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A FAMILY OF HYBRID LINEAR MULTI-STEP METHODS TYPE FOR SPECIAL THIRD ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, we derive a family of three step hybrid linear multi-step method type with one to three off-step points. Orders and error constants and convergence analysis of the proposed method are established. Numerical experiments on special third order initial and boundary value problems (IVPs, BVPs) are performed to show the efficiency and accuracy of the proposed methods over existing method found in the literature.

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1. INTRODUCTION

The mathematical formulation of some physical phenomena in science and engineering [1, 2, 3] leads to special third-order initial and boundary value problems of the type:

$$y''' = f(x,y), y(a) = y_0, y'(a) = \eta_0, y''(a) = \eta_1, \tag{1}$$

$$y''' = f(x,y), y(a) = y_0, y'(a) = \delta_0, y(b) = y_M, \tag{2}$$

$$y''' = f(x,y), y(a) = y_0, y'(a) = \delta_0, y'(b) = y_M. \tag{3}$$

Various approaches can be used to find the analytical solutions of third-order ordinary differential equations (ODEs). However, only a limited number of numerical methods are available for solving Equations (1)-(3) directly without reducing to an equivalent first-order system of differential equations. Some authors have proposed a solution to third-order of ODE using different analytical techniques, for instance, the linearizing tangent transformation [4] leads Equation (1) into the second-order ODE, the extension of Stäckel transform [5].

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For the numerical integration of general third-order of ODEs, Awoyemi et al. [6, 7, 8] derived a p-stable linear multistep method (LMM) which is in form of predictor-corrector and like most LMMs, which requires one-step method to get starting values. The predictors are also developed in the same way as correctors. Moreover, the block methods in Fatunla [9] are discrete and are proposed for non-stiff special second-order ODEs in form of a predictor-corrector integration process. Zainuddin et al. [10] solved third-order ODEs directly by using the block backward differentiation formula. Also like other LMMs, they are usually applied to the IVPs as a single formula but they are not self-starting; and they advance the numerical integration of the ODEs in one-step at a time, which leads to overlapping of the piecewise polynomials solution model.

Approaches for finding approximation solutions of IVP and BVP are actively developing, for example, the geometric representation concept was proposed by Hairer et al. [11] have been developed in more details as systematic tri-colored tree theory [12].

There is the need to develop a method which is self-starting, eliminating the use of predictors with better accuracy and efficiency. Recently, several researchers [13, 14, 15, 16, 17] proposed LMMs for the direct solution of the general second-order IVPs, which were shown to be zero-stable and implemented without the need for either predictors or starting values from other methods. Jator used the LMMs developed for IVPs and additional methods obtained from the same continuous k-step LMM to solve third order boundary value problems with Dirichlet and Neumann boundary conditions and also Awoyemi et al. [8], Olabode and Yusuph [18] developed a LMM for the direct solution of IVP for a special third order of ODEs. Also many authors have solved third order BVP by transforming them to IVPs and then solving using Runge-Kutta method (see, [12, 19]). Many methods like single finite difference method, spline method proposed by Khan and Aziz [20], non-polynomial spline proposed by Islam and Tirmizi [21], quartic splines studied by Pandey [22] and high order difference method by Salama and Monsor [23] are used in solving third order BVP. These methods were applied by reducing the BVP to an equivalent system of first order ODEs which consume a lot of time and human effort. Biala et al. [24] studied the efficiency of Boundary Value Methods (BVMs) in combination with methods of lines on second order BVPs. Jator [25] derived a LMM for direct solution of third order BVP without reducing it to initial value problem or system of first order equivalent. Sahi et al. [26] derived a continuous forth derivatives method for third order BVP. Fazal-i-Hag et al. [27] proposed a collocation method with the Haar basis functions for the numerical solution both BVPs and IVP without transformation of BVPs into IVPs. Aboiyar et al. [28] proposed the continuous LMM based on Hermite Polynomials as basis functions.

In this research, we extended the works of Jator [25] and Olabode and Yusuph [18] into a hybrid linear multi-step method using collocation and interpolation procedures by considering one-three off-step points in order to solve special third order IVPs and BVPs.

This study, therefore proposes a block hybrid multistep method for the direct solution of third order initial value problems of ordinary differential equations.

The paper is organized as follows. In Section 2, we derive a continuous approximation $Y(x)$ for the exact solution $y(x)$. Section 3 is devoted to the specification of the order and error constant of proposed hybrid linear multi-step methods. In Section 4, stability of proposed hybrid linear multi-step methods is shown. A brief discussion of numerical results is presented in Section 5.

2. DERIVATION METHOD

The main objective here is to derive a modified linear multi-step algorithms. This algorithm shall be in the form shown below

$$\sum_{j=0}^{r-1} \alpha_j y_{n+j} = h^3 \sum_{j=0}^{k} \beta_j y_{n+j} + h^3 \beta_\eta f_{n+\eta} + h^3 \beta_\nu f_{n+\nu} + h^3 \beta_\mu f_{n+\mu} \quad (4)$$

where α_j, β_j, β_η, β_ν and β_μ are unknown constant and η, ν and μ must not be specified as an integer, h is the step size. It is important to note here that $\alpha_k = 1$, $\beta_k \neq 0$, α_0 and β_0 are non-zero. Equation (4) is obtained by assuming the approximate solution $y(x)$ as

$$y(x) = \sum_{j=0}^{r+s-1} a_j x^j \quad (5)$$

where a_j are coefficients and $r = k$ and $s \geq 5$ are distinct interpolation and collocation points. The continuous approximation is then constructed with the imposition of two conditions stated in next equations:

$$y(x_{n+j}) = y_{n+j}, j = 0, 1, 2, \ldots, r-1, \quad (6)$$

$$y'''(x_{n+j}) = f_{n+j}. \quad (7)$$

Equations (6) and (7) result to a $(r+s)$ system of equations which can be evaluated for solution through matrix inversion algorithm. This is with a view to obtaining values for a_j. The construction of final approximation is executed through the substitution of the values of a_j into Equation (5). The method of continuous approximation can be adequately expressed as

$$\begin{aligned} y(x) &= \sum_{j=0}^{r-1}\alpha_j(x)y_{n+j} + h^3\sum_{j=0}^{k}\beta_j(x)y_{n+j} \\ &+ h^3\beta_\eta(x)f_{n+\eta}+h^3\beta_\nu(x)f_{n+\nu}+h^3\beta_\mu(x)f_{n+\mu} \end{aligned} \tag{8}$$

where $\alpha_j(x)$, $\beta_j(x)$, $\beta_\eta(x)$ and $\beta_\mu(x)$ are continuous coefficients. The first and second derivative formulae are as follows:

$$\begin{aligned} y'(x) &= \frac{1}{h}(\sum_{j=0}^{r-1}\alpha'_j(x)y_{n+j} + h^3\sum_{j=0}^{k}\beta'_j(x)y_{n+j} \\ &+ h^3\beta'_\eta(x)f_{n+\eta} + h^3\beta'_\nu(x)f_{n+\nu}+h^3\beta'_\mu(x)f_{n+\mu}), \end{aligned} \tag{9}$$

