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STRONG CONVERGENCE THEOREM SOLUTION BY ITERATION OF NONLINEAR OPERATOR EQUATIONS INVOLVING MONOTONE MAPPINGS

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ABSTRACT. This study focuses on construction of a new explicit iterative scheme for approximation of zeros of nonlinear mappings in reflexive real Banach space with uniformly Gâteaux differentiable norm. In the study, strong convergence of the proposed iterative scheme is proved under mild conditions on the iterative parameters. The scheme does not involve resolvent of the mappings under consideration. Furthermore, applications of results obtained to Dirichlet and Neumann problems are given. Our Theorems improve, extend and unify most of the results that had been proved for this class of mappings.

Keywords and Phrases: Maximal monotone mappings; Zeros; Strong convergence; Reflexive real Banach space; Uniformly Gâteaux differentiable norm.
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1. INTRODUCTION

This research falls within the general area of nonlinear functional analysis and application, an area which has been of increasing research interest to numerous mathematicians in recent years. Within the last three decades or so, many results had been recorded on construction of approximation methods for fixed points and zeros of several classes of nonlinear mappings. In this direction, several authors had introduced numerous iterative algorithms, and many convergence results had been obtained. To appreciate the quantum of work already done in this area of research, interested reader(s) may see the references at the end of this paper and references therein. As shall be seen towards the end of next section, this paper is primarily motivated by the work of Zegeye [37].

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2. PRELIMINARIES

In order to 'water the ground' for appreciation and comprehension of what follows in the sequel, we present some preliminary facts and concepts which could be found in Chidume [9].

Definition 1. Let E be a real normed space and let $S := \{x \in E : \|x\| = 1\}$. The space E is said to have a *Gâteaux differentiable* norm (and E is called *smooth*) if and only if the limit

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for each $x, y \in S$; E is said to have a *uniformly Gâteaux differentiable* norm if and only if for each $y \in S$ the limit is attained uniformly for $x \in S$. Further, E is said to be *uniformly smooth* if and only if the limit exists uniformly for $(x, y) \in S \times S$.

Definition 2. The *modulus of smoothness* of E is defined by

$$\rho_E(\tau) := sup\Big\{\frac{\|x+y\| + \|x-y\|}{2} - 1 : \|x\| = 1,\ \|y\| = \tau\Big\};\ \tau > 0.$$

The space E is equivalently said to be *smooth* if $\rho_E(\tau) > 0, \forall \tau > 0$. Let $q > 1$, then the space E is said to be q-uniformly smooth (or to have a modulus of smoothness of power type q) if and only if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$.

Hilbert spaces, $L_p (and l_p)$ spaces, $1 < p < \infty$, and the Sobolev spaces, W^p_m, $1 < p < \infty$, are p-uniformly smooth. Hilbert spaces are 2-uniformly smooth while

$$L_p \text{or } \ell_p \text{ or } W^{m,p} \text{ is} \begin{cases} p-\text{uniformly smooth if } 1 < p \leq 2 \\ 2-\text{uniformly smooth if } p \geq 2. \end{cases}$$

Definition 3. Let E be a real normed space and let J_q, $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by

$$J_q(x) = \{f \in E^* : \langle x, f\rangle = ||x||^q \text{ and } \|f\| = \|x\|^{q-1}\},$$

where E^* denotes the dual space of E and $\langle ., .\rangle$ denotes the duality pairing between elements of E and E^*. For $q = 2$, the mapping $J = J_2$ from E to 2^{E^*} is called normalized duality mapping. It is well known (see, for example, Xu [34]) that $J_q(x) = ||x||^{q-2}J(x)$, and that if E is uniformly smooth, then J is single-valued (see, e.g., [34, 35]). In the sequel, we shall denote the single-valued normalized duality mapping by j.

Definition 4. A mapping $A : E \to E$ is called *accretive* if and only if for all $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that

$$\big\langle Ax - Ay, j(x-y)\big\rangle \geq 0. \tag{1}$$

As a result of Kato [19], it follows from (1) that the mapping A is accretive if and only if the inequality

$$\|x-y\| \leq \|x-y+s(Ax-Ay)\| \tag{2}$$

holds for each $x, y \in D(A)$ and for all $s > 0$,

Definition 5. A mapping $A : E \to E^*$ is said to be *monotone* if and only if for each $x, y \in D(A)$,

$$\langle x-y, Ax-Ay\rangle \geq 0.$$

We note immediately that accretive and monotone operators coincide in Hilbert spaces.

