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λ-SEQUENCES AND FIXED POINT THEOREMS G-METRIC TYPE SPACES

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ABSTRACT. In this article, we prove a common fixed point theorem for a family of self mappings in G-metric type spaces using the idea of λ-sequences.

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1. INTRODUCTION AND PRELIMINARIES

In this section, we give the basic concepts and corresponding notations attached to the ideas of G-metric type spaces and λ-sequences. The first well known generalization of metric spaces is due to Dhage, who introduced the so-called D-metric space in [1]. Later on and more recently, the works done by Mustafa and Sims [8, 9] established a more general type of metric space, namely the G-metric spaces. In this article, we introduce a slightly modified version of G-metric that we call G-metric type in analogy to what appears in [3, 4, 6] with respect to metric spaces. We give and prove fixed point theorems in this setting. The results presented here generalize similar ones already proved in [2, 5, 7, 8, 10].

Definition 1: (Compare [4]) Let X be a nonempty set, and let the function $D : X \times X \to [0, \infty)$ satisfy the following properties:

(D1) $D(x, x) = 0$ for any $x \in X$;
(D2) $D(x, y) = D(y, x)$ for any $x, y \in X$;
(D3) $D(x, y) \leq \alpha[D(x, z_1) + D(z_1, z_2) + \cdots + D(z_n, y)]$ for any points $x, y, z_i \in X,\ i = 1, 2, \ldots, n$ where $n \geq 1$ is a fixed natural number and α some constant such that $\alpha \geq 1$.

The triplet (X, D, α) is called a **metric type space**.

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Definition 2: (Compare [9]) Let X be a nonempty set, and let the function $D : X \times X \times X \to [0, \infty)$ satisfy the following properties:

(G1) $D(x, y, z) = 0$ if $x = y = z$ whenever $x, y, z \in X$;
(G2) $D(x, x, y) > 0$ whenever $x, y \in X$ with $x \neq y$;
(G3) $D(x, x, y) \leq D(x, y, z)$ whenever $x, y, z \in X$ with $z \neq y$;
(G4) $D(x, y, z) = D(x, z, y) = D(y, z, x) = \ldots$, (symmetry in all three variables);
(G5) $D(x, y, z) \leq K[D(x, a, a) + D(a, y, z)[$ whenever $x, y, z, a \in X$ and K some constant such that $K \geq 1$, (rectangle inequality)[1].

The triplet (X, D, K) is called a G**-metric type space**.

It is therefore trivial to observe that, as metric type spaces generalize metric spaces, G-metric type spaces generalize G-metric spaces. Moreover, for $K = 1$, we recover the classical G-metric. Furthermore, if (X, D, K) is a G-metric type space, then for any $L \geq K$, (X, D, L) is also a G-metric type space. The concepts of Cauchy sequence and convergence for a sequence in a G-metric type space are defined in the same way as defined for a G-metric spaces and this can be read in [9]. Nevertheless, for the convenient of the ready, we shall recall the below definitions.

Definition 3: (Compare [9]) Let (X, D, K) be a G-metric type space. Then a sequence $(x_n) \subset X$

(i) is D-convergent to $x \in X$ if $\lim_{n,m\to\infty} D(x, x_n, x_m) = 0$, that is, for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $D(x, x_n, x_m) < \varepsilon$ for all $m, n \geq N$;
(ii) is G-Cauchy if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $D(x_n, x_m, x_m) < \varepsilon$ for all $m, n \geq N$;

Definition 4: (Compare [9])
A G-metric type space (X, D, K) is G-complete if every D-Cauchy sequence in (X, D, K) is D-convergent.

We complete this section by recalling the definition of a λ-sequence.

Definition 5: (Compare [2]) A sequence $(x_n)_{n\geq 1}$ in a metric type space (X, d, K) is a λ-sequence if there exist $0 < \lambda < 1$ and

[1]Following property D3 of Definition 1, this can be generalized as

$$D(x, y, z) \leq K^n[D(x, z_1, z_1) + D(z_1, z_2, z_2) + \cdots + D(z_n, y, z)]$$

for any points $x, y, z, z_i \in X,\ i = 1, 2, \ldots, n$ where $n \geq 1$.

$n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L-1} d(x_i, x_{i+1}) \leq \lambda L \text{ for each } L \geq n(\lambda) + 1.$$

Remark 1: We know (see [9]) that given a metric type space (X, d, K), we can build a G-metric type (X, D, K) space by setting

$$D(x, y, z) = \max\{d(x, y), d(y, z), d(x, z)\},$$

whenever $x, y, z \in X$.