$$\begin{aligned} y''(x) &= \frac{1}{h^2}(\sum_{j=0}^{r-1}\alpha''_j(x)y_{n+j} + h^3\sum_{j=0}^{k}\beta''_j(x)y_{n+j} \\ &+ h^3\beta''_\eta(x)f_{n+\eta}+h^3\beta''_\nu(x)f_{n+\nu}+h^3\beta''_\mu(x)f_{n+\mu}), \end{aligned} \tag{10}$$

to obtain additional equation and derivative by imposing that

$$y'(x) = \delta(x), y''(x) = \gamma(x), \tag{11}$$

$$y'(a) = \delta_0, y''(a) = \gamma_0. \tag{12}$$

2.1. THREE-STEP HYBRID LINEAR METHOD WITH ONE OFF-STEP COLLOCATION POINT (3SHLM1)

We use Equation (8) to obtain a 3-step HLM with the following specification: $r = 3$, $s = 5$, $\eta = \frac{8}{3}$, $k = 3$, and $\alpha_j(x)$, $\beta_j(x)$, $\beta_\eta(x)$ can be expressed as functions of t, given that $t = \frac{x-x_n}{h}$ to obtain the continuous form as follows

$$\begin{aligned} y(x) &= \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} \\ &+ h^3[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_{\frac{8}{3}} f_{n+\frac{8}{3}} + \beta_3 f_{n+3}]. \end{aligned} \tag{13}$$

Initial value problem. Evaluate Equation (13) at $x = x_{n+3}$, $x = x_{n+\frac{8}{3}}$ to get the following

$$\begin{aligned} y_{n+3} &= y_n - 3y_{n+1} + 3y_{n+2} \\ &+ \frac{h^3}{800}[5f_n + 376f_{n+1} + 460f_{n+2} - 81f_{n+\frac{8}{3}} + 40f_{n+3}], \end{aligned} \tag{14}$$

$$\begin{aligned} y_{n+\frac{8}{3}} &= \frac{5}{9}y_n - \frac{16}{9}y_{n+1} + \frac{20}{9}y_{n+2} \\ &+ \frac{h^3}{8748}[31f_n + 2268f_{n+1} + 2436f_{n+2} - 675f_{n+\frac{8}{3}} + 260f_{n+3}] \end{aligned} \tag{15}$$

to start the IVP for $n = 0$, the additional method can be obtained from Equation (12) as follows

$$\begin{aligned} h\delta_0 &= -\frac{3}{2}y_0 + 2y_1 - \frac{1}{2}y_2 \\ &+ \frac{h^3}{16800}[975f_0 + 5596f_1 - 2300f_2 + 2349f_{\frac{8}{3}} - 1020f_3], \end{aligned} \tag{16}$$

$$\begin{aligned} h^2\gamma_0 &= y_0 - 2y_1 + y_2 \\ &+ \frac{h^3}{1440}[-451f_0 - 1308f_1 + 732f_2 - 729f_{\frac{8}{3}} + 316f_3]. \end{aligned} \tag{17}$$

Boundary value problem. We need two additional methods which can be combined with Equations (14)-(15) to simultaneously solve third order BVPs. Hence, we assume that $\delta(x)$ and $\gamma(x)$ are continuous at $x = x_{n+3}$. Hence, the following two additional methods were obtained

$$\begin{aligned} &y_{n+5} - 4y_{n+4} + 3y_{n+3} + 5y_{n+2} - 8y_{n+1} + 3y_n \\ &= h^3[-\frac{61}{3360}f_n - \frac{599}{420}f_{n+1} - \frac{353}{168}f_{n+2} + \frac{27}{1120}f_{n+\frac{8}{3}} - \frac{7}{240}f_{n+3} \\ &+\frac{1399}{2100}f_{n+4} - \frac{23}{84}f_{n+5} + \frac{783}{2800}f_{n+\frac{17}{3}} - \frac{17}{140}f_{n+6}], \end{aligned} \tag{18}$$

$$\begin{aligned} &y_{n+5} - 2y_{n+4} + y_{n+3} - y_{n+2} + 2y_{n+1} - y_n \\ &= h^3[\frac{1}{180}f_n + \frac{73}{150}f_{n+1} + \frac{23}{24}f_{n+2} + \frac{81}{200}f_{n+\frac{8}{3}} + \frac{659}{1440}f_{n+3} \\ &+\frac{109}{120}f_{n+4} - \frac{61}{120}f_{n+5} + \frac{81}{160}f_{n+\frac{17}{3}} - \frac{79}{360}f_{n+6}] \end{aligned} \tag{19}$$

where

$$\begin{aligned} \delta(x) = \frac{1}{h}\Bigg[&\frac{3}{2}y_n - 4y_{n+1} + \frac{5}{2}y_{n+2} \\ &+ h^3\left(\frac{61}{6720}f_n + \frac{599}{840}f_{n+1} + \frac{353}{336}f_{n+2}\right. \\ &\left.- \frac{27}{2240}f_{n+\frac{8}{3}} + \frac{61}{840}f_{n+3}\right)\Bigg], \quad x_n \le x \le x_{n+3}, \end{aligned} \tag{20a}$$

$$\begin{aligned} \delta(x) = \frac{1}{h}\Bigg[&-\frac{3}{2}y_{n+3} + 2y_{n+4} - \frac{1}{2}y_{n+5} \\ &+ h^3\left(\frac{13}{224}f_{n+3} + \frac{1399}{4200}35f_{n+4} - \frac{23}{168}f_{n+5}\right. \\ &\left.+ \frac{783}{5600}f_{n+\frac{17}{3}} - \frac{17}{280}f_{n+6}\right)\Bigg], \quad x_{n+3} < x \le x_{n+6}. \end{aligned} \tag{20b}$$

and

$$\gamma(x) = \frac{1}{h^3}\Bigg[y_n - 2y_{n+1} + y_{n+2} + \left(\frac{h^3}{1800}\left(10f_n + 876f_{n+1} + 1725f_{n+2}\right)\right] + 729f_{n+\frac{8}{3}} + 260f_{n+3}\Bigg)\Bigg], \quad x_n \le x \le x_{n+3}, \tag{21a}$$

$$\gamma(x) = \frac{1}{h^2}\Bigg[y_{n+3} - 2y_{n+4} + y_{n+5} + \left(\frac{h^3}{360}\left(-451f_{n+3} - 1308f_{n+4} + 732f_{n+5}\right)\right] - 729f_{n+\frac{17}{3}} + 316f_{n+6}\Bigg)\Bigg], \quad x_{n+3} < x \le x_{n+6}. \tag{21b}$$

2.2. THREE-STEP HYBRID LINEAR METHOD WITH TWO OFF-STEP COLLOCATION POINTS (3SHLM2)

Using Equation (8) to obtain a 3-step HLM with the following specification as: $r = 3$, $s = 6$, $\nu = \frac{5}{2}$, $\mu = \frac{8}{3}$, $k = 3$, $\beta_\eta(x) = 0$, and $\alpha_j(x)$, $\beta_j(x)$, $\beta_\nu(x)$, $\beta_\mu(x)$ can be expressed as function of t, given that $t = \frac{x-x_n}{h}$ to obtain the continuous form as follows

$$\begin{aligned} y(x) &= \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} \\ &+ h^3[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_{\frac{8}{3}} f_{n+\frac{8}{3}} + \beta_3 f_{n+3}], \end{aligned} \tag{22}$$

