Journal of the Nigerian Mathematical Society
Vol. 35, pp. 282-302, 2016

APPROXIMATION OF COMMON FIXED POINTS FOR FINITE FAMILIES OF BREGMAN QUASI-TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. In this paper we establish necessary and sufficient conditions for the convergence of a multistep iterative scheme to a common fixed point of a finite family of Bregman quasi-total asymptotically nonexpansive mappings in a real Banach space. We then establish strong convergence theorems for finite families of Bregman quasi-total asymptotically nonexpansive mappings in a real uniformly convex Banach spaces. The results presented generalize and improve some recently announced ones.

Keywords and phrases: Uniformly L-Lipschitzian maps; Asymptotically nonexpansive mappings; Bregman quasi-total asymptotically nonexpansive mappings; Banach spaces , Fixed point
2010 Mathematical Subject Classification: 47H09, 47J25

1. INTRODUCTION

Let E be a real Banach space and K a nonempty closed convex subset of E. The normalized duality map from E to 2^{E^*} (E^* is the dual space of E) denoted by J is defined by
$J(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^2, \|x\| = \|f\|\}$.
Let $T : K \to K$ be a map, a point $x \in K$ is called a fixed point of T if $Tx = x$, and the set of all fixed points of T is denoted by $F(T) = \{x \in K : Tx = x\}$. The mapping T is called L-Lipschitzian or simply Lipschitz if there exists $L > 0$, such that $\|Tx - Ty\| \leq L\|x - y\|$, $\forall x, y \in K$ and if $L = 1$, then T is nonexpansive. T is asymptotically nonexpansive if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, \infty)$ with $\lim_{n\to\infty} \mu_n = 0$, such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| \quad \forall n \geq 1.$$

In 1967, Bregman [5] discovered an elegant and effective technique for the use of so-called Bregman distance function in the process

Received by the editors November 05, 2015; Revised: October 04, 2016; Accepted: October 31, 2016
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of designing and analyzing feasibility and optimization algorithms. This opened a growing area of research in which Bregman's technique is applied in various ways to design and analyze not only iterative algorithms for solving feasibility and optimization problems, but also algorithms for solving variational inequalities, for approximating equilibria and for computing fixed points of nonlinear mappings. A mapping T is said to be Bregman firmly nonexpansive [17], if for all $x, y \in C$.

$$\langle \bigtriangledown f(Tx) - \bigtriangledown f(Ty), Tx - Ty\rangle \leq \langle \bigtriangledown f(x) - \bigtriangledown f(y), Tx - Ty\rangle.$$

Or equivalently,

$$\begin{aligned} D_f(Tx, Ty) \ &+ \ D_f(Ty, Tx) + D_f(Tx, x) \\ &+ \ D_f(Ty, y) \leq D_f(Tx, y) + D_f(Ty, x). \end{aligned}$$

Let K be a nonempty, closed, and convex subset of E and T a mapping from K into itself. Let $f : E \to (-\infty, +\infty]$ be an admissible function (i.e, a proper, convex and lower semicontinuous on E and Gâteaux differentiable on int dom$f$). Then T is said to be:

(i) $quasi-Bregman\ relatively\ nonexpansive$ if $F(T) \neq \emptyset$ and

$$D_f(p, Tx) \leq D_f(p, x)\ \forall x \in K \text{ and } p \in F(T).$$

In [16] quasi-Bregman relatively nonexpansive is called left quasi-Bregman relatively nonexpansive.

(ii) $Bregman\ quasi-asymptotically\ nonexpansive$ if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, \infty)$ with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x \in K$ and $p \in F(T)$

$$D_f(T^n x, p) \leq (1 + \mu_n) D_f(x, p)\ \forall\ n \geq 1.$$

(iii) $Bregman\ quasi - asymptotically\ nonexpansive$ in the intermediate sense [18] if $F(T)) \neq \emptyset$ and

$$\limsup_{n\to\infty} \sup_{x\in K, p\in F(T)} \big(D_f(T^n x, p) - D_f(x, p)\big) \leq 0. \tag{1}$$

Put

$$\sigma_n = max\left\{0, \sup_{x\in K, p\in F(T)} \big(D_f(T^n x, p) - D_f(x, p)\big)\right\}$$

then $\sigma_n \to 0$ as $n \to \infty$ and (1) reduces to

$$D_f(T^n x, p) \leq D_f(x, p) + \sigma_n \tag{2}$$

(iv) *Bregman quasi−totalasymptotically nonexpansive* if there exists nonnegative real sequences $\{\mu_n\}$ and $\{l_n\}$ with $\mu_n \to 0$, $l_n \to 0$ as $n \to \infty$ and strictly increasing continuous function
$\phi : \mathbb{R}^+ \to \mathbb{R}^+$ with $\phi(0) = 0$ such that for all $x \in K$ and $p \in F(T)$.

$$D_f(T^n x, p) \ \leq \ D_f(x, p) + \mu_n \phi\big(D_f(x, p)\big) + l_n, \quad n \geq 1 \qquad (3)$$

Example of Bregman quasi-total asymptotically nonexpansive mappings is given in [9].

Remark 1: If $\phi(\lambda) = \lambda$, then (3) reduces to

$$D_f(T^n x, p) \ \leq \ (1 + \mu_n) D_f(x, p) + l_n, \quad n \geq 1$$

In addition, if $l_n = 0$ for all $n \geq 1$, then Bregman quasi-total asymptotically nonexpansive mappings coincide with Bregman quasi- asymptotically nonexpansive mappings. If $\mu_n = 0$ and $l_n = 0$ for all $n \geq 1$, we obtain from (3) the class of mappings that includes the class of Bregman quasi-nonexpansive mappings. If $\mu_n = 0$ and $l_n = \sigma_n = max\{0, a_n\}$, where $a_n := \sup_{x \in C, p \in F(T)} \big(D_f(p, T^n x) - D_f(p, x)\big)$ for all $n \geq 1$, then (3) reduces to (2) which has been studied as Bregman quasi-asymptotically nonexpansive mappings in the intermediate sense. In 2007, Chidume and Ofoedu [10] constructed the following iterative sequence, for approximation of common fixed points of finite families of total asymptotically nonexpansive mappings.

$$\begin{cases} x_1 \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + T_1^n x_n, \quad \text{if } m = 1, \ n \geq 1, \\ x_1 \in K, \\ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_1^n y_{1n}, \\ y_{1n} = (1 - \alpha_n) x_n + \alpha_n T_2^n y_{2n}, \\ \quad \vdots \\ y(m-2)_n = (1 - \alpha_n) x_n + \alpha_n T_{m-1}^n y_{(m-1)n}, \\ y(m-1)_n = (1 - \alpha_n) x_n + \alpha_n T_m^n x_n, \quad \text{if } m \geq 2, \ n \geq 1, \end{cases} \qquad (4)$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1] bounded away from 0 and 1.