Definition 6. A mapping $A : E \to E^*$ is called *maximal monotone* if and only if its graph $G(A) = \{(x, y) : y \in Ax\}$ is not properly contained in the graph of any other monotone mapping defined on E. That is, monotone mapping A is maximal if and only if for $(x, y^*) \in E \times E^*$ such that $\langle x-y, u^*-y^*\rangle \geq 0$, for every $x \in D(A)$ and $u^* \in Ax$ implies that $y \in D(A)$ and $y^* \in Ay$. We know that if A is a maximal monotone mapping, then the set of zeros of $A, A^{-1}(0) := \{x \in E : 0 \in Ax\}$, is closed and convex. If E is reflexive, strictly convex and smooth Banach space, then a monotone mapping A from E into E^* is maximal if and only if $R(J+\lambda A) = E^*$ for each $\lambda > 0$, where $R(J+\lambda A)$ denotes the range of the mapping $J+\lambda A$ (see [30] for more details).

For a proper lower semicontinuous convex function $f : E \to (-\infty, \infty]$, Rockafellar [30] proved that the *subdifferential mapping* $\partial f \in E \times E^*$ of f defined by

$$\partial f(x) := \{x^* \in E^* : f(x) + \langle y-x, x^*\rangle \leq f(y),\ y \in E\},$$

for all $x \in E$, is a maximal monotone mapping.

Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm and let $A : E \to E^*$ be a maximal monotone mapping. Our focus in this paper is to consider the problem of finding a point $v \in E$ satisfying $A(v) = 0$. Such a problem is connected with the convex minimization problem. In fact, if $f : E \to (-\infty, \infty]$ is a proper lower semi-continuous convex function, then we have that the equation $0 \in \partial f(v)$ is equivalent to $f(v) = \min_{x \in E} f(x)$ (see, e.g., [38] for more details).

A well-known method for solving the equation $A(v) = 0$ in a Hilbert space H is the proximal point algorithm generated from arbitrary $x_1 = x \in H$ by

$$x_{n+1} = J_{r_n} x_n, \ n \geq 1, \tag{3}$$

where $\{r_n\} \subset (0, \infty)$ and $J_{r_n} = (I + r_n A)^{-1}$ for $n \geq 1$. This algorithm was first introduced by Martinet [24]. In 1976, Rockafellar [31] proved that if $\liminf_{n\to\infty} r_n > 0$ and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ defined by (3) converges weakly to an element of $A^{-1}(0)$. Many researchers have studied the convergence of the sequence defined by (3) in a Hilbert space (see, for instance, [5, 6, 20, 21] and a host of other authors). In particular, Kamimura and Takahashi [20] obtained the following strong convergence theorem.

Theorem 1. *Let H be a real Hilbert space, let $A \subset H \times H$ be a maximal monotone mapping and let $J_r = (I + rA)^{-1}$ for $r > 0$. For $u \in H$, let $\{x_n\}$ be a sequence defined by*

$$x_{n+1} = \alpha_n u + (1 - \alpha_n) J_{r_n} x_n, \ n \geq 1,$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty)$ satisfy $\lim_{n\to\infty} \alpha_n = 0$, $\sum_{n=1}^{\infty} \alpha_n = \infty$ and $\lim_{n\to\infty} r_n = \infty$. If $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to Pu, where P is the metric projection of H onto $N(A)$.

In the case when the space is a Banach space, to find a zero point of a maximal mapping using the proximal point algorithm, Kohsaka and Takahashi [21] introduced the following iterative sequence for a monotone mapping $A \subset E \times E^*$: $x_1 = u \in E$ and

$$x_{n+1} = J^{-1}(\alpha_n J u + (1 - \alpha_n) J J_{r_n} x_n), \ n \geq 1,$$

where $J_r = (J + rA)^{-1}$ for $r > 0$, and J is the duality mapping from E into E^*, $\{\alpha_n\} \subset [0, 1]$ such that $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty$ and $\{r_n\} \subset (0, +\infty)$, such that $\lim_{n\to\infty} r_n = \infty$. They proved that if E is smooth and uniformly convex, $A \subset E \times E^*$ is maximal monotone and $A^{-1}(0) \neq \emptyset$, then the sequence $\{x_n\}$ converges strongly to an element of $N(A)$. This result extends Theorem 1 to Banach spaces. However, the sequence involves the resolvent mappings $J_r = (J + rA)^{-1}$ for $r > 0$, which is not easily obtainable in applications since it requires computation of inverses of mappings.