Initial value problem. Evaluating Equation (22) at $x = x_{n+3}$, $x = x_{n+\frac{8}{3}}$, $x = x_{n+\frac{5}{2}}$, we get the following

$$\begin{aligned} y_{n+3} &= y_n - 3y_{n+1} + 3y_{n+2} \\ &+ \frac{h^3}{1200}[11f_n + 536f_{n+1} + 900f_{n+2} - 896f_{n+\frac{5}{2}} + 729f_{n+\frac{8}{3}} - 80f_{n+3}], \end{aligned} \tag{23}$$

$$\begin{aligned} y_{n+\frac{8}{3}} &= \frac{5}{9}y_n - \frac{16}{9}y_{n+1} + \frac{20}{9}y_{n+2} \\ &+ \frac{h^3}{52488}[271f_n + 81304f_{n+1} + 19716f_{n+2} - 21760f_{n+\frac{5}{2}} \\ &+ 16605f_{n+\frac{8}{3}} - 1840f_{n+3}], \end{aligned} \tag{24}$$

$$\begin{aligned} y_{n+\frac{5}{2}} &= \frac{3}{8}y_n - \frac{5}{4}y_{n+1} + \frac{15}{8}y_{n+2} \\ &+ \frac{h^3}{491520}[1729f_n + 81304f_{n+1} + 115740f_{n+2} - 135424f_{n+\frac{5}{2}} \\ &+ 101331f_{n+\frac{8}{3}} - 11080f_{n+3}]. \end{aligned} \tag{25}$$

In order to start IVP for $n = 0$, the additional method can be obtained from Equation (12) as follows

$$\begin{aligned} h\delta_0 &= -\frac{3}{2}y_0 + 2y_1 - \frac{1}{2}y_2 \\ &+ h^3[\frac{5039}{100800}f_0 + \frac{626}{1575}f_1 - \frac{347}{560}f_2 + \frac{3244}{1575}f_{\frac{5}{2}} - \frac{20331}{11200}f_{\frac{8}{3}} + \frac{47}{180}f_3], \end{aligned} \quad (26)$$

$$\begin{aligned} h^2\gamma_0 &= y_0 - 2y_1 + y_2 \\ &+ h^3[-\frac{5723}{20160}f_0 - \frac{8}{7}f_1 + \frac{3809}{1680}f_2 - \frac{788}{105}f_{\frac{5}{2}} + \frac{14823}{2240}f_{\frac{8}{3}} - \frac{1201}{1260}f_3]. \end{aligned} \quad (27)$$

Boundary value problem. We need two additional methods which can be combined with Equations (23)-(25) to simultaneously solve third order BVPs. Hence, we imposed that $\delta(x)$ and $\gamma(x)$ are continuous at $x = x_{n+3}$. Hence, the following two additional methods were obtained

$$\begin{aligned} & y_{n+5} - 4y_{n+4} + 3y_{n+3} + 5y_{n+2} - 8y_{n+1} + 3y_n \\ &= h^3[-\frac{271}{10080}f_n - \frac{1709}{1260}f_{n+1} - \frac{21}{8}f_{n+2} + \frac{704}{315}f_{n+\frac{5}{2}} - \frac{2349}{1120}f_{n+\frac{8}{3}} \\ &+ \frac{15319}{50400}f_{n+3} + \frac{1252}{1575}f_{n+4} - \frac{347}{280}f_{n+5} - \frac{6488}{1575}f_{n+\frac{11}{2}} - \frac{20331}{5600}f_{n+\frac{17}{3}} - \frac{47}{90}f_{n+6}], \end{aligned} \quad (28)$$

$$\begin{aligned} & y_{n+5} - 2y_{n+4} + y_{n+3} - y_{n+2} + 2y_{n+1} - y_n \\ &= h^3[\frac{869}{100800}f_n + \frac{647}{1400}f_{n+1} + \frac{1919}{1680}f_{n+2} - \frac{412}{525}f_{n+\frac{5}{2}} + \frac{12879}{11200}f_{n+\frac{8}{3}} \\ &+ \frac{6163}{20160}f_{n+3} + \frac{8}{7}f_{n+4} - \frac{3809}{1680}f_{n+5} + \frac{788}{105}f_{n+\frac{11}{2}} - \frac{14823}{2240}f_{n+\frac{17}{3}} + \frac{1201}{1260}f_{n+6}]. \end{aligned} \quad (29)$$

2.3. THREE-STEP HYBRID LINEAR METHOD WITH THREE OFF-STEP COLLOCATION POINTS (3SHLM3)

Using Equation (8) to obtain a 3-step HLM with the following specifications as: $r = 3$, $s = 7$, $\eta = \frac{7}{3}$, $\nu = \frac{5}{2}$, $\mu = \frac{8}{3}$, $k = 3$, and $\alpha_j(x)$, $\beta_j(x)$, $\beta_\eta(x)$, $\beta_\nu(x)$, $\beta_\mu(x)$ can be expressed as functions of t, given that $t = \frac{x-x_n}{h}$ to obtain the continuous form as follows

$$\begin{aligned} y(x) &= \alpha_0 y_n + \alpha_1 y_{n+1} + \alpha_2 y_{n+2} \\ &+ h^3[\beta_0 f_n + \beta_1 f_{n+1} + \beta_2 f_{n+2} + \beta_{\frac{7}{3}} f_{n+\frac{7}{3}} + \beta_{\frac{5}{2}} f_{n+\frac{5}{2}} + \beta_{\frac{8}{3}} f_{n+\frac{8}{3}} + \beta_3 f_{n+3}]. \end{aligned} \quad (30)$$

Initial value problem. Evaluating Equation (30) at $x = x_{n+3}$, $x = x_{n+\frac{8}{3}}$, $x = x_{n+\frac{5}{2}}$, $x = x_{n+\frac{7}{3}}$, we get the following

$$\begin{aligned} y_{n+3} &= y_n - 3y_{n+1} + 3y_{n+2} \\ &+ h^3[\frac{433}{47040}f_n + \frac{1499}{3360}f_{n+1} + \frac{429}{560}f_{n+2} - \frac{729}{7840}f_{n+\frac{7}{3}} - \frac{64}{105}f_{n+\frac{5}{2}} \\ &+ \frac{243}{448}f_{n+\frac{8}{3}} - \frac{103}{1680}f_{n+3}], \end{aligned} \quad (31)$$

$$\begin{aligned} y_{n+\frac{8}{3}} &= \frac{5}{9}y_n - \frac{16}{9}y_{n+1} + \frac{20}{9}y_{n+2} \\ &+ h^3[\frac{1189943}{231472080}f_n + \frac{4077487}{16533720}f_{n+1} + \frac{201851}{551124}f_{n+2} + \frac{5167}{95256}f_{n+\frac{7}{3}} \\ &- \frac{1022144}{2066715}f_{n+\frac{5}{2}} + \frac{48217}{136080}f_{n+\frac{8}{3}} - \frac{63127}{1653372}f_{n+3}], \end{aligned} \quad (32)$$

