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NUMERICAL SOLUTION OF GENERALIZED EMDEN-FOWLER EQUATIONS BY SOME APPROXIMATION TECHNIQUES

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ABSTRACT. In this paper, we provide reliable approximations to the generalized Emden - Fowler equation using two semi - analytic methods; Adomian decomposition method and variational iteration method, and the recursive Tau method that employed Newton-Kantorovich approach. The three methods give very close results, with the semi - analytic methods giving results that agree completely with some existing results in the literature when certain parameters are fixed. The results are presented in both tabular and graphical forms.

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1. INTRODUCTION

The family of equation studied in this paper has received commendable attentions of many researchers in the field of applied Mathematics due to its wide applications in fluid mechanics and relativistic mechanics. The generalized form of the equation is in the form

$$\begin{aligned} &y''(x) + \frac{\alpha}{x} y'(x) + \beta f(x) g(y) = 0, \\ &y(0) = a, \quad y'(0) = 0, \quad \alpha \geq 0 \end{aligned} \tag{1}$$

where α is the shape factor and β is a real constant, $f(x)$ is a polynomial function of x and $g(y)$ is the nonlinear component of (1) which is a function of the dependent variable y.
The equation described in (1) has been used by many scientists to model several phenomenona in physics and astrophysics, such as the theory of Stellar structure, the thermal behavior of a spherical

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cloud, especially when (1) is reduced to the form

$$y''(x) + \frac{\alpha}{x}y'(x) + e^{y(x)} = 0 \tag{2}$$

known as Lane - Emden equation of the second kind, Wazwaz [14] and Hermann and Saravi [12] (and the references therein). The formations and solutions of higher order Lane-Emden-Fowler type equations were carried out in [13, 14]. The effort in that work was on the formation and solution of Emden-Fowler type equation of third order by using variational iteration method [8, 14, 16], Biazar and Hosseini [3, 17] solved equation (1) using modified Adomian decomposition method, Homotopy method was adopted in [5, 7, 15], Waheed, Youssri and Eid [24] adopted the construction of ultraspherical operational matrices of derivatives and [6, 22] adopted Hermite functions collocation method to solve (1).
In the present work, we adopt the approach contained in Hermann and Serran [12] to solve (1) using both Adomian decomposition method (ADM) and variational iteration method (VIM). Meanwhile, we deviated completely from Hermann and Sarravi [12] by using some of the techniques developed in [1, 2, 10, 21] which bothers on the solution of nonlinear variable coefficients ODEs by Tau method to solve (1).

2. SOLUTION OF EMDEN-FOWLER EQUATION

In this section, we are generalizing an algorithms to the solution Emden - Fowler equation using two semi - analytic methods; Adomian decomposition method and variational iteration method, and the recursive Tau method.

2.1 SOLUTION OF EMDEN-FOWLER EQUATION BY ADOMIAN DECOMPOSITION METHOD

In this section, we present the solution of (1) by adopting the approach presented by Hermann and Seravi in [12].
The linear operator L in (1) consists of two derivatives in the first two terms $y'' + \frac{\alpha}{x}y'$. Then (1) can be written as

$$Ly(x) = -\beta f(x)g(y), \tag{3}$$

with

$$L \equiv x^{-\alpha}\frac{d}{dx}\left(x^{\alpha}\frac{d}{dx}\right)$$

The corresponding inverse operator L^{-1} is given as

$$L^{-1}(.) = \int_0^x \tau^{-\alpha} \int_0^\tau t^\alpha (.) dt d\tau$$

Berkorich [9] did a group classification of (1) and introduced certain Lemma, Preposition and Theorems to establish the said group classification. Also, [13] solved Two - Dimensional Lane-Emden type equation using Adomian Decomposition Method. Major discussion in that paper was centered on the partial differential form of the equation.

Applying L^{-1} to the first and second derivative terms in (1), we have

$$\begin{aligned}
L^{-1}(y''(x) + \frac{\alpha}{x} y'(x)) &= \int_0^x \tau^{-\alpha} \int_0^\tau t^\alpha (y''(t) + \frac{\alpha}{t} y'(t)) dt d\tau \\
&= \int_o^x \tau^{-\alpha} \left[\int_0^\tau t^\alpha y''(t) dt + \int_0^e \alpha t^{\alpha-1} y'(t) dt \right] d\tau \\
&= \int_0^x t^{-\alpha} \left[t^\alpha y'(t) \Big|_0^\tau - \int_0^\tau \alpha t^{\alpha-1} y'(t) dt \right. \\
&\left. + \int_0^\tau \alpha t^{\alpha-1} y'(t) dt \right] d\tau \\
&= \int_0^x t^{-\alpha} (\tau^\alpha y'(\tau)) d\tau \\
&= \int_0^x y'(\tau) d\tau \\
&= y(x) - y(0) \\
&\Rightarrow L^{-1}(y''(x) + \frac{\alpha}{x} y'(x)) = y(x) - a
\end{aligned}$$