In [37], Zegeye studied the sequence $\{x_n\}_{n\geq 1}$ of iterates which does not involve resolvents of an operator A in real Banach spaces and generated by $u, x_1 \in E$,

$$x_{n+1} = \beta_n u + (1-\beta_n)(x_n - \alpha_n AJx_n),\ n \geq 1,$$

where J is the normalized duality mapping from E into E^*; A is maximal monotone; and $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ satisfying mild conditions. Zegeye [37], first of all, proved that the sequence $\{x_n\}_{n\geq 1}$ is bounded; and in order to obtain strong convergence of $\{x_n\}_{n\geq 1}$ to a point in $(AJ)^{-1}(0)$, he made the following remark:

Remark 1. (**See Remark 3.3 of** [37]) "Since $\{x_n\}_{n\geq 1}$ is bounded, there exists $R > 0$ sufficiently large such that $u, x_n \in B := B(x^*)$ $\forall\ n \in \mathbb{N}$ (for some $x^* \in (AJ)^{-1}(0)$). Furthermore, the set B is a bounded closed and convex nonempty subset of E. If we define a map $\phi : E \to \mathbb{R}$ by

$$\phi(y) = \mu_n \|x_{n+1} - y\|^2,$$

where μ_n is Banach limt, then ϕ is continuous, convex and $\phi(y) \to +\infty$ as $\|y\| \to \infty$. Thus, if E is a reflexive Banach space, then there exists $x_0 \in B$ such that

$$\phi(x_0) = \min_{y\in B} \phi(y).$$

So, the set

$$B_{min} := \Big\{x \in B : \phi(x) = \min_{y\in B} \phi(y)\Big\} \neq \emptyset."$$

Zegeye [37] then proved the following theorem:

Theorem 2. *Let E be a uniformly convex and 2-uniformly smooth real Banach space with dual E^*. Let $A : E^* \to E$ be a Lipschitz continuous monotone mapping with Lipschitz constant $L > 0$ and $A^{-1}(0) \neq \emptyset$. For given $u, x_1 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = \beta_n u + (1-\beta_n)(x_n - \alpha_n AJx_n),\ n \geq 1,$$

where J is the normalized duality mapping from E into E^*; and $\{\alpha_n\}_{n\geq 1}$, $\{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that (i) $\lim_{n\to\infty} \beta_n = 0$, (ii) $\sum_{n=1}^{\infty} \beta_n = \infty$ and (iii) $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$. Suppose that $B_{\min} \cap$

$(AJ)^{-1}(0) \neq \emptyset$. Then $\{x_n\}$ converges strongly to $Ru := x^* \in (AJ)^{-1}(0)$, with $Jx^* \in A^{-1}(0)$, where R is a sunny generalized nonexpansive retraction of E onto $(AJ)^{-1}(0)$.

3. THE HEART OF THE MATTER

A gap is observed in Theorem 2. To see this gap, the following Lemma is needed:

Lemma 1. *(see, e.g., [4, 33]) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \leq (1-\gamma_n)a_n + \sigma_n,\ n \geq 1,$$

where $\{\gamma_n\}_{n=1}^{\infty}$ and $\{\sigma_n\}_{n=1}^{\infty}$ satisfy the conditions:
(i) $\{\gamma_n\}_{n=1}^{\infty} \subset [0,1]$, $\sum_{n=1}^{\infty}\gamma_n = \infty$; (ii) $\sigma_n = o(\gamma_n)$, that is $\lim\limits_{n\to\infty}\dfrac{\sigma_n}{\gamma_n} = 0$. Then, $a_n \to 0$ as $n \to \infty$.

We now return to our discussion on Theorem 2.

Remark 2. A close look at the algorithm in Theorem 2 showed that if $AJ = I$, the identity mapping of E, and we choose the iterative parameters $\alpha_n = \frac{1}{(n+1)^2}$ and $\beta_n = \frac{1}{n+1}$, then $\{\alpha_n\}_{n\geq 1}$ and $\{\beta_n\}_{n\geq 1}$ both satisfy the conditions imposed on the parameters in Theorem 2; and for $u \in E$, $u \neq 0$ (of course with $AJ = I$), we obtain that

$$\begin{aligned}
x_{n+1} &= \frac{1}{n+1}u + \Big(1-\frac{1}{n+1}\Big)\Big(x_n - \frac{1}{(n+1)^2}x_n\Big)\\
&= \Big(1-\frac{1}{n+1}\Big)\Big(1-\frac{1}{(n+1)^2}\Big)x_n + \frac{1}{n+1}u\\
&= \Big(1-\frac{1}{(n+1)^2} - \frac{1}{n+1} + \frac{1}{(n+1)^3}\Big)x_n + \frac{1}{n+1}u\\
&= \Big(1-\frac{1}{(n+1)^2} - \frac{1}{n+1} + \frac{1}{(n+1)^3}\Big)(x_n-u)\\
&\quad +\Big(1-\frac{1}{(n+1)^2} - \frac{1}{n+1} + \frac{1}{(n+1)^3}\Big)u + \frac{1}{n+1}u.
\end{aligned}$$