$$\begin{aligned} y_{n+\frac{5}{2}} &= \frac{3}{8}y_n - \frac{5}{4}y_{n+1} + \frac{15}{8}y_{n+2} \\ &+ h^3[\frac{167729}{48168960}f_n + \frac{570841}{3440640}f_{n+1} + \frac{25293}{114688}f_{n+2} + \frac{138753}{1605632}f_{n+\frac{7}{3}} \\ &- \frac{677}{1680}f_{n+\frac{5}{2}} + \frac{611631}{2293760}f_{n+\frac{8}{3}} - \frac{9469}{344064}f_{n+3}], \end{aligned} \quad (33)$$

$$\begin{aligned} y_{n+\frac{7}{3}} &= \frac{2}{9}y_n - \frac{7}{9}y_{n+1} + \frac{14}{9}y_{n+2} \\ &+ h^3[\frac{979}{472392}f_n + \frac{231251}{2361960}f_{n+1} + \frac{21853}{196830}f_{n+2} + \frac{877}{9720}f_{n+\frac{7}{3}} \\ &- \frac{85408}{295245}f_{n+\frac{5}{2}} + \frac{347}{1944}f_{n+\frac{8}{3}} - \frac{20797}{1180980}f_{n+3}]. \end{aligned} \quad (34)$$

In order to start IVP for $n = 0$, the additional method can be obtained from Equation (12) as

$$\begin{aligned} h\delta_0 &= -\frac{3}{2}y_0 + 2y_1 - \frac{1}{2}y_2 \\ &+ h^3[\frac{8611}{201600}f_0 + \frac{50333}{100800}f_1 - \frac{4117}{1120}f_2 + \frac{39609}{2240}f_{\frac{7}{3}} - \frac{37832}{1575}f_{\frac{5}{2}} \\ &+ \frac{236601}{22400}f_{\frac{8}{3}} - \frac{1091}{1440}f_3], \end{aligned} \quad (35)$$

$$\begin{aligned} h^2\gamma_0 &= y_0 - 2y_1 + y_2 \\ &+ h^3[-\frac{72481}{282240}f_0 - \frac{487}{320}f_1 + \frac{45823}{3360}f_2 - \frac{206307}{3136}f_{\frac{7}{3}} + \frac{1880}{21}f_{\frac{5}{2}} \\ &- \frac{176661}{4480}f_{\frac{8}{3}} + \frac{28597}{10080}f_3]. \end{aligned} \quad (36)$$

Boundary value problem. We need two additional methods which can be combined with Equations (31)-(34) to simultaneously solve third order BVPs. Hence, we imposed that $\delta(x)$ and $\gamma(x)$ are continuous at $x = x_{n+3}$. Hence, the following two additional methods were obtained

$$\begin{aligned} & y_{n+5} - 4y_{n+4} + 3y_{n+3} + 5y_{n+2} - 8y_{n+1} + 3y_n \\ = h^3[&-\frac{1207}{44100}f_n - \frac{34009}{25200}f_{n+1} - \frac{99}{35}f_{n+2} + \frac{4617}{3920}f_{n+\frac{7}{3}} + \frac{112}{225}f_{n+\frac{5}{2}} \\ &- \frac{891}{700}f_{n+\frac{8}{3}} + \frac{22331}{100800}f_{n+3} + \frac{50333}{50400}f_{n+4} - \frac{4117}{560}f_{n+5} \\ &+ \frac{39609}{1120}f_{n+\frac{16}{3}} - \frac{75664}{1575}f_{n+\frac{11}{2}} + \frac{236601}{11200}f_{n+\frac{17}{3}} - \frac{1091}{720}f_{n+6}], \end{aligned} \quad (37)$$

$$\begin{aligned}&y_{n+5}-2y_{n+4}+y_{n+3}-y_{n+2}+2y_{n+1}-y_n\\ =h^3[&\frac{505}{56448}f_n+\frac{205}{448}f_{n+1}+\frac{4297}{3360}f_{n+2}-\frac{12393}{15680}f_{n+\frac{7}{3}}+\frac{8}{21}f_{n+\frac{5}{2}}\\ &+\frac{2673}{4480}f_{n+\frac{8}{3}}+\frac{91493}{282240}f_{n+3}+\frac{487}{320}f_{n+4}-\frac{45823}{3360}f_{n+5}\\ &+\frac{206307}{3136}f_{n+\frac{16}{3}}-\frac{1880}{21}f_{n+\frac{11}{2}}+\frac{176661}{4480}f_{n+\frac{17}{3}}-\frac{28597}{10080}f_{n+6}]. \quad (38)\end{aligned}$$

3. ORDER AND ERROR CONSTANT OF HYBRID LINEAR MULTI-STEP METHOD

With specific reference to the works of Fatunla [9] and Lambert [29], the local truncation error attributed to the conventional form of Equation (8) is defined by the linear difference operator

$$\begin{aligned}L[y(x);h]&=\sum_{j=0}^{k}\left\{\alpha_j y(x+jh)-h^3\beta_j y'''(x+jh)\right\}\\ &-h^3\beta_\eta y'''(x+\eta h)-h^3\beta_v y'''(x+vh)-h^3\beta_\mu y'''(x+\mu h). \quad (39)\end{aligned}$$

Suppose it is assumed that $y(x)$ can be adequately differentiated. It is possible to expand Equation (39) in the form of Taylor series about the point x to arrive at the expression

$$L[y(x);h]=C_0y(x)+C_1y'(x)+\ldots+C_qh^qy^q(x)+\ldots \quad (40)$$

where the constant coefficients C_q are given as shown below:

$$\begin{aligned}C_0&=\sum_{j=0}^{k}\alpha_j, C_1=\sum_{j=1}^{k}j\alpha_j,\ldots,\\ C_q&=\frac{1}{q!}\sum_{j=1}^{k}j^q\alpha_j-q(q-1)(q-2)\left(\sum_{j=1}^{k}j^{q-3}\beta_j+\eta^{q-3}\beta_\eta+v^{q-3}\beta_v+\mu^{q-3}\beta_\mu\right),\\ q&=2,3,\ldots.\end{aligned}$$

According to Henrici [30], the method (8) has the order p if

$$C_0=C_1=\ldots=C_p=C_{p+1}=0, C_{p+2}=0, C_{p+3}\neq 0.$$

Therefore, C_{p+3} is the error constant as shown in Table 1.In order to analyze the methods for zero stability, we normalize the schemes and write them as a block method from which we obtain the first characteristic polynomial $\rho(R)$ given by

$$\rho(R)=\det(R\cdot A^{(0)}-A^{(1)})=R^k(R-1)$$

where $A^{(0)}=\mathbf{1}_{(k+1)\times(k+1)}$ is the identity matrix of dimension $(k+1)$, $A^{(1)}=\mathbf{1}_{(k+1)\times 1}\cdot\mathbf{i}^{\mathrm{T}}_{(k+1),(k+1)}$ is the matrix of dimension $(k+1)$, here $\mathbf{i}_{n,k}$ is the k-th column of an $n\times n$ identity matrix.

Case 1. 3SHLM1. It is easily shown that Equations (14)-(17) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(R \cdot A^{(0)} - A^{(1)}) = R^3(R-1)$$

where $A^{(0)} = \mathbf{1}_{4\times4}$, $A^{(1)} = \mathbf{1}_{4\times1} \cdot \mathbf{i}^{\mathrm{T}}_{4,4}$.