Applying L^{-1} on (1) generally gives

$$y(x) = a - L^{-1}(\beta f(x) g(y)) \tag{4}$$

Now to apply the ADM, the given nonlinear function $g(y)$ is represented by an infinite series of Adomian polynomials (details on how to generate Adomian polynomials can be found in [3, 13, 17]).

$$g(y(x)) = \sum_{n=0}^{\infty} A_n(x), \tag{5}$$

where

$$A_n(x) = A_n(y_0(x), y_1(x)), \cdots, y_{n-1})$$

The Solution $y(x)$ in Adomian decomposition method is represented as

$$y(x) = \sum_{n=0}^{\infty} y_n(x) \tag{6}$$

using (5) and (6) in (4) gives

$$\sum_{n=0}^{\infty} y_n(x) = a - \beta L^{-1}\left(f(x)\sum_{n=0}^{\infty} A_n(x)\right) \tag{7}$$

Now, the successive $y_n(x), n = 0, 1, 2, \cdots$ are obtained recursively as

$$\begin{aligned} y_0(x) &= a \\ y_k(x) &= -\beta L^{-1}(f(x)A_{k-1}(x)), k = 1, 2, \cdots \end{aligned}$$

or equivalently as

$$y_k(x) = -\beta\left(\int_0^x \tau^{-\alpha}\right)\int_0^\tau t^{\alpha}(f(t)A_{k-1}(t)A_{k-1}(t)dt)d\tau\right), \ k = 1, 2, \cdots \tag{8}$$

2.2 SOLUTION OF EMDEN-FOWLER EQUATION BY VARIATIONAL ITERATION METHOD

It is immediately observed that (1) has a singularity at $x = 0$. To overcome the said singularity, we shall adopt a change of variable as follows:
Let $y = \frac{z}{x}$, and this implies

$$z' = xy' + y,$$

and

$$z'' = xy'' + 2y$$

using these in (1), we have

$$z''(x) + \beta x f(x) = 0, \quad z(0) = 0, \quad z'(0) = a \tag{9}$$

where $g(z)$ represents the non linearity expressed in terms of z. From (9) we derive

$$Lz \equiv z'', \quad N(z) \equiv \beta x f(x) g(z) \quad \text{and} \quad f(x) \equiv 0$$

The corresponding non linearity is

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda(\tau)(z_n''(\tau) + \beta\tau f(\tau)g(\overline{z}_n(\tau)))d\tau \tag{10}$$

The optimal value of the Lagrange multiplier can be obtained from the recursive relation

$$\lambda(x) = \frac{(-1)^m}{(m-1)!}(\tau - x)^{m-1},\ m = 1,,2,\cdots \tag{11}$$

where m is the order of the differential equations. The starting value of $z_n(x)$, that is $z_0(x)$ is also derived from

$$z_0(x) = z(0) + xz'(0) + \frac{x^2}{2!}z''(0) + \cdots \tag{12}$$

For details on (11) and (12) see [13].
Using (11) and (12), the optimal value of $\lambda(x)$ is obtained as $\lambda(x) = \tau - x$ and $z_0(x) = ax$ respectively. Using the two and the restricted variation $\delta z_n(0) = 0$, we have the recurrence form of (10) as

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda(\tau)(z_n''(\tau) + \beta\tau f(\tau)g(z_n(\tau)))d\tau \tag{13}$$

the final result is derived from

$$z(x) = \lim_{n\to\infty} z_n(x)$$

and consequently $y(x) = \dfrac{z(x)}{x}$.

2.3 EXTENSION OF RECURSIVE FORMULATION OF TAU METHOD TO EMDEN-FOWLER EQUATIONS

In this section, we extend Tau approximant reported in [1, 2] that handle differential equations of the form:

$$Ly(x) \equiv \sum_{r=0}^{m}\left(\sum_{k=0}^{N_r} P_{rk}x^k\right) y^{(r)}(x) = \sum_{r=0}^{\sigma} f_r x^r, a \le x \le b \tag{14}$$

$$L^*y(x_{rk}) \equiv \sum_{r=0}^{m-1} a_{rk}y^{(r)}(x_{rk}) = \alpha_k, k = 1(1)m \tag{15}$$

by seeking an approximant derived in [1]

$$y_n(x) = \sum_{r=s}^{\sigma} f_r q_r(x) + \sum_{i=0}^{m+s-1}\tau_{i+1}\sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)}q_r(x), \tag{16}$$

$$y_n^{\lambda}(x) = \sum_{r=s}^{\sigma} f_r Q_r^{\lambda}(x) + \sum_{i=0}^{m+s-1}\tau_{i+1}\sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)}Q_r^{\lambda}(x) = \alpha_\lambda, \lambda = 0(1)(m-1) \tag{17}$$