In this paper, we shall call the G-metric type space (X, D, K) (given above), the **d-induced G-metric type space**. It is an immediate consequence of the above definition that the d-induced G-metric type space (X, D, K) is G-complete if and only if (X, d, K) is complete[2]. Hence we extend λ-sequence to G-metric type space in the following way:

Definition 6: (Compare [2]) A sequence $(x_n)_{n\geq 1}$ in a G-metric type space (X, D, K) is a λ-sequence if there exist $0 < \lambda < 1$ and $n(\lambda) \in \mathbb{N}$ such that

$$\sum_{i=1}^{L-1} D(x_i, x_{i+1}, x_{i+1}) \leq \lambda L \text{ for each } L \geq n(\lambda) + 1.$$

Remark 2: Let (X, d, K) be a metric type space, then any λ-sequence in (X, d, K) is also a λ-sequence in the d-induced G-metric type space (X, d, K).

We now state and prove the main results of this article.

2. MAIN RESULTS

Let Φ be the class of continuous, non-decreasing, sub-additive and homogeneous functions $F : [0, \infty) \to [0, \infty)$ such that $F^{-1}(0) = \{0\}$. The following result is analogous to the main result of Vetro [10] and the proof is similar.

Theorem 1: (Copmare [10, Theorem 2.1.]) Let (X, d, K) be a complete metric type, (X, D, K) the d-induced G-complete G-metric type space and $\{T_n\}$ be a sequence of self mappings on X.

[2]The reverse construction is given in [5]. In that reference the authors build a metric space out of a G-metric space and prove that the completeness of the two spaces are equivalent.

Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$F(D(T_i(x),T_j(y),T_j(z))) \leq F(\delta_{i,j}[D(x,T_i(x),T_i(x))+D(y,T_j(y),T_j(z))]) \quad (1)$$
$$+ F(\gamma_{i,j}D(x,y,z))$$

for $x,y,z \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i,j = 1,2,\cdots$, and some $F \in \Phi$, homogeneous with degree s, where $\delta_{i,j} = D(a_i,a_j,a_j)$ and $\gamma_{i,j} = D(b_i,b_j,b_j)$. If the sequence (s_n) where $s_i = \frac{\delta^s_{i,i+1}+\gamma^s_{i,i+1}}{1-\delta^s_{i,i+1}}$ is a non-increasing λ-sequence of $\mathbb{R}^+$ endowed with the max[3] metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof: We split the proof into two steps. For any $x_0 \in X$, we construct the sequence (x_n) by setting $x_n = T_n(x_{n-1})$, $n = 1,2,\cdots$. Using (1) and the homogeneity of F, we obtain

$$\begin{aligned}
F(D(x_1,x_2,x_2)) &= F(D(T_1(x_0),T_2(x_1),T_2(x_1))) \\
&\leq \delta^s_{1,2}F([D(x_0,T_1(x_0),T_1(x_0)))+D(x_1,T_2(x_1),T_2(x_1))]) \\
&\quad + \gamma^s_{1,2}F(D(x_0,x_1,x_1)) \\
&= \delta^s_{1,2}F([D(x_0,x_1,x_1)+D(x_1,x_2,x_2)])+\gamma^s_{1,2}F(D(x_0,x_1,x_1)).
\end{aligned}$$

By the sub-additivity of F, we have

$$(1-\delta^s_{1,2})F(D(x_1,x_2,x_2)) \leq (\delta^s_{1,2}+\gamma^s_{1,2})F(D(x_0,x_1,x_1)),$$

i.e.

$$F(D(x_1,x_2,x_2)) \leq \left(\frac{\delta^s_{1,2}+\gamma^s_{1,2}}{1-\delta^s_{1,2}}\right) F(D(x_0,x_1,x_1)).$$

Also, we get

$$\begin{aligned}
F(D(x_2,x_3,x_3)) &= F(D(T_2(x_1),T_3(x_2),T_3(x_2))) \\
&\leq \left(\frac{\delta^s_{2,3}+\gamma^s_{2,3}}{1-\delta^s_{2,3}}\right) F(D(x_1,x_2,x_2)) \\
&\leq \left(\frac{\delta^s_{2,3}+\gamma^s_{2,3}}{1-\delta^s_{2,3}}\right)\left(\frac{\delta^s_{1,2}+\gamma^s_{1,2}}{1-\delta^s_{1,2}}\right) F(D(x_0,x_1,x_1)).
\end{aligned}$$

By repeating the above process, we have

$$F(D(x_n,x_{n+1},x_{n+1})) \leq \prod_{i=1}^{n}\left(\frac{\delta^s_{i,i+1}+\gamma^s_{i,i+1}}{1-\delta^s_{i,i+1}}\right) F(D(x_0,x_1,x_1)). \quad (2)$$