Case 2. 3SHLM2. It is easily shown that Equations (23)-(27) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(R \cdot A^{(0)} - A^{(1)}) = R^4(R-1)$$

where $A^{(0)} = \mathbf{1}_{5\times5}$, $A^{(1)} = \mathbf{1}_{5\times1} \cdot \mathbf{i}^{\mathrm{T}}_{5,5}$.

Case 3. 3SHLM3. It is easily shown that Equations (31)-(36) are normalized to give the first characteristic polynomial $\rho(R)$ given by

$$\rho(R) = \det(R \cdot A^{(0)} - A^{(1)}) = R^5(R-1)$$

where $A^{(0)} = \mathbf{1}_{6\times6}$, $A^{(1)} = \mathbf{1}_{6\times1} \cdot \mathbf{i}^{\mathrm{T}}_{6,6}$.

Table 1. Order and Error Constants
for the Modified Linear Multi-step Methods.

Methods, Equation	Order, p	Error Constant, C_{p+3}
(14)	5	$-\frac{7}{7200}$
(15)	5	$-\frac{85}{157464}$
(23)	6	$\frac{1}{201600}$
(24)	6	$-\frac{5167}{1785641768}$
(25)	6	$-\frac{571}{123863040}$
(31)	7	$-\frac{61}{1270080}$
(32)	7	$-\frac{1028869}{37498476960}$
(33)	7	$\frac{1696721}{650280960}$
(34)	7	$-\frac{17489}{1530550080}$

4. STABILITY OF HYBRID LINEAR MULTI-STEP METHOD

To evaluate and plot the region of absolute stability of HLM, the methods were reformulated as general linear method expressed as:

$$\begin{bmatrix} Y \\ y_{i+1} \end{bmatrix} = \begin{bmatrix} A & U \\ B & V \end{bmatrix} \begin{bmatrix} hf(Y) \\ y_{i+1} \end{bmatrix} \tag{41}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1s} \\ \vdots & \vdots & \ddots & \\ a_{s1} & a_{s2} & \dots & a_{ss} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1s} \\ \vdots & \vdots & \ddots & \\ b_{s1} & b_{s2} & \dots & b_{ss} \end{bmatrix},$$

$$Y = \begin{bmatrix} y_n \\ y_{n+1} \\ y_{n+k} \end{bmatrix}, y_{i+1} = \begin{bmatrix} y_{n+k} \\ y_{n+k-1} \end{bmatrix}, y_{i-1} = \begin{bmatrix} y_{n+k-1} \\ y_{n+k-2} \end{bmatrix}$$

Also the elements of the matrices A, B, U, and V were obtained from interpolation and collocation points and then substituted into the stability matrix as

$$M(z) = V + ZB(1 - ZA)^{-1}U \tag{42}$$

and the stability matrix (42) was substituted into the stability function

$$\rho(\eta, z) = \det(\eta I - M(z)) \tag{43}$$

and then computed with Maple software to yield the stability polynomial.

Case 1. 3SHLM1. The coefficients of Equations (14)-(17) are shown below:

$$\left[\begin{array}{ccccc:ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{13}{448} & -\frac{1399}{8400} & \frac{23}{336} & -\frac{783}{11200} & \frac{17}{560} & \frac{1}{4} & 0 & \frac{3}{4} \\
\frac{451}{1440} & \frac{109}{120} & -\frac{61}{120} & \frac{81}{800} & -\frac{79}{360} & 0 & 2 & -1 \\
\frac{31}{8748} & \frac{7}{27} & \frac{203}{729} & -\frac{25}{324} & \frac{65}{2187} & \frac{20}{9} & -\frac{16}{9} & \frac{5}{9} \\
\frac{1}{160} & \frac{47}{100} & \frac{23}{40} & -\frac{81}{800} & \frac{1}{20} & 3 & -3 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{1}{160} & \frac{47}{100} & \frac{23}{40} & -\frac{81}{800} & \frac{1}{20} & 3 & -3 & 1 \\
\frac{451}{1440} & \frac{109}{120} & -\frac{61}{120} & \frac{81}{800} & -\frac{79}{360} & 0 & 2 & -1 \\
-\frac{13}{448} & -\frac{1399}{8400} & \frac{23}{336} & -\frac{783}{11200} & \frac{17}{560} & 0 & 0 & 1
\end{array}\right]$$

By substituting the entries of the above matrices into Equations (42)-(43), the stability polynomial of 3SHLMS1 is

$$\begin{aligned}
f(z) \;&=\; (11520\eta^3 z^4 - 4560\eta^2 z^4 + 234919\eta^3 z^3 + 434649\eta z^3 - 3472783\eta^2 \\
&+\; z^3 + 5925795\eta^3 z^2 + 23165565\eta z^2 - 1154220 z^2 - 100249140\eta^2 z^2 \\
&+\; 678927600\eta z - 21388950 z + 143300700\eta^3 z - -528679350\eta^2 z \\
&-\; 969570000 + 204120000\eta^3 - 1173690000\eta^2 + 1939140000\eta) \\
&/\; (11520 z^4 + 234919 z^3 + 5925795 z^2 + 143300700 z + 204120000).
\end{aligned}$$

The region of absolute stability for 3SHLM1, 3SHLM2, and 3SHLM3 are shown in Figures 1-3 respectively. From the Figures 1-2 it was found that the interval of absolute stability for 3SHLM1 is $(-27, 0)$, for 3SHLM2 is $(-1, 0)$. Thus, the methods have a moderate wide interval of stability. While from Figure 3 it was found that the 3SHLM3 is $A(\alpha)$-stable.

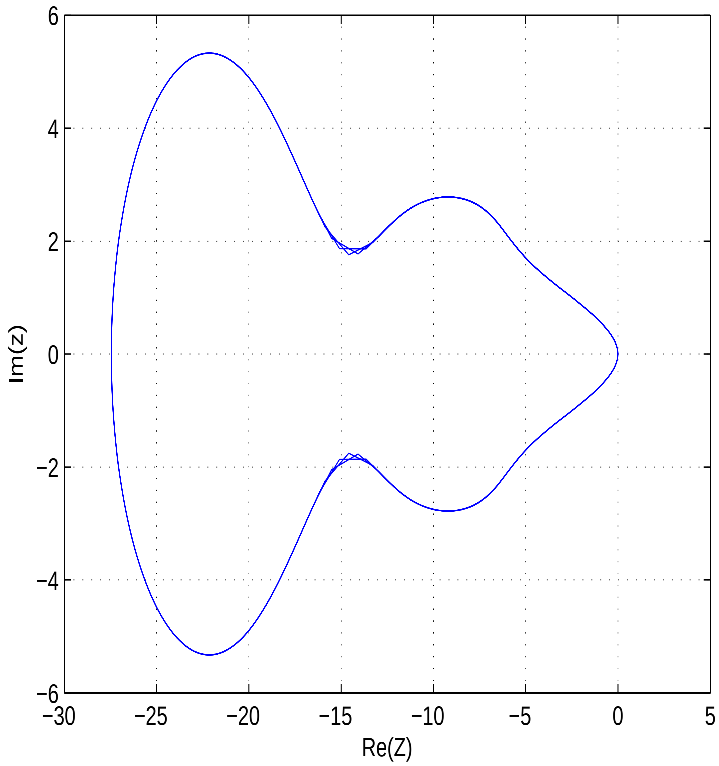

Fig. 1. Stability region of 3SHLM1.