$$\sum_{i=0}^{m+s-1}\tau_{i+1}\sum_{r=0}^{n-m+i+1} C_r^{(n-m+i+1)}P_r + \sum_{r=0}^{\sigma} f_r P_r = 0 \tag{18}$$

equation (18) is the equation of undetermined canonical polynomial as reported in [1, 2] and

$$Q_n(x) = \frac{1}{\sum_{k=0}^{m} k!\binom{n-s}{k} P_{k,k+s}} \left\{ x^{n-s} - \left[\sum_{k=1}^{m} \left(\sum_{j=k}^{m} j!\binom{n-s}{j} P_{j,j-k} \right) Q_{n-s-k}(x) + \sum_{k=0}^{s-1} \left(\sum_{j=0}^{m} j!\binom{n-s}{j} P_{j,j+k} \right) Q_{n-s+k}(x) \right] \right\} \tag{19}$$

is the generalized Canonical polynomial, reported in [11], to solve Emden-Fowler equation (1), we shall adopt Newton-Kantorovich linearization approach reported in [23], that is

$$\Omega(x, y(x), y'(x), \ldots, y^{(m)}(x)) = \sum_{r=0}^{\sigma} f_r x^r. \tag{20}$$

This process was derived from the Taylor series expansion in several variables of Ω, which is given by:

$$\Omega + \triangle y \frac{\partial \Omega}{\partial y} + \triangle y' \frac{\partial \Omega}{\partial y'} + \triangle y'' \frac{\partial \Omega}{\partial y''} + \cdots + \triangle y^{(m)} \frac{\partial \Omega}{\partial y^{(m)}} = \sum_{r=0}^{\sigma} f_r x^r, \tag{21}$$

where $\triangle y_k^i = y_{k+1}^i - y_k^i$, $i = 0, 1 \ldots, m$
We seek $k - th$ iterative approximate solution of the form:

$$y_{n,k}(x) = \sum_{r=s}^{\sigma} f_r q_r(x) + \sum_{i=0}^{m+s-1} \tau_{i+1} \sum_{r=s}^{n-m+i+1} C_r^{(n-m+i+1)} q_r(x), \tag{22}$$

3. NUMERICAL EXPERIMENTS

In this section, we present two problems that are solved using the algorithms discussed in the earlier sections for Adomian decomposition method, variational iteration method and recursive Tau method

3.1 SOLUTIONS BASED ON ADM

Problem 1
Consider a class of the generalized Emden - Fowler equation that takes the form

$$y''(x) + \frac{2}{x} y'(x) + \alpha x^m e^{y(x)} = 0, \quad y(0) = y'(0) = 0. \tag{23}$$

It is easily noticed that when compared to the generalized Emden - Fowler equation studied in section 2, $\alpha = 2, \beta = \alpha, f(x) = x^m, g(y) = e^{y(x)}$ and $a = 0$

Now using the Adomian decomposition algorithm discussed in the earlier section, we proceed as shown below.

$$y(x) = a - \beta \int_0^x \tau^{-\alpha} \int_0^\tau (f(t)g(y(t))dtd\tau \tag{24}$$

Making the appropriate substitutions, (6) reduces to

$$y(x) = -\alpha \int_0^x d\tau^{-2} \int_0^\tau t^2(t^m g(y(t)))dtdz.$$

Also,

$$y(x) = -\alpha \int_0^x \tau^{-2} \int_0^\tau t^{m+2} g(y(t))dtd\tau$$

where $g(y(t))$ is the non linearity term.
The Adomian polynomials corresponding to the non linearity $g(y) = e^{y(x)}$ are: $A_0 = e^{y_0}, A_1 = y_1 e^{y_0}, A_2 = y_2 e^{y_0} + \frac{1}{2!} y_1^2 e^{y_0}, A_3 = y_3 e^{y_0} + y_1 y_2 e^{y_0} + \frac{1}{3!} y_1^3 e^{y_0}$.
Hence (6) can be written as

$$y_0(x) = 0, \tag{25}$$

$$y_k(x) = -\alpha \int_0^x \tau^{-2} \int_0^\tau t^{m+2} A_{k-1}(t) d\tau, \quad k = 1, 2, 3. \cdots \tag{26}$$

using the appropriate Adomian polynomials in (8), we have

$$y_1(x) = -\alpha \int_0^x \tau^{-2} \int_0^\tau t^{m+2} e^{y_0(t)} dtd\tau$$

$$y_1(x) = -\alpha \int_0^x \tau^{-2} \int_0^\tau t^{m+2} e^0 dtd\tau$$

$$y_1(x) = -\frac{x^{m+2}\alpha}{(m+2)(m+3)}, \quad m \geq -2$$

$$y_2(x) = -\alpha \int_0^x \tau^{-2} \int_0^\tau t^{m+2} y_1(t) e^{y_0(t)} dtd\tau$$

$$y_2(x) = \frac{x^{2m+4}}{(2m+4)(2m+5)(m+2)(m+3)}, \quad m > -2$$

Following the same procedure, we have subsequent results as:

$$y_3(x) = \frac{(3m+8)x^{3m+6}}{6(m+2)^3 + (m+3)^2(2m+5)(3m+7)}, \qquad m > -2$$

$$y_4(x) = \frac{[3m(m+6)+26]x^{4m+8}\alpha^4}{3(m+2)^3(m+3)^3(2m+5)(3m+7)(4m+8)(4m+9)}, \quad m \geq -2$$

and so on. The solution $y(x)$ is given by

$$y(x) = y_0(x) + y_1(x) + y_2(x) + y_3(x) + y_4(x) + \cdots$$

$$\begin{aligned} y(x) = &-\frac{x^{m+2}}{(m+2)(m+3)} + \frac{x^{2m+4}\alpha^2}{(2m+4)(2m+5)(m+2)(m+3)} \\ &- \frac{(3m+8)x^{3m+6}\alpha^3}{6(m+2)^3(m+3)^2(2m+5)(3m+7)} \\ &+ \frac{[3m(m+6)+26]x^{4m+8}\alpha^4}{3(m+2)^3(m+3)^3(2m+5)(3m+7)(4m+8)(4m+9)} \end{aligned}$$

Taking $m = 0$ and $\alpha = 1$, we have

$$y(x) = -\frac{x^2}{6} + \frac{x^4}{120} - \frac{x^6}{1890} + \cdots,$$

which generalizes the result in Hermann and Saravi (2016).

Problem 2

Consider another class of Emden - Fowler equation of the form

$$y''(x) + \frac{2}{x}y'(x) + \alpha x^m y(x) = 0, \quad y(0) = 1, \quad y'(0) = 0. \tag{27}$$

When the given DE is compared with the generalized Emden - Fowler equation (1), it is obvious that $\alpha = 2$, $\beta = \alpha$, $f(x) = x^m$ and $a = 1$.
The ADM algorithm for the generalized Emden - Fowler equation is given as

$$y(x) = a - \beta \int_0^x \tau^{-\alpha} t^{\alpha}(f(t)g(y(t))))dtd\tau$$

is used here as

$$y(x) = 1 - \alpha \int_0^x \tau^{-2} \int_0^\tau t^2(t^m g(y(t)))dtd\tau$$

which can as well be written as

$$y(x) = 1 - \alpha \int_0^x \tau^{-2} \int_0^\tau t^{m+2}(g(y(t)))dtd\tau \tag{28}$$

$$y_0(x) = 1, \tag{29}$$

$$y_k(x) = -\alpha \int_0^x \tau^{m+2} A_{k-1}(t)dt\tau, \quad k = 1, 2, 3\cdots \tag{30}$$

The Adomian Polynomials for the class of non linearity

$$g(y) = y^\mu$$

are

$$A_0 = y_0^\mu, \quad A_1 = \mu y_1 y_0^{\mu-1}$$

$$A_2(x) = \mu y_2 y_0^{\mu-1} + \frac{\mu(\mu-1)}{2!}y_1^2 y_0^{\mu-2},$$

$$A_3 = \mu y_3 y_0^{\mu-1} + \mu(\mu-1)y_1y_2 + \frac{1}{3!}\mu(\mu-1)(\mu-2)y_1^3y_0^{\mu-3}$$

etc. These polynomials shall be used in (30) to get the following results:

$$y_1(x) = -\alpha\int_0^x \tau\int_0 \tau t^{m+2}(1^\mu)dtd\tau$$

$$y_1(x) = -\frac{x^{m+2}\alpha}{(m+2)(m+3)}, \quad m > -2.$$

$$y_2(x) = -\alpha\int_0^x \tau^{-2}\int t^{m+2}\mu\left(-\frac{t^{m+2}\alpha}{(m+2)(m+3)}\right)dtd\tau, \quad m > -2.$$

$$y_2(x) = \frac{x^{2m+4}\alpha^2\mu}{(2m+4)(2m+5)(m+2)(m+3)}, \quad m > -2.$$

Similar procedure is followed to obtain the subsequent results as:

$$y_3(x) = -\frac{x^{3m+6}\alpha^3\mu[m(3\mu-2)+8\mu-5]}{2(3m+6)(m+2)^2(m+3)^2(2m+5)(3m+7)},$$

$$y_4(x) = \frac{x^{4m+8}\alpha^4\mu[2m[\mu(47\mu-73)+29]+\mu^2[\mu(18\mu-29)+12]+61\mu(2\mu-3)+70]}{6(m+2)^3(m+3)^3(2m+5)(3m+7)(4m+8)(4m+9)},$$

and so on.