They studied the convergence of this scheme to a common fixed point of finite families of total asymptotically nonexpansive mappings in a uniformly convex Banach spaces.

Let $\{\alpha_n\}$ be a real sequence in $[\epsilon, 1-\epsilon]$, $\epsilon \in (0,1)$. Let $T_1, T_2, \cdots, T_m : E \to E$ be a family of mappings. Define a sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in E, \\ x_{n+1} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_1^n x_n)\big), \text{ if } m = 1,\ n \geq 1, \\ x_1 \in E, \\ x_{n+1} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_1^n y_{1n})\big), \\ y_{1n} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_2^n y_{2n})\big), \\ \quad \vdots \\ y_{(m-2)n} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_{m-1}^n y_{(m-1)n}\big), \\ y_{(m-1)n} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_m^n x_n)\big), \text{ if } m \geq 2,\ n \geq 1, \end{cases} \tag{5}$$

It is our purpose in this paper to prove necessary and sufficient conditions for the strong convergence of the scheme defined by (5) to a common fixed point of finite family $T_1, T_2, ..., T_m$ of uniformly-L-Lipschzian Bregman total asymptotically quasi-nonexpansive mappings. We also prove sufficient condition for strong convergence of the scheme to a common fixed point of mapping in real uniformly convex Banach spaces.

2. PRELIMINARY

Throughout this paper, we shall assume $f : E \to (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\text{dom} f := \{x \in E : f(x) < +\infty\}$ the domain of f. Let $x \in \text{intdom} f$, the subdifferential of f at x is the convex set defined by

$$\partial f(x) = \{x^* \in E^* : f(x) + \langle x^*, y - x\rangle \leq f(y),\ \forall y \in E\},$$

where the fenchel conjugate of f is the function $f^* : E^* \to (-\infty, +\infty]$ defined by

$$f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in E\}.$$

It is known that the young-Fenchel inequality holds:

$$\langle x^*, x\rangle \leq f(x) + f^*(x^*), \quad \forall x \in E.$$

A function f on E is coercive [11] if the sublevel set of f is bounded; equivalently,

$$\lim_{\|x\| \to +\infty} f(x) = +\infty.$$

A function f on E is said to be strongly coercive [19] if

$$\lim_{\|x\|\to+\infty} \frac{f(x)}{\|x\|} = +\infty.$$

For any $x \in \text{intdom}f$ and $y \in E$, the right-hand derivative of f at x in the direction y is defined by

$$f^{\circ}(x,y) := \lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}.$$

The function f is said to be Gâteaux differentiable at x if $\lim_{t\to 0^+} \frac{f(x+ty)-f(x)}{t}$ exists for any y. In this case, $f^{\circ}(x,y)$ coincides with $\bigtriangledown f(x)$, the value of the gradient $\bigtriangledown f$ of f at x. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable at every point $x \in \text{intdom}f$. The function f is said to be Fréchet differentiable at x if this limit is attained uniformly in $\|y\| = 1$. Finally, f is said to be uniformly Fréchet differentiable on a subset K of E if the limit is attained uniformly for $x \in K$ and $\|y\| = 1$. It is well known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{intdom}f$, then f is continuous and its Gâteaux derivative $\bigtriangledown f$ is norm-to-weak* continuous (resp. uniformly continuous) on $\text{intdom}f$ (see also [1, 4]). We will need the following results.

Lemma 1 [15] : If $f : E \to \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E, then $\bigtriangledown f$ is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^*.

Definition 1 [2]: The function f is said to be:

(1) Essentially smooth, if ∂f is both locally bounded and single-valued on its domain;
(ii) Essentially strictly convex, if $(\partial f)^{-1}$ is locally bounded on its domain and f is strictly convex on every subset of $\text{dom}f$;
(iii) Legendre, if it is both essentially smooth and essentially strictly convex.

Remark 2: Observe that if E is reflexive Banach space, Then we have:

(i) f is essentially smooth if and only if f^* is essentially strictly convex (see [2] Theorem 5.4);
(ii) $(\partial f)^{-1} = \partial f^*$ (see [4])
(iii) f is Legendre if and only if f^* is Legendre (see [2],Corollary 5.5)

(iv) If f is Legendre, then $\bigtriangledown f$ is a bijection satisfying $\bigtriangledown f = (\bigtriangledown f^*)^{-1}$, ran$\bigtriangledown f$ = dom$\bigtriangledown f^*$ = intdomf^* and ran $\bigtriangledown f^*$ = dom$f$ = intdom$f$ (see [2], Theorem 5.10).

Examples of Legendre functions were given in [2, 3]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ $(1 < p < \infty)$ when E is a smooth and strictly convex Banach space. In this case the gradient $\bigtriangledown f$ of f coincides with the generalized duality mapping of E, i.e, $\bigtriangledown f = J_p$ $(1 < p < \infty)$. In particular, $\bigtriangledown f = I$ the identity mapping in Hilbert spaces. In the rest of this paper, we always assume that $f : E \to (-\infty, +\infty]$ is Legendre.

Let $f : E \to (-\infty, +\infty]$ be a convex and Gâteaux differentiable function. The function D_f : dom$f\times$ intdom$f \to (-\infty, +\infty]$, defined by:

$$D_f(x, y) := f(y) - f(x) - \langle \bigtriangledown f(x), y - x\rangle, \tag{6}$$

is called the Bregman distance of x to y with respect to f (see [8]). It is obvious from the definition of D_f that

$$D_f(z, x) := D_f(z, y) + D_f(y, x) + \langle \bigtriangledown f(y) - \bigtriangledown f(x), z - y\rangle. \tag{7}$$

Recall that the Bregman projection [5] of $x \in$ intdom$f$ onto nonempty, closed and convex set $K \subset$ dom$f$ is the unique vector $P_K^f(x) \in K$ satisfying

$$D_f(P_K^f(x), x) = \inf\{D_f(y, x) : y \in K\}.$$

Concerning the Bregman projection, the following are well known.