Case 2. 3SHLM2. The coefficients of Equations (24)-(28) are shown below:

$$\left[\begin{array}{rrrrrrcrrr}
0 & 0 & 0 & 0 & 0 & 0 & \vdots & 0 & 0 & 1 \\
-\frac{5039}{201600} & -\frac{313}{1575} & \frac{347}{1120} & -\frac{1622}{1575} & \frac{20331}{22400} & -\frac{47}{360} & \vdots & \frac{1}{4} & 0 & \frac{3}{4} \\
\frac{5723}{20160} & \frac{8}{7} & -\frac{3809}{1680} & \frac{788}{105} & -\frac{14823}{2240} & \frac{1201}{1260} & \vdots & 0 & 2 & -1 \\
\frac{1729}{491520} & \frac{10163}{61440} & \frac{1929}{8192} & -\frac{529}{1920} & \frac{33777}{163840} & -\frac{277}{12288} & \vdots & \frac{15}{8} & -\frac{5}{4} & \frac{3}{8} \\
\frac{271}{52488} & \frac{1616}{6561} & \frac{1613}{4374} & -\frac{2720}{6561} & \frac{205}{648} & -\frac{230}{6561} & \vdots & \frac{20}{9} & -\frac{16}{9} & \frac{5}{9} \\
\frac{11}{1200} & \frac{67}{150} & \frac{3}{4} & -\frac{56}{75} & \frac{243}{400} & -\frac{1}{5} & \vdots & 3 & -3 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{11}{1200} & \frac{167}{150} & \frac{3}{4} & -\frac{56}{75} & \frac{243}{400} & -\frac{1}{5} & \vdots & 3 & -3 & 1 \\
\frac{5723}{20160} & \frac{8}{7} & -\frac{3809}{1680} & \frac{788}{105} & -\frac{14823}{2240} & \frac{1201}{1260} & \vdots & 0 & 2 & -1 \\
-\frac{5039}{201600} & -\frac{313}{1575} & \frac{347}{1120} & -\frac{1622}{1575} & \frac{20331}{22400} & -\frac{47}{360} & \vdots & \frac{1}{4} & 0 & \frac{3}{4}
\end{array}\right].$$

By substituting the entries of the above matrices into Equations (42)-(43), the stability polynomial of the 3SHLM2 is

$$
\begin{aligned}
f(z) &= \frac{1}{12}(1579355625\eta^2 z^5 + 4731567000\eta^3 z^5 - 236871890802\eta^2 z^4 \\
&+ 37665885040\eta z^4 - 340956065868\eta^3 z^4 - 10579345534728\eta^2 z^3 \\
&+ 6242912605296\eta^3 z^3 + 2888996401800\eta z^3 \\
&- 154780934400 z^3 - 183509896632480\eta^2 z^2 \\
&+ 73185538176000\eta z^2 + 35312810034240\eta^3 z^2 \\
&- 12679492217760 z^2 - 714971363500800\eta^2 z \\
&+ 207411219993600\eta^3 z + 861035226508800\eta z \\
&- 250763205369600 z - 454302535680000\eta^2 \\
&+ 7505867980000\eta + 79009136640000\eta^3 + 6584094720000) \\
&/ \ (394297250 z^5 + 28413005489 z^4 + 520242717108 z^3 \\
&+ 2942734169520 z^2 + 17284268332800 z + 6584094720000).
\end{aligned}
$$

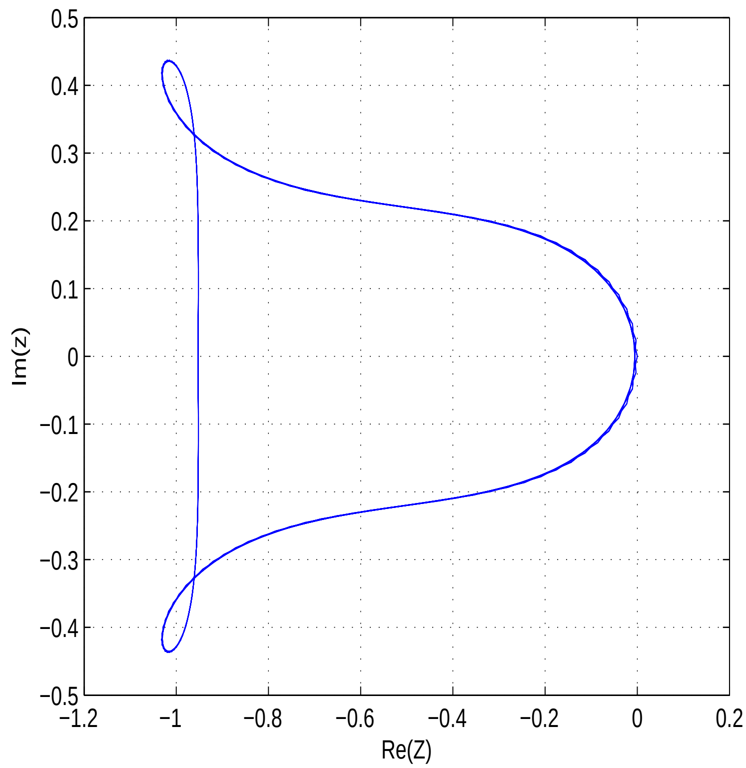

Fig. 2. Stability region of 3SHLM2.

Case 3. 3SHLM3. The coefficients of Equations (32)-(37) are shown below:

$$
\left[\begin{array}{ccccccc:ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-\frac{8611}{403200} & -\frac{50333}{201600} & \frac{4117}{2240} & -\frac{39609}{4480} & \frac{18916}{1575} & -\frac{236601}{44800} & \frac{1091}{2880} & \frac{1}{4} & 0 & \frac{3}{4} \\
\frac{72481}{282240} & \frac{478}{320} & -\frac{45823}{3360} & \frac{206307}{3136} & -\frac{1880}{21} & \frac{176661}{4480} & -\frac{28597}{10080} & 0 & 2 & -1 \\
\frac{979}{472392} & \frac{231251}{2361960} & \frac{21853}{196830} & \frac{877}{9720} & -\frac{85408}{295245} & \frac{347}{1944} & -\frac{20797}{1180980} & \frac{14}{9} & -\frac{7}{9} & \frac{2}{9} \\
\frac{167729}{48168960} & \frac{570841}{3440640} & \frac{25293}{114688} & \frac{138753}{1605632} & -\frac{677}{1680} & \frac{611631}{2293760} & -\frac{9469}{344064} & \frac{15}{8} & -\frac{5}{4} & \frac{3}{8} \\
\frac{1189943}{23472080} & \frac{4077487}{16533720} & \frac{201851}{551124} & \frac{5167}{95256} & -\frac{1022144}{2066715} & \frac{48217}{136080} & -\frac{63127}{1653372} & \frac{20}{9} & -\frac{16}{9} & \frac{5}{9} \\
\frac{433}{47040} & \frac{1499}{3360} & \frac{429}{560} & -\frac{729}{7840} & -\frac{64}{105} & \frac{243}{448} & -\frac{103}{1680} & 3 & -3 & 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\
\frac{433}{47040} & \frac{1499}{3360} & \frac{429}{560} & -\frac{729}{7840} & -\frac{64}{105} & \frac{243}{448} & -\frac{103}{1680} & 3 & -3 & 1 \\
\frac{72481}{282240} & \frac{487}{320} & -\frac{45823}{3360} & \frac{206307}{3136} & -\frac{1880}{21} & \frac{176661}{4480} & -\frac{28597}{10080} & 0 & 2 & -1 \\
-\frac{8611}{403200} & -\frac{50333}{201600} & \frac{4117}{2240} & -\frac{39609}{4480} & \frac{18916}{1575} & -\frac{236601}{44800} & \frac{1091}{2880} & \frac{1}{4} & 0 & \frac{3}{4}
\end{array}\right]
$$

