases. Figure 3 depicts the displacement response of the beam to moving distributed masses for various values of foundation modulli K and fixed rotatory inertia correction factor $R_0 = 5$. It is shown that the response amplitude decreases as the value of K increases. Furthermore, for various time t,the displacement of the Rayleigh beam under the action of distributed masses for various values of rotatory inertia correction factor R_0 and for fixed foundation modulli K=4000 is shown in figure 4 Clearly, the response amplitude of the beam decreases as the value of R_0 increases. Finally, figure 5, depicts the comparison of the displacement response to moving distributed force and moving distributed mass for the simply supported finite non-uniform Rayleigh beam for K=4000 and $R_0 = 5$. It is seen that the response amplitudes of the moving distributed mass is higher than that of the moving distributed force. Thus resonance is reached earlier in the moving distributed mass system than in the moving distributed force system; this agrees with the results obtained in the analytical solutions.

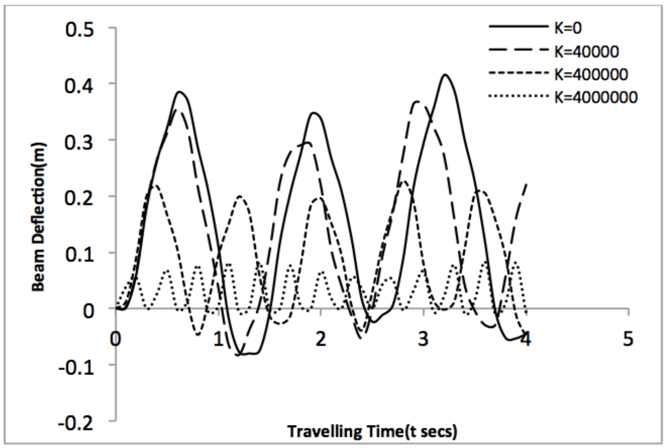

Fig. 1. Displacement response to moving distributed forces of simply supported non-uniform Rayleigh beam for various values of foundation moduli K

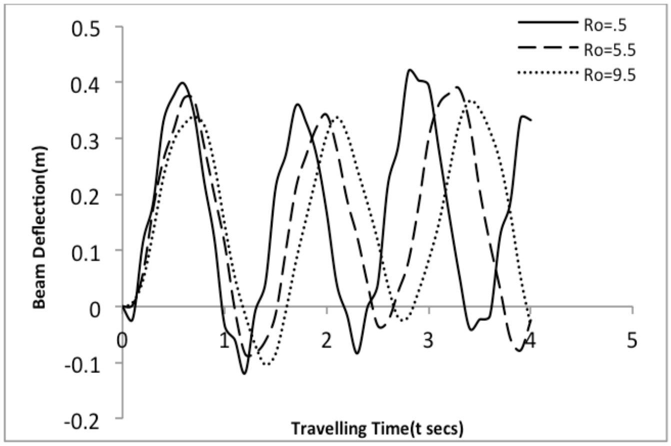

Fig. 2. Displacement response to distributed forces of simply supported non-uniform Rayleigh beam for various values of rotatory inertia factor R_0

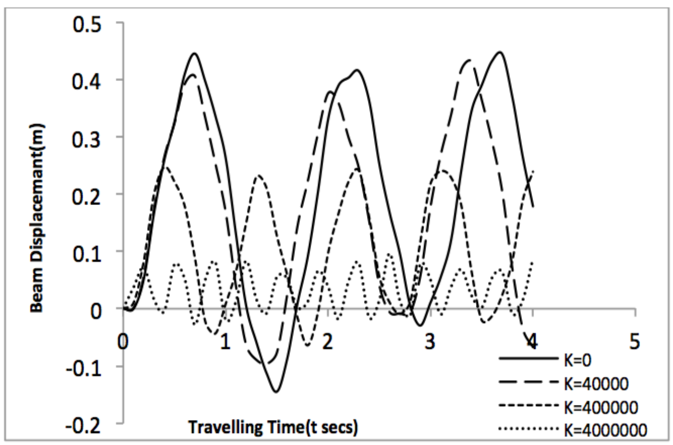

Fig. 3. Displacement response to distributed masses of simply supported non-uniform Rayleigh beam for various values of foundation moduli K

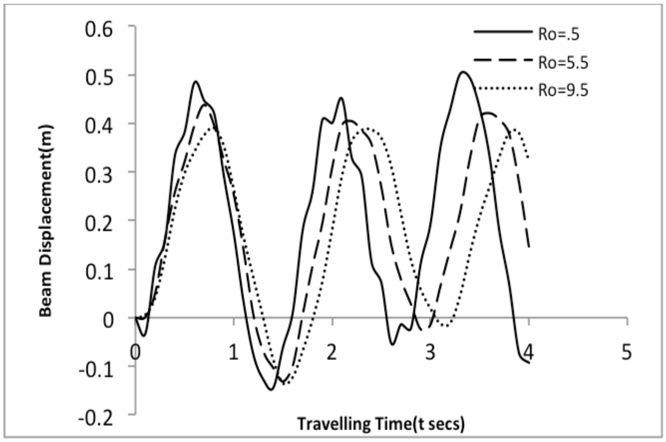

Fig. 4. Displacement response to distributed masses of simply supported non-uniform Rayleigh beam for various values of rotator inertia factor R_0

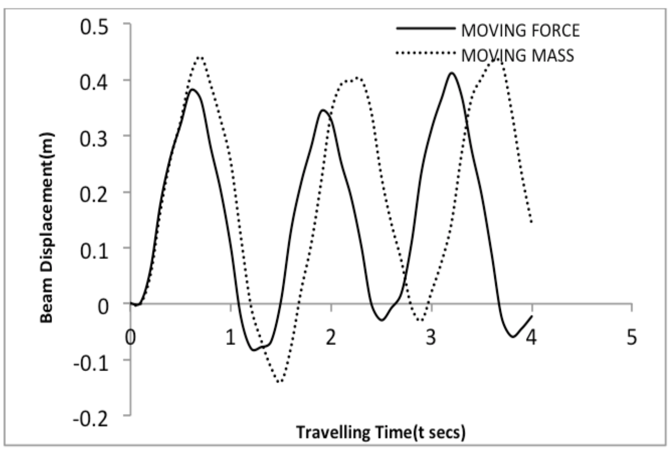

Fig. 5. Comparison of the displacement response of moving distributed force and moving distributed mass cases for simply supported non-uniform Rayleigh beam for R_0=5 and K=4000

5. CONCLUSION

The problem of a non-uniform simply supported Rayleigh beam under travelling distributed loads is investigated in this paper. Both gravity and inertia effects of the distributed loads are taken into consideration. The technique due to Galerkin is employed to reduce the governing fourth order partial differential equation with variable coefficients to a sequence of second order ordinary differential equations. These series of equations are simplified using a modification of the asymptotic method of Struble. The resulting equations are then solved using the method of integral transformations. Numerical analysis is carried out and results show that the response amplitudes of both moving distributed force and moving distributed mass problems decrease with increase in the values of foundation moduli. Similarly, higher values of rotatory inertia correction factor reduce the response amplitudes of both moving distributed force and moving distributed mass problems. Finally, it is observed that for the non-uniform Rayleigh beam having simple supports at both ends, for fixed values of foundation moduli and rotatory inertia correction factor, the transverse deflection under the actions of moving distributed masses is higher than that under the actions of moving distributed force. Thus, moving distributed force solution is not always an upper bound to moving distributed mass problems. This is in agreement with existing results where the moving load is modelled as moving concentrated load.

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