Thus,

$$\begin{aligned}
x_{n+1}-u &= \Big(1-\frac{1}{(n+1)^2} - \frac{1}{n+1} + \frac{1}{(n+1)^3}\Big)(x_n-u)\\
&\quad -\Big(\frac{1}{(n+1)^2} + \frac{1}{(n+1)^3}\Big)u.
\end{aligned}$$

Therefore,

$$\begin{aligned} ||x_{n+1}-u|| &\leq \Big(1-\frac{1}{(n+1)^2}-\frac{1}{n+1}+\frac{1}{(n+1)^3}\Big)||x_n-u|| \\ &\quad +\Big(\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}\Big)||u||. \end{aligned} \tag{4}$$

It is easy to check that if we set $\gamma_n = \frac{1}{(n+1)^2}+\frac{1}{n+1}-\frac{1}{(n+1)^3}$ and $\sigma_n = \Big(\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}\Big)||u||$, then (4) becomes

$$||x_{n+1}-u|| \leq (1-\gamma_n)||x_n-u|| + \sigma_n$$

and $\frac{\sigma_n}{\gamma_n} \to 0$ so that by Lemma 1, $x_n \to u$ as $n \to \infty$. But $u \notin I^{-1}(0) = \{0\}$. Thus the scheme studied by Zegeye [37] is wanting. Moreover, the assumption that $A^{-1}(0) \cap B_{\min} \neq \emptyset$ in the result obtained in [37] is rather strong.

Motivated by the result of Zegeye [37], it is our purpose in this paper to study an explicit iterative algorithm which will enable us to correct the anomalies pointed out in Remark 2. As an interesting corollary from the main Theorem obtained in this paper, a replica of the result obtained in [37] is presented. Furthermore, we give some examples where our assumed conditions are fulfilled and give applications of our results in solving Dirichlet and Neumann problems.

We shall make use of the following lemmas in the sequel.

Lemma 2. *(Morales and Jung, [26]) Let K be a closed convex subset of a reflexive Banach space E with a uniformly Gâteaux differentiable norm. Let $T : K \to K$ be continuous pseudo-contractive mapping with $F(T) \neq \emptyset$. Suppose that every closed convex and bounded subset of K has the fixed point property for nonexpansive self-mappings. Then for $u \in K$, the path $t \to y_t \in K$, $t \in [0, 1)$, satisfying $y_t = tTy_t + (1-t)u$, converges strongly to a fixed point Qu of T as $t \to 1$, where Q is the unique sunny nonexpansive retraction from K onto $F(T)$.*

The following is an immediate consequence of Lemma 2.

Corollary 1. *Let K be a closed convex subset of a reflexive Banach space E with a uniformly Gâteaux differentiable norm such that $0 \in K$. Let $T : K \to K$ be continuous pseudo-contractive mapping with $F(T) \neq \emptyset$. Suppose that every closed convex and bounded subset of K has the fixed point property for nonexpansive self-mappings.*

Then the path $t \to y_t \in K,\ t \in [0,1)$, satisfying $y_t = tTy_t$, converges strongly to a fixed point $Q(0)$ of T as $t \to 1$, where Q is the unique sunny nonexpansive retraction from K onto $F(T)$.

Lemma 3. *(Moore and Nnoli, [27]) Let $\{\lambda_n\}_{n=1}^{\infty}$, $\{\alpha_n\}_{n=1}^{\infty}$ and $\{\gamma_n\}_{n=1}^{\infty}$ be a sequences of non-negative real numbers satisfying the following relation: $\lim\limits_{n\to\infty} a_n = 0,\ \sum\limits_{n=1}^{\infty} \alpha_n$ and $\frac{\gamma_n}{\alpha_n} \to 0,\ n \to \infty$. Suppose that*

$$\lambda_{n+1} \leq \lambda_n - \alpha_n \varphi(\lambda_{n+1}) + \gamma_n,\ n \geq 1$$

be given where $\varphi : [0,\infty) \to [0,\infty)$ is a strictly increasing function such that is positive on $(0,\infty)$ with $\varphi(0) = 0$. Then $\lambda_n \to 0$, as $n \to \infty$.

Lemma 4. *Let E be a real normed space and J the normalized duality mapping on E. Then, for any $x, y \in E$, the following inequality holds:*

$$||x+y||^2 \leq ||x||^2 + 2\langle y, j(x+y)\rangle,\ \forall j(x+y) \in J(x+y).$$

Remark 3. For the rest of this paper, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $(0,1)$ satisfying the following conditions:
(i) $\lim\limits_{n\to\infty} \beta_n = 0$; *(ii)* $\alpha_n(1+\beta_n) \leq 1$, $\sum \alpha_n\beta_n = \infty$, $\lim\limits_{n\to\infty} \dfrac{\alpha_n}{\beta_n} = 0$;
(iii) $\lim\limits_{n\to\infty} \alpha_n^{-1}\beta_n^{-2}(\beta_{n-1} - \beta_n) = 0$. Examples of real sequences which satisfy these conditions are $\alpha_n = \frac{1}{(n+1)^a}$ and $\beta_n = \frac{1}{(n+1)^b}$, where $0 < b < a$ and $a + b < 1$. Also $\{y_n\}$ denotes the sequence defined by $y_n := y_{t_n} = t_n(I - AJ)y_{t_n},\ t_n = \frac{1}{1+\beta_n},\ \forall\ n \geq 1$ guaranteed by Corollary 1.