By substituting the entries of the above matrices into Equations (42)-(43), the stability polynomial of the 3SHLM3 is

$$
\begin{aligned}
f(z) \quad = \quad & \frac{1}{172519788}(-34225982757724058416281600z - 7592739196573921094190z^4 \\
- \quad & 3103088016498671459420082z^3 + 2243731545396242968489920z^2 \\
- \quad & 6118453306997029232640000 + 1288095433052006154240000\eta^3 \\
+ \quad & 30253836428031506675112\eta^3 z^3 + 1363073230742807416966560\eta^3 z^2 \\
+ \quad & 17913796064214199501363200\eta^3 z + 85210973688960000\eta^3 z^6 \\
+ \quad & 13079596712913157980\eta^3 z^5 + 451137491873027922396\eta^3 z^4 \\
+ \quad & 211294379598616125\eta^2 z^6 - 125615347262244692990\eta^2 z^5 \\
- \quad & 21173744633227501132884\eta^2 z^4 - 886243423227056863597566\eta^2 z^3 \\
- \quad & 14216346532520201226326400\eta^2 z^2 - 63913156371539340735744000\eta^2 z \\
- \quad & 7406548740049035386880000\eta^2 + 366816659022365322995\eta z^5 \\
+ \quad & 11280330928349493317988\eta z^4 - 49333862358114245675064\eta z^3 \\
+ \quad & 6178669396758571686412320\eta z^2 + 81942803642451874522982400\eta z \\
+ \quad & 12236906613994058465280000\eta) \\
/ \quad & (493920000z^6 + 75815052085z^5 + 2614989834517z^4 \\
+ \quad & 175364442414174z^3 + 7900967457384120z^2 \\
+ \quad & 103836181761446400z + 7466363412480000).
\end{aligned}
$$

5. NUMERICAL EXPERIMENTS AND DISCUSSION

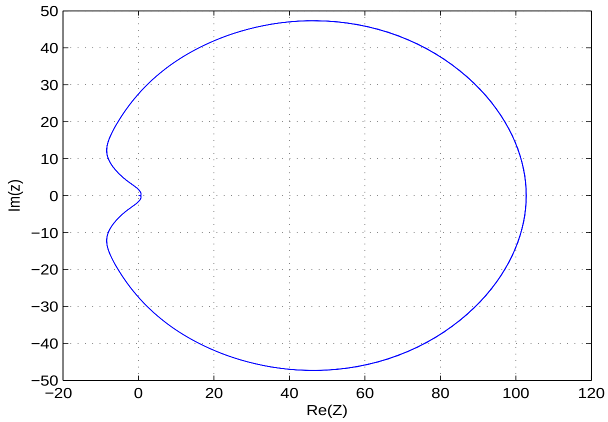

Fig. 3. Stability region of 3SHLM3.

In this section, the results of the proposed method developed in Section 3 are presented for some IVP and BVP of third order differential equations.

Our method is implemented efficiently by combining the MFDMs as simultaneous numerical integrator for IVPs and BVPs. For instance, the method (14)-(17) are combined as instantaneous numerical integrators for IVPs without looking for any other methods to provide the starting values by explicitly obtaining the initial conditions at x_{n+3}, $n = 0, 3, \ldots, N-3$ using computed values $y(x_{n+3}) = y_{n+3}$, $\delta(x_{n+3}) = \delta_{n+3}$ and $\gamma(x_{n+3}) = \gamma_{n+3}$ over sub-intervals $[x_0, x_3], \ldots,$ $[x_{n-3}, x_n]$. On the other hand, the methods (14)-(19) are combined to give the single matrix of finite difference equation which simultaneously solves BVPs for both linear and non-linear differential equation.

Problem 1. Linear non-homogeneous problem [18]

$$y''' = 3\sin x, \; y(0) = 1, \; y'(0) = 0, \; y''(0) = -2.$$

Exact solution is $y(x) = 3\cos x + \dfrac{x^2}{2} - 2$.

Problem 2. Non-linear homogeneous problem [12]

$$y''' + \exp(-y) - 3\exp(-2y) + 2\exp(-3y) = 0,$$

$$y(0) = \ln 2, \; y'(0) = 1/2, \; y''(0) = 1/4.$$

Exact solution is $y(x) = \ln(\exp(x) + 1)$.

Problem 3. System of third order non-homogeneous equations

$$\begin{cases} y''' = \dfrac{1}{68}(817y + 1393z + 448w), y(0) = -2, y'(0) = -12, y''(0) = 20, \\ z''' = -\dfrac{1}{68}(1141y + 2837z + 896w), z(0) = -2, z'(0) = 28, z''(0) = -52, \\ w''' = \dfrac{1}{136}(3059y + 4319z + 1592w), w(0) = -12, w'(0) = -33, w''(0) = 5. \end{cases}$$

The analytical solution of the problem 3 is given by

$$\begin{cases} y = \exp(x) - 2\exp(2x) + 3\exp(-3x), \\ z = 3\exp(x) + 2\exp(2x) - 7\exp(-3x), \\ w = -11\exp(x) - 5\exp(2x) + 4\exp(-3x). \end{cases}$$

Problem 4. Application problem [27]

$$y^n y''' = 1, y(0) = 1, y'(0) = 0, y''(0) = \lambda.$$

The above third order initial value problem was derived by Fazal-i-Hag et al. [27] to investigate wave solution of the form: $h(x,t) = y(x)$, $x = \bar{x} - Vt$, where V is the wave velocity and y is the height of a thin film on a solid surface.

Numerical results of problems 1-4 are represented in Figures 1–4.

Problem 5. Linear non-homogeneous problem [26]

$$y''' = xy + (x^3 - 2x^2 - 5x - 3)e^x, y(0) = 0, y(1) = 0, y'(0) = 1, 0 \leq x \leq 1.$$

Exact solution is $y(x) = x(1-x)e^x$.

Problem 6. Non-linear third order BVP

$$y''' = -2\exp(-3y) + \frac{4}{(1+x)^3}, y(0) = 0, y(1) = \ln 2, y'(0) = 1, 0 \leq x \leq 1.$$

Exact solution is $y(x) = \ln(1+x)$.