Lemma 2 [7] : Let K be a nonempty, closed and convex subset of a reflexive Banach space E. Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function and let $x \in E$ Then

(a) $z = P_K^f(x)$ if and only if $\langle \bigtriangledown f(x) - \bigtriangledown f(z), y - z\rangle \leq 0, \quad \forall y \in K$;

(b) $D_f(y, P_K^f(x)) + D_f(P_K^f(x), x) \leq D_f(y, x), \quad \forall x \in E,\ y \in K.$

Let $f : E \to (-\infty, +\infty]$ be a Gâteaux differentiable function. The modulus of total convexity of f at $x \in$ intdom$f$ is the function $v_f(x, \cdot) : [0, +\infty] \to [0, +\infty]$ defined by

$$v_f(x, t) := \inf\{D_f(x, y) : y \in \text{dom} f, \|y - x\| = t\}.$$

The function f is called totally convex at x if $v_f(x, t) > 0$ whenever $t > 0$. The function f is called convex if it is totally convex at any point $x \in$ intdom$f$ and is said to be totally convex on bounded set if $v_f(B, t) > 0$ for any nonempty bounded subset B of E and $t > 0$, where the modulus of the total convexity of the function f on the set B is the function v_f : $\text{intdom} f \times [0, +\infty) \to [0, +\infty)$ defined by

$$v_f(B, t) := \inf\{v_f(x, t) : x \in B \cap \text{dom} f\}.$$

The following result was proved in [14]

Lemma 3[14]: Let E be a Banach space and $f : E \to \mathbb{R}$ be a Gâteaux differentiable function which is uniformly convex on bounded subset of E. Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be bounded sequences in E. Then

$$\lim_{n\to\infty} D_f(x_n, y_n) = 0 \quad \Leftrightarrow \quad \lim_{n\to\infty} ||x_n - y_n|| = 0.$$

Lemma 4 [16] : Let $f : E \to \mathbb{R}$ be Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded the sequence $\{x_n\}$ is bounded too.

Lemma 5 [6] : The function f is totally convex on bounded set if and only if the function f is sequentially consistent.

The following definition is slightly different from that in Butnariu and Iusem [6].

Definition 2[12]: Let E be a Banach space. The function $f : E \to \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

(i) f is continuous, strictly convex and Gâteaux differentiable;
(ii) the set $\{y \in E : D_f(x, y) \le r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [6] and Zălinescu [19].

Lemma 6: Let E be a reflexive Banach space and $f : E \to \mathbb{R}$ be a strongly coercive Bregman function. Then

(i) $\nabla f : E \to E^*$ is one-to-one, onto and norm-to-weak* continuous;
(ii) $\langle x - y, \nabla f(x) - \nabla f(y)\rangle = 0$ if and only if $x = y$;
(iii) $\{x \in E : D_f(x, y) \le r\}$ is bounded for all $y \in E$ and $r > 0$;
(iv) *dom* $f^* = E^*$, f^* is Gâteaux differentiable and $\nabla f^* = (\nabla f)^{-1}$.

The following two results were proved in [19]

Theorem 1: Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a convex function which is bounded on bounded subsets of E. Then the following assertions are equivalent:

(1) f is strongly coercive and uniformly convex on bounded subsets of E;

(2) $dom f^* = E^*$, f^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^*;
(3) $dom f^* = E^*, f^*$ is Frechet differentiable and ∇f is uniformly norm-to-norm continuous on bounded subsets of E^*.

Theorem 2: Let E be a reflexive Banach space and let $f : E \to \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

(1) f is bounded on bounded subsets and uniformly smooth on bounded subsets of E;
(2) f^* is Frechet differentiable and f^* is uniformly norm-to-norm continuous on bounded subsets of E^*;
(3) $dom f^* = E^*, f^*$ is strongly coercive and uniformly convex on bounded subsets of E^*.

The following result was first proved in [7] (see also [12]).

Lemma 7: Let E be a reflexive Banach space, let $f : E \to \mathbb{R}$ be a strongly coercive Bregman function and let V_f be the function defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*),\ x \in E,\ x^* \in E^*.$$

Then the following assertions hold:

(1) $D_f(x, \nabla f(x^*)) = V_f(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.
(2) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 8[14]: Let E be a Banach space and $f : E \to \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E. Let $r > 0$ be a constant and ρ_r be the gauge of uniform convexity of f . Then

(i) For any $x, y \in B_r$ and $\alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) - \alpha(1 - \alpha)\rho_r(||x - y||).$$

(ii) For any $x, y \in B_r$,

$$\rho_r(||x - y||) \leq D_f(x, y)$$

(iii) If, in addition, f is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E, y^*, z^* \in B_r$ and $\alpha \in (0, 1)$,

$$V_f(x, \alpha y^* + (1-\alpha)z^*) \leq \alpha V_f(x, y^*) + (1-\alpha)V_f(x, z^*) - \alpha(1-\alpha)\rho_r^*(||y^* - x^*||).$$

Lemma 9 [16] : Let $f : E \to \mathbb{R}$ be a Gâteaux differentiable and totally convex function, $x_0 \in E$ and let K be a nonempty, closed and convex subset of E. Suppose that the sequence $\{x_n\}$ is bounded and any weak subsequential limit of $\{x_n\}$ belongs to K. If $D_f(x_n, x_0) \leq$

$D_f(P_K^f(x_0), x_0)$ for any $n \in \mathbb{N}$, then $\{x_n\}$ converges strongly to $P_K^f(x_0)$.

Lemma 10 [13] : Let E be a real reflexive Banach space, $F : E \to (-\infty, +\infty]$ be a proper lower semi-continuous function, then $F : E \to (-\infty, +\infty]$ is a proper weak* lower semi-continuous and convex function. Thus, for all $z \in E$, we have

$$D_f\left(z, \bigtriangledown f^*\left(\sum_{i=1}^{N} t_i \bigtriangledown f(x_i)\right)\right) \le \sum_{i=1}^{N} t_i D_f(z, x_i). \tag{8}$$

Lemma 11 (see Lemma 4 in [10]) : Let $\{a_n\}, \{\alpha_n\}$ *and* $\{b_n\}$ be sequences of nonnegative real numbers such that
$a_{n+1} \le (1 + \alpha_n)a_n + b_n$.
Suppose that $\sum_{n=1}^{\infty} \alpha_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$. Then a_n is bounded and $\lim_{n\to\infty} a_n$ exists. Moreover, if in addition, $\liminf_{n\to\infty} a_n = 0, then \lim_{n\to\infty} a_n = 0$.