The solution is given as

$$y(x) = y_0(x) + y_2(x) + y_3(x) + y_4(x) + \cdots$$

This,

$$y(x) = 1 - \frac{x^{m+2}\alpha}{(m+2)(m+3)} + \frac{x^{2m+4}\alpha^2\mu}{(2m+4)(2m+5)(m+2)} - \frac{x^{3m+6}\alpha^3\mu[m(3\mu-2)+8\mu-5]}{2(3m+6)(m+2)^2(m+3)^2(2m+5)(3m+7)} + \ldots$$

To show that the present result generalizes the earlier one in [12], we take $m = 0$, $\alpha = 1$ and $\mu = 5$ so that the problem narrows down to Emden - Lane - Fowler problem.

The result now becomes

$$y(x) = 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \frac{35x^8}{10368} \quad \ldots \quad .$$

3.2 SOLUTION BASED ON VIM

In this section, variational iteration method (VIM) is applied to the same set of the problems that were solved in 3.1.

Problem 1

Consider the following IVP for the Emden - Fowler ODE

$$y''(x) + \frac{2}{x}y'(x) + \alpha x^m e^{y(x)} = 0, \qquad y(0) = y'(0) = 0.$$

Change of variable is inevitable here in order to overcome the singularity in the given problem.
Let $y = \frac{z}{x}$, so that the problem now becomes

$$z''(x) + \alpha x^{m+1} e^{\frac{z}{x}} = 0, \qquad z(0) = z'(0) = 0$$

Thus,

$$Lz \equiv z'', \quad N(z) = \alpha x^{m+1} e^{\frac{z}{x}} \quad \text{and} \quad f(x) = 0$$

The corresponding correction functional is obtained after the expansion of $e^{y(x)}$, stopping at the sixth term, and replacing $y(x)$ by $\frac{z(x)}{x}$ as

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda(\tau)(z_n''(\tau)) + \alpha\tau^{m+1}\Bigg(1 + \frac{\overline{z}_n(\tau)}{\tau} + \frac{1}{2}\frac{\overline{z}_n(\tau)^2}{\tau^2} + \frac{1}{6}\frac{\overline{z}_n(\tau)^3}{\tau^3} + \frac{1}{24}\frac{\overline{z}_n(\tau)^4}{\tau^4} + \frac{1}{120}\frac{\overline{z}_n(\tau)^5}{\tau^5}\Bigg)\, d\tau$$

Application of restricted variation, $\delta z_n(0) = 0$ gives the recurrence formula

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda(\tau)\Bigg[z_n''(\tau) + \alpha\tau^{m+1}\Bigg(1 + \frac{z_n(\tau)}{\tau} + \frac{1}{2}\frac{z_n(\tau)^2}{\tau^2} + \frac{1}{6}\frac{z_n(\tau)^3}{\tau^3} + \frac{1}{24}\frac{z_n(\tau)^4}{\tau^4} + \frac{1}{120}\frac{z_n(\tau)^5}{\tau^5}\Bigg)\Bigg]\, d\tau$$

Since the problem is a second order problem, the optimal value of the Lagrange multiplier is $\lambda(x) = \tau - x$, also from the initial condition $z_0(x) = 0$. The recurrence now becomes

$$z_{n+1}(x) = z_n(x) + \int_0^x (\tau - x)\Bigg[z_n''(\tau) + \alpha\tau^{m+1}\Bigg(1 + \frac{z_n(\tau)}{\tau} + \frac{1}{2}\frac{z_n(\tau)^2}{\tau^2} + \frac{1}{6}\frac{z_n(\tau)^3}{\tau^3} + \frac{1}{24}\frac{z_n(\tau)^4}{\tau^4} + \frac{1}{120}\frac{z_n(\tau)^5}{\tau^5}\Bigg)\Bigg]\, d\tau$$

$z_0(x) = 0,$

$$z_1(x) = z_0(x) + \int_0^x (\tau - x)\left[z_0''(\tau) + \alpha\tau^{m+1}\left(1 + \frac{z_0(\tau)}{\tau} + \frac{1}{2}\frac{z_0(\tau)^2}{\tau^2} + \frac{1}{6}\frac{z_0(\tau)^3}{\tau^3} + \frac{1}{24}\frac{z_0(\tau)^4}{\tau^4} + \frac{1}{120}\frac{z_0(\tau)^5}{\tau^5}\right)\right] d\tau$$

$z_1(x) = -\frac{x^{m+3}\alpha}{(m+3)(m+2)}$

$$z_2(x) = z_1(x) + \int_0^x (\tau - x)\left[z_1''(\tau) + \alpha\tau^{m+1}\left(1 + \frac{z_1(\tau)}{\tau} + \frac{1}{2}\frac{z_1(\tau)^2}{\tau^2} + \frac{1}{6}\frac{z_1(\tau)^3}{\tau^3} + \frac{1}{24}\frac{z_1(\tau)^4}{\tau^4} + \frac{1}{120}\frac{z_1(\tau)^5}{\tau^5}\right)\right] d\tau$$

This gives

$$z_2(x) = -\frac{x^{m+3}\alpha}{(m+3)(m+2)} + \frac{648x^{5+2m}\alpha^2}{(m+2)^6(m+3)^5(2m+5)}$$