Remark 4. Some of the ideas displayed in the proof of our theorems in this section are borrowed from the methods of proof used in the results of Chidume and Zegeye [12].

Next, we prove our main theorems:

Theorem 3. *Let E be a real Banach space with dual E^* such that the normalized duality mapping J from E to E^* is single-valued. Let $A : E^* \to E$ be a mapping. For given $x_1 \in E$, let $\{x_n\}$ be generated by the algorithm*

$$x_{n+1} = x_n - \alpha_n AJx_n - \alpha_n\beta_n x_n,\ n \geq 1, \tag{5}$$

where J is the normalized duality mapping from E to E^*. Suppose that $(AJ)^{-1}(0) \neq \emptyset$ and that AJ is an accretive Lipschitz with Lipschitz constant $L \geq 0$, then $\{x_n\}$ is bounded.

Proof. Since $\frac{\alpha_n}{\beta_n} \to 0$ as $n \to \infty$, there exists $N_0 \in \mathbb{N}$ such that $\forall n \geq N_0$, $\frac{\alpha_n}{\beta_n} \leq d := \frac{1}{2(2+L)^2}$. Let $x^* \in (AJ)^{-1}(0)$ and let $r > 0$ be sufficiently large such that $x_{N_0} \in B_r(x^*)$ and $x^* \in B_{\frac{r}{2(2+L)}}(0)$. It suffices to show that $\{x_n\}_{n \geq N_0}$ is in $B := \overline{B_r(x^*)}$. Now, $x_{N_0} \in B$ by construction. Hence we may assume $x_k \in B$ for any $n = k \geq N_0$ and prove that $x_{k+1} \in B$. Suppose x_{k+1} is not in B, then $||x_{k+1} - x^*|| > r$ and thus from the recursion formula (5) and Lemma 4 we get that

$$\begin{aligned}
||x_{k+1} - x^*||^2 &= ||x_k - x^* - \alpha_k(AJx_k + \beta_k x_k)||^2 \\
&\leq ||x_k - x^*||^2 - 2\alpha_k \langle AJx_k + \beta_k x_k, J(x_{k+1} - x^*) \rangle \\
&= ||x_k - x^*||^2 - 2\alpha_k\beta_k ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k \langle \beta_k(x_{k+1} - x_k) - AJx_k - \beta_k x^* \\
&\quad + AJx_{k+1} - AJx_{k+1}, J(x_{k+1} - x^*) \rangle. \qquad (6)
\end{aligned}$$

Since AJ is accretive, we have $\langle -AJx_{k+1}, J(x_{k+1} - x^*) \rangle \leq 0$. Thus, (6) gives

$$\begin{aligned}
||x_{k+1} - x^*||^2 &\leq ||x_k - x^*||^2 - 2\alpha_k\beta_k ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k \Big[2||x_{k+1} - x_k|| \\
&\quad + \|AJx_{k+1} - AJx_k\| \Big] . ||x_{k+1} - x^*|| \\
&\quad - 2\alpha_k\beta_k \langle x^*, J(x_{k+1} - x^*) \rangle \\
&\leq ||x_k - x^*||^2 - 2\alpha_k\beta_k ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k(1 + L)||x_{k+1} - x_k||.||x_{k+1} - x^*|| \\
&\quad + 2\alpha_k\beta_k ||x_{k+1} - x^*||.||x^*|| \\
&= ||x_k - x^*||^2 - 2\alpha_k\beta_k ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k(1 + L)\Big[\alpha_k ||\beta_n(-x^* + x^* - x_k) \\
&\quad + AJx_k - AJx^*|| \Big] ||x_{k+1} - x^*|| \\
&\quad + 2\alpha_k\beta_k ||x_{k+1} - x^*||.||x^*|| \\
&\leq ||x_k - x^*||^2 - 2\alpha_k\beta_k ||x_{k+1} - x^*||^2 \\
&\quad + 2\alpha_k^2(1 + L)^2 ||x_k - x^*||.||x_{k+1} - x^*|| \\
&\quad + 2\alpha_k^2\beta_k(2 + L)||x^*||.||x_{k+1} - x^*||. \qquad (7)
\end{aligned}$$