Table 2. Maximum Absolute Error, $EMAX = \max_{i=1,2,\ldots,N} |y(x_i) - y_i|$ for Problem 5.

h	3SHLM1	3SHLM2	3SHLM3	Sahi [26]
1/6	1.079×10^{-06}	9.13×10^{-08}	8.33×10^{-08}	1.52×10^{-5}
1/9	1.210×10^{-7}	6.80×10^{-09}	3.30×10^{-09}	2.93×10^{-6}
1/12	2.770×10^{-8}	1.10×10^{-09}	9.53×10^{-10}	9.26×10^{-7}
1/15	8.900×10^{-9}	2.00×10^{-10}	1.13×10^{-10}	3.85×10^{-7}

Table 3. Maximum Absolute Error, $EMAX = \max_{i=1,2,\ldots,N} |y(x_i) - y_i|$ for Problem 6.

h	3SHLM1	3SHLM2	3SHLM3
1/6	1.93×10^{-06}	7.783×10^{-07}	8.62×10^{-08}
1/9	2.93×10^{-07}	8.16×10^{-08}	5.95×10^{-09}
1/12	8.18×10^{-08}	1.68×10^{-08}	1.82×10^{-09}
1/15	3.05×10^{-08}	5.10×10^{-09}	1.03×10^{-10}

The proposed schemes were applied to both initial value problem and boundary value problems arising from third order differential equations. These problems were also characterized variously by linearity, homogeneity and coefficient-wise (variable and constant).

All the proposed three-step hybrid schemes with one, two and three off-grid points (3SHLM1, 3SHLM2, 3SHLM3) at collocation were more

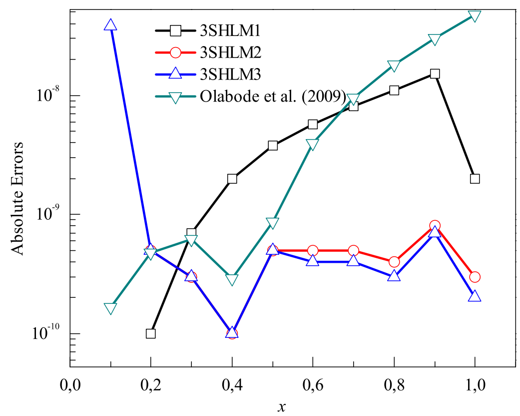

Fig. 4. Absolute Errors of Methods for Problem 1 [18].

accurate than the schemes of Olabode and Yusuph [18] for the linear non-homogeneous problem (problem 1, Fig. 4) due to the number of off-grid points that were considered.

Problem 2 considered a non-linear homogeneous problem solved by You and Chen [12]. The results for this problem were shown in Fig. 5. Comparison was made when the same problem was reduced to the system of first order ODEs and the results of the proposed method performed better than that of fourth-order Runge-Kutta method.

Problem 3 involved system of third order non-homogeneous equations and the results were displayed in Fig. 6. Component-wise, the errors of the proposed method with three step and one off-grid point are of low order indicating good performance for problem 3 (Fig. 6, a). The three component of solution are more accurate for the proposed method with three step and two off-grid points than those of the three step with one off-grid point for problem 3 (Fig. 6, b). Also, the proposed schemes of three steps with three off-grid points at collocation, perform better than the three step with two off-grid points for problem 3 (Fig. 6, c).

Problem 4 considered a third order initial value problem derived by Fazal-i-Hag et al. [27] to investigate wave solution. It was noticed that

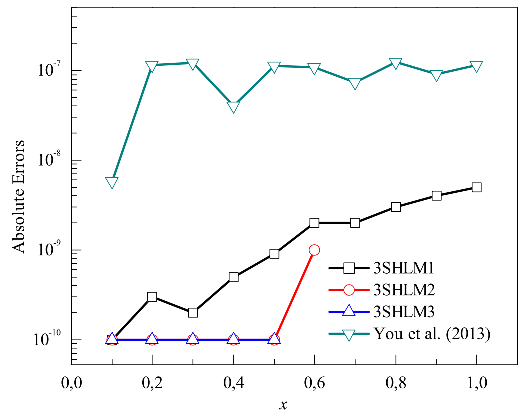

Fig. 5. Absolute Errors of Methods for Problem 2 [12].

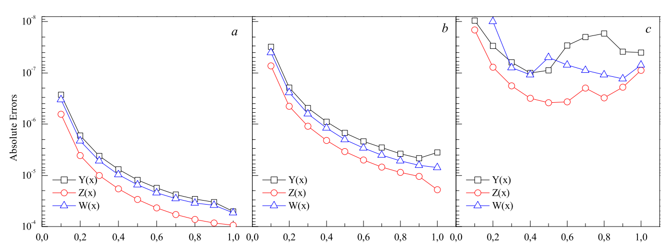

Fig. 6. Absolute Errors of Methods for Problem 3:
a) 3SHLM1, b) 3SHLM2, c) 3SHLM3.

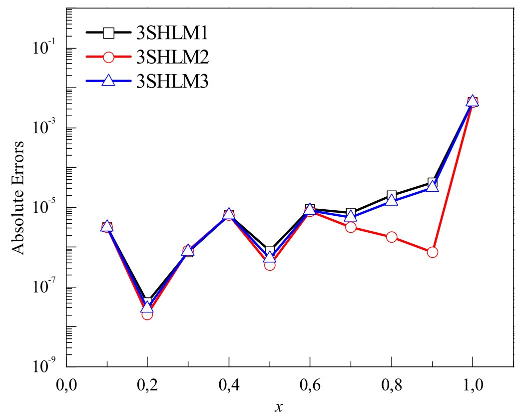

Fig. 7. Absolute Errors of Methods for Problem 4 [27].

the proposed schemes compared favorably with the existing method as shown in in Fig. 7.

Problem 5 considered special third order BVP which was also solved by Al-Said [31], Islam [21] and Jator [25] with a smaller step size $h = \frac{1}{8}$. However, the proposed method used lager step size $h = \frac{1}{6}$ and it was found that error was better than the existing methods found in the literature. Furthermore, it has the advantage of estimating the solution and it derivatives at every point within the range of integration as presented in Table 2.

Problem 6 considered the non-linear third order BVP. It was observed from Table 3 that the maximum errors of the proposed method reduces as the hybrid point increases.

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Thanks to the editor and referees for several comments and suggestions that were instrumental in improving the paper.

CONCLUSION

We have derived a three-step continuous hybrid linear multi-step (HLM) method from which multiple finite difference methods (MFDMs) are obtained and applied to solve third-order ordinary differential equations (ODEs) without first adapting the ODE to an equivalent first-order system. The proposed method is universal and it is possible to apply it for solving directly third-order initial value problems as well as boundary value problems.

The proposed MFDMs are applied as simultaneous numerical integrators over sub-intervals which do not overlap and hence proposed methods are more accurate than corresponding to single finite difference methods (SFDMs) which are generally applied as single formulas over overlapping intervals. We consider three numerical examples to test the efficiency of the derived hybrid linear multi-step method. Numerical results are presented which show that the new method is more efficient in terms of approximation in solving the third-order ODEs compared to the existing MFDMs.

We have shown that the proposed methods are stable convergent. Our approach is quite general and has the potential to design methods for solving high-orders ODEs.

The further research of our study can be continued in the following directions. At first, it is an adapting the proposed MFDMs to solve third-order partial differential equations. At second, it is a using our approach to derive methods for the solution of high-orders initial value problems as well as boundary value problems.

The implementation of the method was coded using Maple software environment.

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