3. MAIN RESULTS

Let E a real uniformly convex Banach space and $T_1, T_2, \cdots, T_m : E \to E$ be m Bregman quasi-total asymptotically nonexpansive mappings. We define the iterative sequence $\{x_n\}$ by

$$\begin{cases} x_1 \in E, \\ x_{n+1} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n x_n)\big), \text{ if } \ m = 1, \ n \ge 1, \\ x_1 \in E, \\ x_{n+1} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_1^n y_{1n})\big), \\ y_{1n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_2^n y_{2n})\big), \\ \quad \vdots \\ y_{(m-2)n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_{m-1}^n y_{(m-1)n}\big), \\ y_{(m-1)n} = \bigtriangledown f^*\big((1-\alpha_n) \bigtriangledown f(x_n) + \alpha_n \bigtriangledown f(T_m^n x_n)\big), \text{ if } \ m \ge 2, \ n \ge 1, \end{cases} \tag{9}$$

where $\{\alpha_n\}_{n=1}^{\infty}$ is a sequence in [0,1].

Theorem 3 : Let E be a real uniformly convex Banach space, and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : E \to E \quad i = 1, 2, \ldots, m$ be Bregman quasi-total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ $n \ge 1$ and mappings $\phi_i : [0, \infty) \to [0, \infty) \quad i = 1, 2, \ldots, m$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty, \quad \sum_{n=1}^{\infty} l_{in} < \infty, \quad i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined as in (9) and suppose there exists M_i, $M_i^* > 0$ such

that $\phi(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$, then the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty} D_f(x_n, p)$ exist, where $p \in F$.
Proof:For $m = 1$ we have from (9) that $x_1 \in E$, and

$$x_{n+1} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_1^n x_n)\big),$$

then

$$\begin{aligned} D_f(x_{n+1}, p) &= D_f\big(\triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_1^n x_n)\big), p\big) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n D_f(T_1^n x_n, p) \\ &\leq (1-\alpha_n) D_f(x_n, p) \\ &\quad + \alpha_n \big[D_f(x_n, p) + \mu_{1n} \phi_1\big(D_f(x_n, p)\big) + l_{1n}\big] \\ &= D_f(x_n, p) + \alpha_n \mu_{1n} \phi_1\big(D_f(x_n, p)\big) + \alpha_n l_{1n}. \end{aligned}$$

Since ϕ_1 is an increasing function, it follows that $\phi_1(\lambda_1) \leq \phi_1(M_1)$ whenever $\lambda_1 \leq M_1$ and by hypothesis $\phi_1(\lambda_1) \leq M_1^* \lambda_1$, if $\lambda_1 \geq M_1$. In either case, we have

$$\phi_1\big(D_f(x_n, p)\big) \leq \phi_1(M_1) + M_1^* D_f(x_n, p)$$

for some $M_1 > 0,\ M_1^* > 0$. Thus,

$$\begin{aligned} D_f(x_{n+1}, p) &\leq D_f(x_n, p) + \alpha_n \mu_{1n} \phi_1(M_1) \\ &\quad + \alpha_n \mu_{1n} M_1^* D_f(x_n, p) + \alpha_n l_{1n} \\ &= (1 + \mu_{1n} Q_1) D_f(x_n, p) + (\mu_{1n} + l_{1n}) Q_1, \end{aligned}$$

for some constant $Q_1 \geq 1$.
Next, for $m = 2$, we obtain from (9) that

$$\begin{cases} x_1 \in E, \\ x_{n+1} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_1^n y_{1n})\big), \\ y_{1n} = \triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_2^n x_n)\big), \end{cases}$$

from this, we have

$$\begin{aligned} D_f(x_{n+1}, p) &= D_f\big(\triangledown f^*\big((1-\alpha_n) \triangledown f(x_n) + \alpha_n \triangledown f(T_1^n y_{1n})\big), p\big) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n D_f(T_1^n y_{1n}, p) \\ &\leq (1-\alpha_n) D_f(x_n, p) + \alpha_n \big[D_f(y_{1n}, p) \\ &\quad + \mu_{1n} \phi_1\big(D_f(y_{1n}, p)\big) + l_{1n}\big] \end{aligned}$$

and

$$\begin{aligned}
D_f(y_{1n},p) &= D_f\big(\bigtriangledown f^*\big((1-\alpha_n)\bigtriangledown f(x_n)+\alpha_n \bigtriangledown f(T_2^n x_n)\big),p\big)\\
&\leq (1-\alpha_n)D_f(x_n,p)+\alpha_n D_f(T_2^n x_n,p)\\
&\leq (1-\alpha_n)D_f(x_n,p)\\
&\quad+\alpha_n\Big[D_f(x_n,p)+\mu_{2n}\phi_2\big(D_f(x_n,p)\big)+l_{2n}\Big] \qquad (10)
\end{aligned}$$

Again, since ϕ_i is an increasing function for $i=1,2$, it follows that $\phi_i(\lambda_i)\leq\phi_i(M_i)+M_i^*\lambda_i$ for some $M_i>0,\ M_i^*>0,\ i=1,2$.
Hence,

$$\begin{aligned}
D_f(x_{n+1},p) &\leq (1-\alpha_n)D_f(x_n,p)+\alpha_n D_f(y_{1n},p)+\alpha_n\mu_{1n}\phi_1(M_1)\\
&\quad+\alpha_n\mu_{1n}M_1^*D_f(y_{1n},p)+\alpha_n l_i n\\
&\leq (1-\alpha_n)D_f(x_n,p)+\alpha_n\big[D_f(x_n,p)+\alpha_n\mu_{2n}\phi_2(M_2)\\
&\quad+\alpha_n\mu_{2n}M_2^*D_f(x_n,p)+\alpha_n l_{2n}\big]+\alpha_n\mu_{1n}\phi_1(M_1)\\
&\quad+\alpha_n\mu_{1n}M_1^*\big[D_f(x_n,p)+\alpha_n\mu_{2n}\phi_2(M_2)\\
&\quad+\alpha_n\mu_{2n}M_2^*D_f(x_n,p)+\alpha_n l_2 n\big]+\alpha_n l_{1n}\\
&\leq D_f(x_n,p)+\alpha_n\mu_{2n}\phi_2(M_2)+\alpha_n\mu_{2n}M_2^*D_f(x_n,p)\\
&\quad+\alpha_n l_{2n}+\alpha_n\mu_{1n}\phi_1(M_1)+\alpha_n\mu_{1n}M_1^*D_f(x_n,p)\\
&\quad+\alpha_n\mu_{1n}\mu_{2n}\phi_2(M_2)M_1^*+\alpha_n\mu_{1n}\mu_{2n}M_1^*M_2^*D_f(x_n,p)\\
&\quad+\alpha_n\mu_{1n}l_{2n}M_1^*+\alpha_n l_{1n}\\
&\leq \big(1+(\mu_{1n}+\mu_{2n})Q_2\big)D_f(x_n,p)\\
&\quad+(\mu_{1n}+\mu_{2n}+l_{1n}+l_{2n})Q_2,
\end{aligned}$$