The next iteration gives

$$z_3(x) = -\frac{2(m-1)x^{m+3}\alpha}{(m+2)^2(m+3)} + \frac{1}{2(m+2)^7(m+3)^5(2m+5)}[(2592 + 11232m + 16848m^2 + 16584m^3 + 1044m^4 + 4361m^5 + 1208m^6 + 214m^7 + 22m^8 + m^9)x^{2m+5}\alpha] - \frac{1}{6(m+2)^7(m+3)^5(2m+5)(3m+7)}[(3456 + 7344m + 10872m^2 + 9152m^3 + 4791m^4 + 1597m^5 + 331m^6 + 39m^7 + 2m^8)x^{3m+7}\alpha^3]$$

Fixing $m = 0$ and $\alpha = 1$, we have

$$z_0(x) = 0, \quad z_1(x) = -\frac{x^3}{6}, \quad z_2(x) = -\frac{x^3}{6} + \frac{x^5}{120}, \quad z_3(x) = -\frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{1890},$$

$$z_4 = -\frac{x^3}{6} + \frac{x^5}{120} - \frac{x^7}{1890} + \frac{61x^9}{1632960}, \quad \text{and so on.}$$

Changing back to the original variable $y(x)$, we have the final results as

$$y_0(x) = 0$$

$$y_1(x) = -\frac{x^2}{6},$$

$$y_2(x) = -\frac{x^2}{6} + \frac{x^4}{12},$$

$$y_3(x) = -\frac{x^2}{6} + \frac{x^4}{12} - \frac{x^6}{1890},$$

$$y_4(x) = -\frac{x^2}{6} + \frac{x^4}{12} - \frac{x^6}{1890} + \frac{61x^8}{1632960}$$

Problem 2

Consider the following IVP for the Emden-Fowler ODE

$$y''(x) + \frac{2}{x}y'(x) + \alpha x^m y(x)^\mu = 0, \quad y(0) = 1, \quad y'(0) = 0.$$

Here, we equally change variable here by using $y = \frac{z}{x}$ to have the problem in the form

$$z''(x) + \alpha x^{m+1-\mu} z(x)^\mu = 0, \quad z(0) = 0, \quad z'(0) = 2$$

From here, we have

$$Lz \equiv z'', \quad N(z) \equiv \alpha x^{m+1-\mu} z^\mu \quad \text{and} \quad f(x) = 0.$$

The corresponding correctional functional is

$$z_{n+1}(x) = z_n(x) + \int_0^x \lambda(\tau)\left(z_n''(\tau) + \alpha\tau^{m+1-\mu}\overline{z}_n(\tau)^\mu\right)d\tau$$

The optimal value of the Lagrange multiplier $\lambda(x) = \tau - x$ and the restricted variation $\delta z_n(0) = 0$, the recurrence relation is given as

$$z_{n+1}(x) = z_n(x) + \int_0^x (\tau - x)\left(z_n''(\tau) + \alpha\tau^{m+1-\mu} z_n(\tau)^\mu\right)d\tau$$

For this problem to be effectively handled by using variational iteration method, there is a need for the constant μ to be fixed. Here, we chose $\mu = 5$ and the recurrence formula now becomes

$$\begin{aligned}
z_{n+1} &= z_n(x) + \int_0^x (\tau - x)(z_n''(z) + \alpha\tau^{m-4} z_n(\tau)^5)d\tau \\
z_0(x) &= z(0) + xz'(0), \\
z_0(x) &= x, \\
z_1(x) &= z_0(x) + \int_0^x (\tau - x)(z_0''(\tau) + \alpha\tau^{m-4} z_0(\tau)^5)d\tau, \\
z_1(x) &= x + \alpha\int_0^x (\tau - x)\tau^{m+1}d\tau \\
z_1(x) &= x - \frac{x^{m+3}\alpha}{(m+2)(m+3)}.
\end{aligned}$$

subsequent iterations give

$$\begin{aligned}
z_2(x) &= x - \frac{x^{m+3}\alpha}{(m+2)(m+3)} + \frac{1620x^{2m+5}\alpha^2}{(m+2)^5(m+3)^5} - \frac{6480x^{2m+5}}{(2m+5)(m+2)^5(m+3)^5}, \\
z_3(x) &= x - \frac{x^{m+3}\alpha}{(m+2)(m+3)} + \frac{1620x^{2m+5}\alpha^2}{(m+2)^5(m+3)^5} - \frac{6480x^{2m+5}\alpha^2}{(2m+5)(m+2)^5(m+3)^5} \\
&\quad - \frac{14396123857747968000000x^{3m+7}}{(m+2)^{26}(m+3)^{25}(2m+5)^5(3m+7)}
\end{aligned}$$

and so on.
For the sake of comparison with the existing results, we fix $m = 0$ and $\alpha = 1$ to get