But $||x_{k+1} - x^*|| > ||x_k - x^*||$. Thus we obtain from (7) that

$$||x_{k+1} - x^*|| \leq \frac{\alpha_k}{\beta_k}(1 + L)^2 ||x_k - x^*|| + (2 + L)||x^*||,$$

and hence $||x_{k+1} - x^*|| \leq r$, since $x_k \in B$ and $x^* \in B_{\frac{r}{2(2+L)}}(0)$ and $\frac{\alpha_k}{\beta_k} \leq \frac{1}{2(1+L)^2}$. But this is a contradiction. Therefore, $x_n \in B$ for all positive integers $n \geq N_0$ and hence the sequence $\{x_n\}$ is bounded. □

Theorem 4. *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let $A : E^* \to E$ be a mapping. For any $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence iteratively generated by*

$$x_{n+1} = x_n - \alpha_n AJx_n - \alpha_n\beta_n x_n, \; n \geq 1, \tag{8}$$

Suppose that $(AJ)^{-1}(0) \neq \emptyset$; and suppose that AJ is an accretive Lipschitz mapping with Lipschitz constant $L \geq 0$, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in (AJ)^{-1}(0)$ with $Ju^* \in A^{-1}(0)$.7

Proof. From the recursion formula (8) and Corollary 1 we have that

$$\begin{aligned} ||x_{n+1} - y_n||^2 &\leq ||x_n - y_n||^2 - 2\alpha_n\beta_n\langle(x_{n+1} - y_n), J(x_{n+1} - y_n)\rangle \\ &\quad +2\alpha_n\langle\beta_n(x_{n+1} - y_n) \\ &\quad -AJx_n - \beta_n x_n, J(x_{n+1} - y_n)\rangle \\ &= ||x_n - y_n||^2 - 2\alpha_n\beta_n||x_{n+1} - y_n||^2 \\ &\quad +2\alpha_n\langle\beta_n(x_{n+1} - x_n) \\ &\quad -[\beta_n(y_n) + AJy_n] - [AJx_{n+1} - AJy_n] \\ &\quad +[AJx_{n+1} - AJx_n], J(x_{n+1} - y_n)\rangle. \end{aligned} \tag{9}$$

Observe that AJ being accretive and by the property of y_n, we have that $\beta_n y_n + AJy_n = 0$ and $\langle AJx_{n+1} - AJy_n, J(x_{n+1} - y_n)\rangle \geq 0$ for all $n \geq 1$. Thus, we have from (9) that

$$\begin{aligned} ||x_{n+1} - y_n||^2 &\leq ||x_n - y_n||^2 - 2\alpha_n\beta_n||x_{n+1} - y_n||^2 \\ &\quad +2\alpha_n\langle\beta_n(x_{n+1} - x_n) \\ &\quad +AJx_{n+1} - AJx_n, J(x_{n+1} - y_n)\rangle \\ &\leq ||x_n - y_n||^2 - 2\alpha_n\beta_n||x_{n+1} - y_n||^2 \\ &\quad +2\alpha_n(1 + L)||x_{n+1} - x_n||.||x_{n+1} - y_n||. \end{aligned} \tag{10}$$

But since $(AJ)^{-1}(0) \neq \emptyset$, we obtain (by Corollary 1 applied to $T = I - AJ$) that $\{y_n\}$ is bounded. Therefore,

$$||x_{n+1} - y_n||.||AJx_n + \beta_n x_n|| \leq M$$

for some $M \geq 0$. Thus, from (10) we get that

$$\begin{aligned} ||x_{n+1} - y_n||^2 &\leq ||x_n - y_n||^2 \\ &\quad -2\alpha_n\beta_n||x_{n+1} - y_n||^2 + 2\alpha_n^2(1 + L)M. \end{aligned} \tag{11}$$

Moreover, since AJ is accretive, we have that

$$\begin{aligned} ||y_{n-1} - y_n|| &\leq \left\| y_{n-1} - y_n + {\beta_n}^{-1}\Big[AJy_{n-1} - AJy_n\Big] \right\| \\ &= |\beta_n^{-1}(\beta_{n-1} - \beta_n)| \|y_{n-1}\|. \end{aligned} \tag{12}$$