for some constant $Q_2>0$.
Following the computation above, we obtain for some $m\in\mathbb{N}$

$$D_f(x_{n+1},p)\leq\left(1+Q\sum_{j=1}^{m}\mu_{jn}\right)D_f(x_n,p)+Q\sum_{j=1}^{m}(\mu_{jn}+l_{jn})$$

for some $Q>0$.
Hence

$$D_f(x_{n+1},p)\leq(1+\delta_n)D_f(x_n,p)+\gamma_n,\quad n\geq1, \qquad (11)$$

where $\delta_n=Q\sum_{n=1}^{\infty}\mu_{jn}$ and $\gamma_n=Q\sum_{n=1}^{\infty}(\mu_{jn}+l_{jn})$. Observed that $\sum_{n=1}^{\infty}\delta_n<\infty$ and $\sum_{n=1}^{\infty}\gamma_n<\infty$. It follows from (11) and Lemma 11 that the sequence $\{x_n\}$ is bounded and $\lim_{n\to\infty}D_f(x_n,p)$ exists for each $p\in F$.

3.1 NECESSARY AND SUFFICIENT CONDITION FOR CONVERGENCE

Theorem 4: Let E be a real uniformly convex Banach space, and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : E \to E \quad i = 1, 2, \ldots, m$ be Bregman quasi-total asymptotically nonexpansive mappings with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ and mapping $\phi_i : [0, \infty) \to [0, \infty) \quad n \geq 1 \quad i = 1, 2, \ldots, m$, such that $\sum_{n=1}^{\infty} \mu_{in} < \infty, \quad \sum_{n=1}^{\infty} l_{in} < \infty, \quad i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be defined as in (9) and suppose there exists $M_i, \ M_i^* > 0$ such that $\phi(\lambda_i) \leq M_i^* \lambda_i$ for all $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_i, \ i = 1, 2, \ldots, m$ if and only if
$\liminf_{n\to\infty} \big(\inf_{y\in F} D_f(x_n, y)\big) = 0, \ n \geq 1.$

Proof: The necessity is trivial. We prove the sufficiency. Let $\liminf_{n\to\infty} \big(\inf_{y\in F} D_f(x_n, y)\big) = 0$, we show $\{x_n\}$ is Cauchy sequence in E. From (11) and Lemma 11, we have that
$\lim_{n\to\infty}(\inf_{y\in F} D_f(x_n, p))$ exists. Since $\liminf_{n\to\infty} \big(\inf_{y\in F} D_f(x_n, y)\big) = 0$, it follows that $\lim_{n\to\infty} \big(\inf_{y\in F} D_f(x_n, y)\big) = 0$. Thus given $\epsilon > 0$ there exists a positive integer N_0 and $p^* \in F$ such that $\forall n \geq N_0$, $D_f(x_n, p^*) < \epsilon$. This shows that $\lim_{n\to\infty} D_f(x_n, p^*) = 0$. Thus in view of Lemma 3, we have $\lim_{n\to\infty} \|x_n - p^*\| = 0$. Then there exists $N_1 \in \mathbb{N}$ such that $\forall n \geq N_1$, $\|x_n - p^*\| < \frac{\epsilon}{2}$.
For any $k \in \mathbb{N}$, we have $\forall n \geq N_1$

$$\begin{aligned} \|x_{n+k} - x_n\| &\leq \|x_{n+k} - p^*\| + \|p^* - x_n\| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon, \end{aligned}$$

and so $\{x_n\}$ is a Cauchy sequence. Let $x_n \to u$, we need to show that $u \in F$. Let $T_i \in \{T_1, T_2, \ldots, T_m\}$. Then for $N^* \in \mathbb{N}$ sufficiently large and $p^* \in F$ such that for all $n \geq N^*$ we have $\|x_n - u\| < \frac{\epsilon}{6(1+\omega_1)}$ and $\|x_n - p^*\| < \frac{\epsilon}{6(1+\omega_1)}$. Hence

$$\|u - p^*\| \leq \|u - x_n\| + \|x_n - p^*\| < \frac{\epsilon}{3(1 + \omega_1)}.$$

Thus we have the following estimates for $n \geq N^*$, arbitrary $T_i, i = 1, 2, \cdots, m$ and $\omega_1 = \max_{1\leq i\leq m}\{u_{i1}\}$:

$$
\begin{aligned}
\|u - T_i u\| &\leq \|u - x_n\| + \|x_n - p^*\| + \|p^* - T_i u\| \\
&\leq \|p - x_n\| + \|x_n - p^*\| + (1 + u_{i1})\|p^* - u\| \\
&\leq \|p - x_n\| + \|x_n - p^*\| + (1 + \omega_1)\|p^* - u\| \\
&\leq \frac{\epsilon}{6(1+\omega_1)} + \frac{\epsilon}{6(1+\omega_1)} + \frac{\epsilon}{3}.
\end{aligned}
$$

This shows that $u \in F(T_i)$ for each $i \in \{1, 2, \cdots, m\}$ and so $u \in F$. This complete the proof.

Lemma 13: Let E be a real uniformly convex Banach space, and $f : E \to \mathbb{R}$ a convex, continuous,strongly coercive and Gâteaux differentiable function which is bounded on bounded subsetsand uniformly convex on bounded subsets of E. Let $T_i : E \to E$, $i = 1, 2, \ldots, m$ be m Bregman quasi-total asymptotically nonexpansive mappings and uniformly L_i-Lipschitzian which is also uniformly asymptotically regular, with sequences $\{\mu_{in}\}, \{l_{in}\}, \subset [0, \infty)$ $n \geq 1$ and mappings $\phi_i : [0, \infty) \to [0, \infty)$ such that $\sum_{n=1}^{\infty} \mu_{in} < \infty$, $\sum_{n=1}^{\infty} l_{in} < \infty$, $i = 1, 2, \ldots, m$, and $F := \cap_{i=1}^{m} F(T_i) \neq \emptyset$. Let $\{\alpha_n\} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$,. From arbitrary $x_1 \in E$ define the sequence $\{x_n\}$ by (9). Suppose that there exist M_i, $M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i$ $\quad i = 1, 2, \ldots, m$ then $\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0$, $i = 1, 2, \ldots, m$.