$$z_0(x) = x, \quad z_1(x) = x - \frac{x^3}{6}, \quad z_2(x) = x - \frac{x^3}{6} + \frac{x^5}{24},$$
$$z_3(x) = x - \frac{x^3}{6} + \frac{x^5}{24} - \frac{5x^7}{432}, \; z_4 = x - \frac{x^3}{6} + \frac{x^5}{34} - \frac{5x^7}{432} + \frac{35x^9}{10368},$$

which when changed back y gives

$$\begin{aligned}
y_0(x) &= 1, \\
y_1(x) &= 1 - \frac{x^2}{6}, \\
y_2(x) &= 1 - \frac{x^2}{6} + \frac{x^4}{24}, \\
y_3(x) &= 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432}, \\
y_4(x) &= 1 - \frac{x^2}{6} + \frac{x^4}{24} - \frac{5x^6}{432} + \frac{35x^8}{10368}.
\end{aligned}$$

3.3 SOLUTIONS BASED ON RECURSIVE TAU METHOD

Solving equation (23) using the method discussed in section 2.3, the linearized form is:

$$xy''_{k+1} + 2y'_k + \alpha x^{m+1} e^{y_k} y_{k+1} = (y_k - 1)\alpha x^{m+1} e^{y_k}, \qquad (31)$$

using the initial approximation $y_0 = 0$. We fix $\alpha = 1$, $m = 0$ and consider solutions of degrees 7 and 8, we obtain the following approximate solutions for first and second iterations

$$\begin{aligned}
y_1 = &- \frac{289991506688352x^2}{1739948055832967} + \frac{2310322560x^3}{1739948055832967} + \frac{762547063040x^4}{91576213464893} + \frac{25793890304x^5}{1739948055832967} \\
&- \frac{376383508480x^6}{1739948055832967} + \frac{19140100096x^7}{1739948055832967}
\end{aligned}$$

$$\begin{aligned}
y_2 = &- 0.1666671658476x^2 + 7.52 \times 10^{-6}x^3 + 0.00829378537597x^4 + 0.000101845270945x^5 \\
&- 0.000671648362947x^6 + 0.0001079891721199x^7
\end{aligned}$$

$$\begin{aligned}
y_1 = &- \frac{8327897250105472x^2}{49967383021015793} + \frac{1464047616x^3}{49967383021015793} + \frac{416385618211328x^4}{49967383021015793} \\
&+ \frac{28219342848x^5}{49967383021015793} - \frac{9960474886144x^6}{49967383021015793} + \frac{40558133248x^7}{49967383021015793} \\
&+ \frac{121843253248x^8}{49967383021015793}
\end{aligned}$$

$$\begin{aligned}
y_2 = &- 0.1666667816545x^2 + 2.13 \times 10^{-6}x^3 + 0.00831963578698x^4 + 0.0000427389618116x^5 \\
&- 0.000601270324491x^6 + 0.0000657579216091 6931`x^7 + 0.0000101071572 8238x^8
\end{aligned}$$

We compare these solutions with those obtained in VIM and ADM and then present our observations in Table 1 and Figure 1.
For the solution of the problem in (27), using the method discussed in section 2.3, we obtain the following results for first and second iterations:

$$
\begin{aligned}
y_1 =& 1 - \frac{7197665547392x^2}{10796040470513} + \frac{4448371200x^3}{10796040470513} + \frac{1776887538688x^4}{10796040470513} + \frac{54787399680x^5}{10796040470513} \\
&- \frac{285644472320x^6}{10796040470513} + \frac{49179852800x^7}{10796040470513}
\end{aligned}
$$

$$
\begin{aligned}
y_2 =& 1 - 0.167837827193x^2 + 0.0170322641749x^3 - 0.0420651093691x^4 + 0.187628728137x^5 \\
&- 0.140646989088x^6 + 0.0319176656167x^7
\end{aligned}
$$

$$
\begin{aligned}
y_1 =& 1 - \frac{255385167751552x^2}{1532303262386687} + \frac{23801088000x^3}{1532303262386687} + \frac{63694581503488x^4}{1532303262386687} \\
&+ \frac{466450841600x^5}{1532303262386687} - \frac{8374711992320x^6}{1532303262386687} + \frac{687964160000x^7}{1532303262386687} \\
&+ \frac{253050060800x^8}{1532303262386687}
\end{aligned}
$$

$$
\begin{aligned}
y_2 =& 1 - 0.16667532340x^2 + 0.000173828128766x^3 + 0.040441033592x^4 \\
&+ 0.00428706091386x^5 - 0.0199149785x^6 + 0.0090938285942x^7 \qquad (32) \\
&- 0.0013792753964x^8
\end{aligned}
$$

The results obtained from VIM (since VIM and ADM give the same results) were compared with equation (32) and presented in the Table 2 and Figure 2.