Thus, we obtain from (11) and (12) that

$$\begin{aligned} ||x_{n+1} - y_n||^2 &\leq ||x_n - y_{n-1}||^2 - 2\alpha_n\beta_n||x_{n+1} - y_n||^2 \\ &\quad + M_1\Big[|\beta_n^{-1}(\beta_{n-1} - \beta_n)| + 2\alpha_n^2(2 + L)\Big], \end{aligned} \tag{13}$$

for some constant $M_1 > 0$. By Lemma 3 and the conditions on $\{\alpha_n\}$ and $\{\beta_n\}$ we get from (13) that $||x_n - y_{n-1}|| \to 0$ as $n \to \infty$. Consequently, $||x_n - y_n|| \to 0$ as $n \to \infty$. Therefore, since by Corollary 1, $\{y_n\}_{n\geq 1}$ converges to some $u^* \in E$, where $(I - AJ)u^* = u^*$ we get that $\{x_n\}_{n\geq 1}$ converges strongly to some $u^* \in (AJ)^{-1}(0)$ with $Ju^* \in A^{-1}(0)$. This completes the proof. □

Corollary 2. *Let E be a reflexive real Banach space with uniformly Gâteaux differentiable norm. Let $A : E^* \to E$ be a monotone mapping. For any $x_1 \in E$, let $\{x_n\}_{n=1}^\infty$ be the sequence iteratively generated by*

$$x_{n+1} = x_n - \alpha_n AJx_n - \alpha_n\beta_n x_n, \ n \geq 1, \tag{14}$$

Suppose that $(AJ)^{-1}(0) \neq \emptyset$; and suppose that AJ is an accretive Lipschitz mapping with Lipschitz constant $L \geq 0$, then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to some $u^* \in (AJ)^{-1}(0)$ with $Ju^* \in A^{-1}(0)$.

It was shown in Remark 5, p. 208 of [35] that if E is 2-uniformly smooth, then for all $x, y \in E$ there exists a constant $L_* > 0$ such that $\forall x, y \in E$, $||Jx - Jy|| \leq L_*||x - y||$. Thus, we obtain the following corollary (in which we drop the assumption that AJ Lipschitz) as a replica of intention of Theorem 2 (Zegeye [37]).

Corollary 3. *Let E be a uniformly convex and 2-uniformly smooth real Banach space with dual E^*. Let $A : E^* \to E$ be a Lipschitz maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. For any $x_1 \in E$, let $\{x_n\}_{n=1}^\infty$ be the sequence iteratively generated by*

$$x_{n+1} = x_n - \alpha_n AJx_n - \alpha_n\beta_n x_n, \ n \geq 1.$$

Suppose that AJ is accretive, then the sequence $\{x_n\}_{n=1}^\infty$ converges strongly to some $u^* \in (AJ)^{-1}(0)$ with $Jx^* \in A^{-1}(0)$.

If E is 2-uniformly smooth Banach space and A is a bounded linear maximal monotone operator on E^*, then the condition AJ is Lipschitz can also be dispensed. Thus, we obtain the following:

Corollary 4. *Let E be a real 2-uniformly smooth Banach space. Let $A : E^* \to E$ be a bounded linear maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. For any $u, x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence iteratively generated by* (14). *Suppose that AJ is accretive, then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in (AJ)^{-1}(0)$ with $Ju^* \in A^{-1}(0)$.*

Proof. Observe that $\forall x, y \in E$,

$$||AJx - AJy|| = ||A(Jx - Jy)|| \leq ||A||||Jx - Jy|| \leq ||A||L_*||x - y||.$$

Hence, AJ is Lipschitz. The remaining follows as in the proof of Theorem 4 □

We know that if E is a real Hilbert space, then the duality mapping J on E becomes an identity mapping on E and monotonicity of an operator A coincides with its accretivity. Hence, we obtain the following corollary from Theorem 4 and Corollary 4.

Corollary 5. *Let H be a real Hilbert space. Let $A : H \to H$ be a Lipschitz maximal monotone mapping with $A^{-1}(0) \neq \emptyset$. For any $x_1 \in E$, let $\{x_n\}_{n=1}^{\infty}$ be the sequence iteratively generated by*

$$x_{n+1} = x_n - \alpha_n Ax_n - \alpha_n\beta_n x_n, \; n \geq 1, \tag{15}$$

then the sequence $\{x_n\}_{n=1}^{\infty}$ converges strongly to some $u^* \in A^{-1}(0)$.

Remark 5. Since any positive linear operator A on a real Hilbert space H (i.e., $\langle Ax, x\rangle \geq 0$, $\forall x \in H$) is maximal monotone, then our Corollary 5 is applicable for any Lipschitz positive linear operator. Thus, we have the following as an example of a mapping $A : H \to H$ satisfying the condition of Corollary 5

Example 1. Let $A : \ell_2 \to \ell_2$ be defined by $Ax = A(x_1, x_2, \ldots) = (x_1, \frac{1}{2}x_2, \ldots)$. Clearly, $A^{-1}(0) \neq \emptyset$, A is linear, $\langle Ax, x\rangle \geq 0$ and $||Ax|| \leq ||x||$, $\forall x \in \ell_2$. Thus, A is a Lipschitz maximal monotone mapping.