Proof: Let $p \in F$ then by Theorem 3, $\lim_{n\to\infty} D_f(x_n, p)$ exists. Let $\lim_{n\to\infty} D_f(x_n, p) = r$. If $r = 0$ then by continuity of T_i, $i = 1, 2, \ldots, m$ we are done. Now suppose $r > 0$, we show $\lim_{n\to\infty} \|T_i^n x_n - x_n\| = 0$, $i = 1, 2, \ldots, m$.

In Theorem 4, we have shown that $\{x_n\}$ is Cauchy. In particular for $m = 1$, we get from (9) that

$$
x_1 \in K \quad x_{n+1} = \nabla f^*((1 - \alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)).
$$

Using this and Lemma 8, we have for some constant $Q_1 > 0$ that

$$\begin{aligned}
D_f(p, x_{n+1}) &= D_f(p, \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n))) \\
&= V_f(p, (1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)) \\
&= f(p) - \langle p, (1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)\rangle \\
&\quad + f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n x_n)) \\
&\leq (1-\alpha_n) f(p) + \alpha_n f(p) - (1-\alpha_n)\langle p, \nabla f(x_n)\rangle \\
&\quad - \alpha_n \langle p, \nabla f(T_1^n x_n) \\
&\quad + (1-\alpha_n) f^*(\nabla f(T_1^n x_n)) + \alpha_n f^*(\nabla f(T_1^n x_n)) \\
&\quad - \alpha_n (1-\alpha_n) \rho_s^*(||\nabla f(y_n) - \nabla f(T_1^n x_n)||) \\
&= (1-\alpha_n) V_f(p, \nabla f(x_n)) + \alpha_n V(p, \nabla f(T_1^n x_n)) \\
&\quad - \alpha_n (1-\alpha_n) \rho_s^*(||\nabla f(y_n) - \nabla f(T_1^n x_n)||) \\
&= (1-\alpha_n) D_f(p, x_n) + \alpha_n D_f(p, T_1^n x_n) \\
&\quad - \alpha_n (1-\alpha_n) \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\
&\leq (1-\alpha_n) D_f(p, x_n) + \alpha_n [D_f(p, x_n) \\
&\quad + \mu_{1n} \phi_1(D_f(p, x_n)) + l_{1n}] \\
&\quad - \alpha_n (1-\alpha_n) \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\
&\leq D_f(p, x_n) + (\mu_{1n} + l_{1n}) Q_1 \\
&\quad - \alpha_n (1-\alpha_n) \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||).
\end{aligned}$$

Thus,

$$\begin{aligned}
\epsilon^2 \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) &\leq D_f(p, x_n) - D_f(p, x_{n+1}) \\
&\quad + (\mu_{1n} + l_{1n}) Q_1
\end{aligned}$$

which implies,

$$\epsilon^2 \sum_{n=1}^{\infty} \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \leq D_f(p, x_1) + Q_1 \sum_{n=1}^{\infty} (\mu_{1n} + l_{1n}).$$

Hence $\epsilon^2 \sum_{n=1}^{\infty} \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) < \infty$. So, $\lim_{n\to\infty} \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) = 0$; and properties of ρ_s^* imply $\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(T_1^n x_n)|| = 0$. Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we arrive at

$$\lim_{n\to\infty} ||x_n - T_1^n x_n|| = 0. \tag{12}$$

For $m = 2$, (9) becomes

$$\begin{cases} x_1 \in E \\ x_{n+1} = \nabla f^*((1-\alpha_n)\nabla f(x_n) + \alpha_n \nabla f(T_1^n y_{1n})) \\ y_{1n} = \nabla f^*(\alpha_n \nabla f(x_n) + (1-\alpha_n)\nabla f(T_2^n x_n)) \end{cases} \tag{13}$$

using (10), (22) and Lemma 8, we obtain, for some $Q_3 > 0$ that

$$\begin{aligned} D_f(p, x_{n+1}) &\leq (1-\alpha_n) D_f(p, x_n) + \alpha_n D_f(p, T_1^n y_{1n}) \\ &\quad -\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||) \\ &\leq (1-\alpha_n) D_f(p, x_n) \\ &\quad + \alpha_n[D_f(p, y_{1n}) + \mu_{1n}\phi_1(D_f(p, y_{1n})) + l_{1n}] \\ &\quad -\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||) \\ &\leq D_f(p, x_n) + (\mu_{2n} + l_{2n} + \mu_{1n} + l_{1n})Q_3 \\ &\quad -\alpha_n(1-\alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||). \end{aligned}$$

Thus,

$$\begin{aligned} \epsilon^2 \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||) &\leq D_f(p, x_n) - D_f(p, x_{n+1}) \\ &\quad + (\mu_{2n} + l_{2n} + \mu_{1n} + l_{1n})Q_3 \end{aligned}$$

which implies ,

$$\begin{aligned} &\epsilon^2 \sum_{n=1}^{\infty} \rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n y_{1n})||) \\ &\leq D_f(p, x_1) + Q_3 \sum_{n=1}^{\infty}(\mu_{2n} + l_{2n} + \mu_{1n} + l_{1n}) < \infty. \end{aligned}$$

So, $\lim_{n\to\infty} \rho_s^* ||\nabla f(x_n) - \nabla f(T_1^n y_{1n})|| = 0$, and properties of ρ_s yield $\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(T_1^n y_{1n})|| = 0$. Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} ||x_n - T_1^n y_{1n}|| = 0. \tag{14}$$

Thus from Lemma 3, we obtain

$$\lim_{n\to\infty} D_f(x_n, T_1^n y_{1n}) = 0. \tag{15}$$

Also, from (7) and (15), we have

$$\begin{aligned} D_f(p, x_n) &\leq D_f(p, T_1^n y_{1n}) + D_f(T_1^n y_{1n}, x_n) \\ &\quad + \langle \nabla f(T_1^n y_{1n}) - \nabla f(x_n), p - T_1^n y_{1n} \rangle \\ &\leq D_f(p, T_1^n y_{1n}) + D_f(T_1^n y_{1n}, x_n) \\ &\quad + ||\nabla f(T_1^n y_{1n}) - \nabla f(x_n)|| ||p - T_1^n y_{1n}|| \\ &\leq D_f(p, y_{1n}) + \mu_{1n} M + l_{1n} + D_f(T_1^n y_{1n}, x_n) \\ &\quad + ||\nabla f(T_1^n y_{1n}) - \nabla f(x_n)|| ||p - T_1^n y_{1n}|| \end{aligned}$$

for some constant $M > 0$. Hence, we deduce from this that

$$r \leq \liminf_{n\to\infty} D_f(p, y_{1n}).$$

Also, since

$$D_f(p, y_{1n}) \leq (1 + \mu_{2n}Q_2)D_f(p, x_n) + (\mu_{2n} + l_{2n})Q_2,$$

this gives

$$\limsup_{n\to\infty} D_f(p, y_{1n}) \leq r.$$

Thus

$$\lim_{n\to\infty} D_f(p, y_{1n}) = r.$$

Also from (22) and Lemma 8, we obtain, for some $Q_2 > 0$ that

$$\begin{aligned} D_f(p, y_{1n}) &\leq (1 - \alpha_n)D_f(p, x_n) + \alpha_n D_f(p, T_2^n x_n) \\ &\quad -\alpha_n(1 - \alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\ &\leq (1 + \alpha_n\mu_{2n}Q_2)D_f(p, x_n) + (\mu_{2n} + l_{2n})Q_2 \\ &\quad -\alpha_n(1 - \alpha_n)\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||). \end{aligned}$$