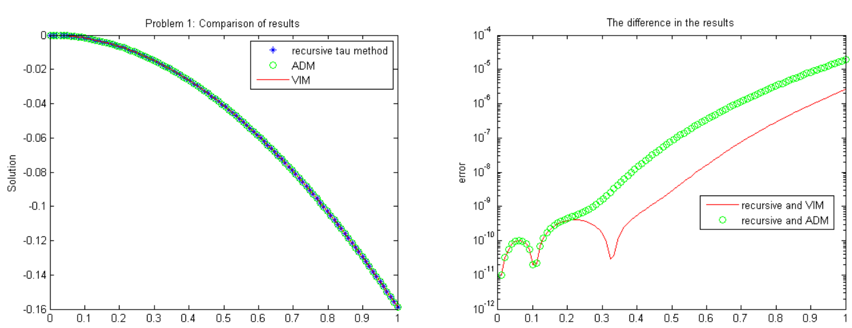

FIGURE 1. Solutions to Problem 1 and their differences

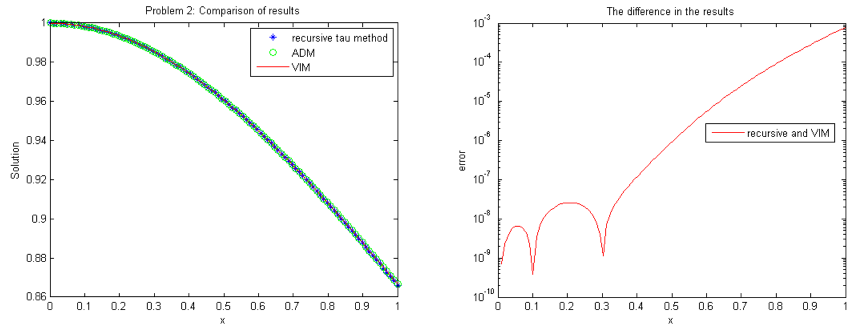

Figure 2. Solutions to Problem 1 and their differences

Table 1: **Problem 1: Differences in the results of Recursive Tau method, ADM and VIM**

x	recursive-ADM	recursive-VIM
0	0.0000	0.0000
0.1	2.3572×10^{-11}	2.3786×10^{-11}
0.2	4.4342×10^{-10}	3.8854×10^{-10}
0.3	1.57134×10^{-9}	1.6509×10^{-10}
0.4	1.3508×10^{-8}	5.3874×10^{-10}
0.5	8.0905×10^{-8}	2.8187×10^{-9}
0.6	3.4378×10^{-7}	1.6222×10^{-8}
0.7	1.1596×10^{-6}	7.5983×10^{-8}
0.8	3.3094×10^{-6}	2.8658×10^{-7}
0.9	8.3075×10^{-6}	9.1894×10^{-7}
1.0	1.8834×10^{-5}	2.5994×10^{-6}

Table 2: **Problem 2: Differences in the results of Recursive Tau method and VIM (or ADM)**

x	recursive-VIM(or ADM)
0	0.0000
0.1	8.9012×10^{-11}
0.2	2.5612×10^{-8}
0.3	5.0581×10^{-10}
0.4	1.1806×10^{-7}
0.5	9.2257×10^{-7}
0.6	5.5005×10^{-6}
0.7	2.4869×10^{-5}
0.8	9.0944×10^{-5}
0.9	2.8300×10^{-4}
1.0	7.7552×10^{-4}

4. DISCUSSION OF RESULTS AND CONCLUSION

4.1 DISCUSSION OF RESULTS

Table 1 and 2 are the tables of results and its corresponding differences in the results when the results are evaluated at selected values of x, with maximum differences of 1.8834×10^{-5} for the recursive - ADM and 2.5994×10^{-6} for recursive - VIM for the solution to problem 1 (that is equation (23)). Also in Problem 2 (equation (27)), it was observed that both ADM and VIM give the same results, with maximum difference of 7.7552×10^{-4} when compare with recursive tau method. It is visually obvious from the tables that the results get closer as the degree of approximation increases and in addition, the recursive tau method improve as the number of iteration increases.

For m positive, it can be observed that ADM and VIM generates polynomials of even powers for both problems, while recursive tau method generates both even and odd powers of x in the approximating polynomials.

The closeness of the results become apparent in the graphical representations. Figure 1 shows the plots of Problem 1 (that is equation

(23)) at degree 8 of approximation and second iteration for the recursive form. Figure 2 present the results of recursive-VIM and its corresponding differences.
The essence of fixing values for some of the constants like the shape factor α, the integer m and μ is for us to confirm whether our results generalize the results for similar problems in the literature or not, but it is gratifying to note that all our results conform with the results obtained in the literature.

4.2 CONCLUSION

The approximate solutions of generalize Emden - Fowler equation have been presented. The problem was narrowed down in certain instances, not for any other reason but for the sake of comparison with existing results in the literature and it was observed that all our results conform with the existing results in the literature. More results can be obtained from the general results we obtained by mere changing the values of certain parameters, like the shape factor, α.

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