3.1. **Application.** In this section, we shall examine the following boundary value problem known as Dirichlet Problem:

$$\begin{cases} -\Delta u + u = f(x), \text{ in } \Omega \\ u = 0, \text{ on } \Gamma \end{cases} \tag{16}$$

where Ω is a bounded conical domain of a Euclidean space $\mathbb{R}^N$ with its boundary $\Gamma \in C^1$ (cf. [1]), $\langle ., .\rangle$ the Euclidean inner-product. As an application of our results in Section 3, we show that the iterative scheme studied could be used to approximate the unique solution of the partial differential equation.

Lemma 5. *(see [2]) Define the mapping $B : H^1(\Omega) \to \Big(H^1(\Omega)\Big)^*$ by*

$$\big\langle v, Bu\big\rangle := \int_\Omega \langle \nabla u, \nabla v\rangle dx + \int_\Omega u(x)v(x)dx - \int_\Omega f(x)v(x)dx$$

for any $u, v \in H^1(\Omega)$. Then, B is everywhere defined, linear, bounded, monotone, hemi-continuous and coercive. Hence, B is Lipschitz maximal monotone operator.

Lemma 6. *([2]) For $f \in L^2(\Omega)$, the partial differential equation* (16) *has a unique solution $u \in H^1(\Omega)$.*

Lemma 7. *$u \in H^1(\Omega)$ is the solution of* (16) *if and only if $u \in H^1(\Omega)$ is the zero point of B.*

Proof. Let u be the solution of (16), then $\forall\ v \in H^1(\Omega)$ by using Green's formula, we have

$$\begin{aligned}\big\langle v, Bu\big\rangle &= \int_\Omega \langle \nabla u, \nabla v\rangle dx + \int_\Omega u(x)v(x)dx - \int_\Omega f(x)v(x)dx \\ &= -\int_\Omega (\Delta u)v dx + \int_\Omega u(x)v(x)dx - \int_\Omega f(x)v(x)dx = 0.\end{aligned}$$

Thus, $u \in B^{-1}(0)$.

Conversely, If $u \in B^{-1}(0)$, then $\forall \varphi \in H^1(\Omega)$, we have

$$0 = \big\langle v, Bu\big\rangle = \int_\Omega \langle \nabla u, \nabla v\rangle dx + \int_\Omega u(x)v(x)dx - \int_\Omega f(x)v(x)dx$$

which implies the result

$$-\Delta u + u = f(x),\ \text{ a.e. } x \in \Omega$$

is true. □

We now apply our Corollary 5 to approximate the solution of (16).

Corollary 6. *For any $u_1 \in H^1(\Omega)$, let $\{u_n\}_{n=1}^\infty$ be the sequence iteratively generated by*

$$u_{n+1} = u_n - \alpha_n B u_n - \alpha_n \beta_n u_n,\ n \geq 1, \tag{17}$$

then the sequence $\{u_n\}_{n=1}^\infty$ converges strongly to some $u^* \in B^{-1}(0)$, where u^* is the unique solution of (16).

Remark 6. We can apply the same method of solution used in solving (16) for solving the following Neumann problem

$$\begin{cases} -\Delta u + u = f(x),\ \text{ in } \Omega \\ \frac{\partial u}{\partial \nu} = 0,\ \text{ on } \Gamma \end{cases} \tag{18}$$

where ν is the outward unit normal at $x \in \Gamma$.

4. CONCLUDING REMARKS

It is worthy to note here that Corollary 5 is of interest on its own because the iterative scheme (15) does not involve the resolvent of the maximal monotone operator A. Hence, our iterative algorithm (15) seems better and more realistic than those of some authors involving the resolvent of the operator A.

We note here that if $A := J^{-1}$, the duality mapping on E^* and E is uniformly smooth Banach space, then A is maximal monotone, $AJ = I_E$ is Lipschitz and accretive. Thus, the assumption that AJ is Lipschitz and accretive in our Theorem is not a vacuous assumption.

In approximating the unique solution of (16), our iterative scheme (17) does not involve the use of resolvent of maximal monotone operator B, whereas in the iterative scheme (4.2) of Corollary B in [1], the use of resolvent was made. Hence, our iterative scheme (17) appears better and more realistic than the scheme (4.2) of Corollary B in [1] for solving problem (16).

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NOMENCLATURE

For a nonnegative integer m, $H^m(\Omega)$ is the Sobolev space $W^{m,2}(\Omega)$ defined by

$$H^m(\Omega) = W^{m,2}(\Omega) = \{u \in L^2(\Omega) : D^\alpha u \in L^2(\Omega) \text{ for all } |\alpha| \leq m\},$$

where $D^\alpha u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$ and $(x_1, x_2, ..., x_n) \in \Omega \subset \mathbb{R}^n$.

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