Therefore

$$\begin{aligned} &\epsilon^2\rho_s^*(||\nabla f(x_n) - \nabla f(T_1^n x_n)||) \\ &\leq (1 + \alpha_n\mu_{2n}Q_2)D_f(p, x_n) - D_f(p, y_{1n}) + (\mu_{2n} + l_{2n})Q_2 \end{aligned}$$

So, $\lim_{n\to\infty} \rho_s^*||\nabla f(x_n) - \nabla f(T_2^n x_n)|| = 0$, and properties of ρ_s yield $\lim_{n\to\infty} ||\nabla f(x_n) - \nabla f(T_2^n x_n)|| = 0$. Since ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^*, we obtain

$$\lim_{n\to\infty} ||x_n - T_2^n x_n|| = 0. \tag{16}$$

Hence, from Lemma 3, we obtain

$$\lim_{n\to\infty} D_f(x_n, T_2^n x_n) = 0. \tag{17}$$

So, from (22) and (17), we obtain

$$\begin{aligned} D_f(x_n, y_{1n}) &\leq (1 - \alpha_n)D_f(x_n, x_n) \\ &+ \alpha_n D_f(x_n, T_2^n x_n) \to 0 \text{ as } n \to \infty. \end{aligned} \tag{18}$$

which implies from Lemma 3, that

$$\lim_{n\to\infty} ||x_n - y_{1n}|| = 0. \tag{19}$$

Also,

$$\begin{aligned} ||T_1^n x_n - x_n|| &\leq ||T_1^n x_n - T_1^n y_{1n}|| + ||T_1^n y_{1n} - x_n|| \\ &\leq L_1||x_n - y_{1n}|| + ||T_1^n y_{1n} - x_n||. \end{aligned}$$

Hence, from (14) and (19), we obtain

$$\lim_{n\to\infty} ||T_1^n x_n - x_n|| = 0.$$

Hence

$$\lim_{n\to\infty} ||T_1^n x_n - x_n|| = \lim_{n\to\infty} ||T_2^n x_n - x_n|| = 0.$$

Continuing, we get

$$\lim_{n\to\infty} ||T_i^n x_n - x_n|| = 0, \quad i = 1, 2, \cdots, m. \tag{20}$$

Furthermore

$$\begin{aligned} ||T_i x_n - x_n|| &\leq ||T_i x_n - T_i(T_i^n x_n)|| \\ &\quad + ||T_i(T_i^n x_n) - T_i^n x_n|| + ||T_i^n x_n - x_n|| \\ &\leq (L_i + 1)||x_n - T_i^n x_n|| + ||T_i^{n+1} x_n - T_i^n x_n||. \end{aligned}$$

From (20) and asymptotic regularity of T_i, for each $i \in \{1, 2. \cdots, m\}$, we obtain

$$\lim_{n\to\infty} ||T_i x_n - x_n|| = 0. \tag{21}$$

This complete the proof.

Theorem 5: Let E be a real uniformly convex Banach space, and $f : E \to \mathbb{R}$ a convex, continuous, strongly coercive and Gâteaux differentiable function which is bounded on bounded subsets and uniformly convex on bounded subsets of E. Let $T_i : E \to E,\ i = 1, 2, \ldots, m$ be m Bregman quasi-total asymptotically nonexpansive mappings and uniformly L_i-Lipschitzian which is also uniformly asymptotically regular with sequences $\{\mu_{in}\}, \{l_{in}\} \subset [0, \infty)$ and mappings $\phi_i : [0, \infty) \to [0, \infty), i = 1, 2, \ldots, m$ and $F := \cap_{i=1}^m F(T_i) \neq \emptyset$. Let $\{\alpha_{in}\} \subset [\epsilon, 1 - \epsilon]$ for some $\epsilon \in (0, 1)$,. From arbitrary $x_1 \in E$ define the sequence $\{x_n\}$ by (9). Suppose that there exist $M_i,\ M_i^* > 0$ such that $\phi_i(\lambda_i) \leq M_i^* \lambda_i$ whenever $\lambda_i \geq M_i \quad i = 1, 2, \ldots, m$ and that at least one of $T_1, T_2, \ldots, T_m$ is semi-compact, then $\{x_n\}$ converges strongly to some $x^* \in F$.

Proof:Without lost of generality, let T_1 be semi-compact. Since T_1 is semi-compact, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that for some $x^* \in E,\ T_1 x_{n_k} \to x^*$ as $k \to \infty$. This implies that $T_1^{n_k} x_{n_k} \to x^*$ as $k \to \infty$. Thus $T_1^{n_k+1} x_{n_k} \to T_1 x^*$ as $k \to \infty$ and from (20) we have $\lim_{k\to\infty} x_{n_k} = x^*$. Also from (20) $T_2^{n_k} x_{n_k} \to x^*,\ T_3^{n_k} x_{n_k} \to x^*, \ldots, T_m^{n_k} x_{n_k} \to x^*$ as $k \to \infty$. Thus $T_2^{n_k+1} x_{n_k} \to T_2 x^*,\ T_3^{n_k+1} x_{n_k} \to T_3 x^*, \ldots, T_m^{n_k+1} x_{n_k} \to T_m x^*$ as $k \to \infty$. Now $\|x_{n_{k+1}} - x_{n_k}\| \leq \|T_1^{n_k} y_{1n_k} - x_{n_k}\|$ from (14), it follows that $x_{n_{k+1}} \to x^*$ as $k \to \infty$. Next, we show that $x^* \in F$. Observed that

$$\begin{aligned} \|x^* - T_1 x^*\| &\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_1^{n_{k+1}} x_{n_{k+1}}\| \\ &\quad + \|T_1^{n_{k+1}} x_{n_{k+1}} - T_1^{n_{k+1}} x_{n_k}\| + \|T_1^{n_{k+1}} x_{n_k} - T_1 x^*\|. \end{aligned}$$

Taking the limit as $k \to \infty$ and using the fact that T_1 is uniformly asymptotically regular we have that $x^* = T_1 x^*$ and so $x^* \in F(T_1)$. Also,

$$\begin{aligned} \|x^* - T_2 x^*\| &\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_2^{n_{k+1}} x_{n_{k+1}}\| \\ &\quad + \|T_2^{n_{k+1}} x_{n_{k+1}} - T_2^{n_{k+1}} x_{n_k}\| + \|T_2^{n_{k+1}} x_{n_k} - T_2 x^*\|. \end{aligned}$$

Taking the limit as $k \to \infty$ and using the fact that T_2 is uniformly asymptotically regular we obtain that $x^* = T_2 x^*$ and so $x^* \in F(T_2)$. Again,

$$\begin{aligned} \|x^* - T_3 x^*\| &\leq \|x^* - x_{n_{k+1}}\| + \|x_{n_{k+1}} - T_3^{n_{k+1}} x_{n_{k+1}}\| \\ &\quad + \|T_3^{n_{k+1}} x_{n_{k+1}} - T_3^{n_{k+1}} x_{n_k}\| + \|T_3^{n_{k+1}} x_{n_k} - T_3 x^*\|. \end{aligned}$$

As $k \to \infty$, we have that $x^* \in F(T_3)$. Eventually, we have that, $x^* \in F$. But by Theorem 3 $\lim_{n\to\infty} D_f(x_n, x^*)$ exists, $x^* \in F$. Hence $\{x_n\}$ converges strongly to $x^* \in F$. This complete the proof.

Below we give example of Bregman total quasi-asymptotically nonexpansive mappings on the real line.

Example 1: Let $E = \mathbb{R}$, $f(x) = x$, and for $i = 1, 2, 3, \ldots, m$, define $T_i : E \to E$ by $T_i x = \frac{1}{3^i} x$. Then f is proper, lower semicontinuous and convex and $0 \in F(T_i)$ for each $i = 1, 2, 3, \ldots, m$. Thus,

$$\begin{aligned} D_f(T_i^n x, 0) &= f(0) - f(T_i^n x) - \langle \nabla f(T_i^n x), 0 - T_i^n x \rangle \\ &= 0 - T_i^n x - \langle \nabla f(T_i^n x), -T_i^n x \rangle \\ &= -\frac{1}{3^{in}} x - \langle \frac{1}{3^{in}}, -\frac{1}{3^{in}} x \rangle \\ &= -\frac{1}{3^{in}} x + \frac{1}{3^{2in}} x = \frac{1}{3^{in}} (-x + \frac{1}{3^{in}} x) \\ &\leq \frac{1}{3^{in}} (-x + x) = \frac{1}{3^{in}} (0 - x - \langle 1, 0 - x \rangle) \\ &= \frac{1}{3^{in}} (f(0) - f(x) - \langle \nabla f(x), 0 - x \rangle) = D_f(x, 0) \\ &\leq (1 + \frac{1}{3^{in}}) D_f(x, 0) + v_{in} \\ &= D_f(x, 0) + \frac{1}{3^{in}} D_f(x, 0) + v_{in} \end{aligned}$$

where $v_{in} = 0$ $\forall n \geq 1$ and $i = 1, 2, 3, \ldots, m$. If $\phi(t) = t$ for $t > 0$ and $u_{in} = \frac{1}{3^{in}}$, then

$$D_f(T_i^n x, 0) \leq D_f(x, 0) + u_{in} \phi(D_f(x, 0)) + v_{in},$$

showing that T_i Bregman total quasi-asymptotically nonexpansive mapping for each $i = 1, 2, 3 \ldots, m$.

4 NUMERICAL EXAMPLE

In this section, we demonstrate the convergence of the iterative scheme (9) on the real line. Let $m = 3$ then $i = 1, 2, 3$. Let $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \frac{2}{3}x^2$, then $\nabla f(x) = \frac{4}{3}x$, since $f^*(x^*) = \sup\{\langle x^*, x\rangle - f(x) : x \in \mathbb{R}\}$, then $f^*(z) = \frac{3}{8}z^2$ and $\nabla f^*(z) = \frac{3}{4}z$. For $T_i : \mathbb{R} \to \mathbb{R}$ defined by $T_i x = \frac{1}{3^i}x$, for $i = 1, 2, 3$. From the scheme we obtain

$$\begin{cases} x_1 \in \mathbb{R} \\ x_{n+1} = (1-\alpha_n)x_n + \frac{1}{3^n}\alpha_n y_{1n} \\ y_{1n} = (1-\alpha_n)x_n + \frac{1}{3^{2n}}\alpha_n y_{2n} \\ y_{2n} = (1-\alpha_n)x_n + \frac{1}{3^{3n}}\alpha_n x_n \end{cases} \tag{22}$$

Hence

$$x_{n+1} = (1-\alpha_n)x_n + \frac{1}{3^n}\alpha_n(1-\alpha_n)x_n + \frac{1}{3^{3n}}\alpha_n^2(1-\alpha_n)x_n + \frac{1}{3^{6n}}\alpha_n^3 x_n.$$

Take the initial point $x_1 = 0.5$ and $\alpha_n = \frac{n+1}{4n}$, the numerical experiment result using MATLAB is given in Figure 1, which shows the iteration process of the sequence $\{x_n\}$ converges to 0.

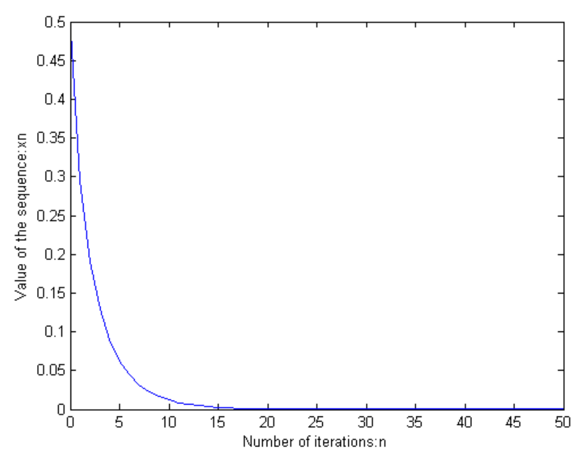

FIGURE 1. $x_1 = 0.5$, the convergence process of the sequence $\{x_n\}$.

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