[3]The max metric m refers to $m(x,y) = \max\{x,y\}$

Hence we derive, by making use of the the rectangle inequality and the properties of F, that for $p > 0$

$$\begin{aligned} F(D(x_n, x_{n+p}, x_{n+p})) &\le K^{s(p-1)}[F(D(x_n, x_{n+1}, , x_{n+1})) + F(D(x_{n+1}, x_{n+2}, x_{n+2})) \\ &\quad + \ldots + F(D(x_{n+p-1}, x_{n+p}, x_{n+p}))] \\ &\le K^{s(p-1)}\left[\prod_{i=1}^{n}\left(\frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}\right) F(D(x_0, x_1, x_1)) \right. \\ &\quad + \prod_{i=1}^{n+1}\left(\frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}\right) F(D(x_0, x_1, x_1)) \\ &\quad + \ldots + \\ &\quad \left. + \prod_{i=1}^{n+p-1}\left(\frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}\right) F(D(x_0, x_1, x_1))\right] \\ &= K^{s(p-1)}\left[\sum_{k=0}^{p-1}\prod_{i=1}^{n+k}\left(\frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}\right) F(D(x_0, x_1, x_1))\right] \\ &= K^{s(p-1)}\left[\sum_{k=n}^{n+p-1}\prod_{i=1}^{k}\left(\frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}\right) F(D(x_0, x_1, x_1))\right]. \end{aligned}$$

Now, let λ and $n(\lambda)$ as in Definition 6, then for $n \ge n(\lambda)$ and using the fact the geometric mean of non-negative real numbers is at most their arithmetic mean, it follows that

$$\begin{aligned} F(D(x_n, x_{n+p}, x_{n+p})) &\le K^{s(p-1)}\left[\sum_{k=n}^{n+p-1}\left[\frac{1}{k}\sum_{i=1}^{k}\left(\frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}\right)\right]^k F(D(x_0, x_1, x_1))\right] \qquad (3) \\ &\le K^{s(p-1)}\left[\left(\sum_{k=n}^{n+p-1}\lambda^k\right) F(D(x_0, x_1, x_1))\right] \\ &\le K^{s(p-1)}\frac{\lambda^n}{1-\lambda} F(D(x_0, x_1, x_1)). \end{aligned}$$

Letting $n \to \infty$ and since $F^{-1}(0) = \{0\}$ and F is continuous, we deduce that $D(x_n, x_{n+p}, x_{n+p}) \to 0$. Thus (x_n) is a Cauchy sequence and, by G-completeness of X, converges to say $x^* \in X$.

Moreover, for any natural number $m \neq 0$, we have

$$\begin{aligned} F(D(x_n, T_m(x^*), T_m(x^*))) &= F(D(T_n(x_{n-1}), T_m(x^*), T_m(x^*))) \\ &\le \delta^s_{n,m}[F(D(x_{n-1}, x_n, x_n)) \\ &\quad + F(D(x^*, T_m(x^*), T_m(x^*)))] \\ &\quad + \gamma^s_{n,m} F(D(x_{n-1}, x^*, x^*)). \end{aligned}$$

Again, letting $n \to \infty$, we get

$$\begin{aligned} F(D(x^*, T_m(x^*), T_m(x^*))) \leq\ & \delta^s_{n,m}[F(D(x^*, x^*, x^*)) \\ & + F(D(x^*, T_m(x^*), T_m(x^*)))] \\ & + \gamma^s_{n,m} F(D(x^*, x^*, x^*)) \\ \leq\ & \delta^s_{n,m} F(D(x^*, T_m(x^*), T_m(x^*))), \end{aligned}$$

and since $0 \leq \delta_{n,m} < 1$, it follows that $F(D(x^*, T_m(x^*), T_m(x^*))) = 0$, i.e. $T_m(x^*) = x^*$. Then x^* is a common fixed point of $\{T_m\}_{m \geq 1}$.

To prove the uniqueness of x^*, let us suppose that y^* is a common fixed point of $\{T_m\}_{m \geq 1}$, that is $T_m(y^*) = y^*$ for any $m \geq 1$. Then, by (1), we have

$$\begin{aligned} F(D(x^*, y^*, y^*)) \leq\ & F(D(T_m(x^*), T_m(y^*), T_m(y^*))) \\ \leq\ & \delta^s_{n,m}[F(D(x^*, T_m(x^*), T_m(x^*)) \\ & + F(D(y^*, T_m(y^*), T_m(y^*))] \\ & + \gamma^s_{n,m} F(D(x^*, y^*, y^*)) \\ =\ & \delta^s_{n,m}[F(D(x^*, x^*, x^*) + F(D(y^*, y^*, y^*)] \\ & + \gamma^s_{n,m} F(D(x^*, y^*)) \\ & = \gamma^s_{n,m} F(D(x^*, y^*, y^*)). \end{aligned}$$

And again, since $0 \leq \gamma_{n,m} < 1$, $x^* = y^*$. So x^* is the unique common fixed point of $\{T_m\}$. As particular cases of Theorem 1, we

have the following two corollaries.

Corollary 1: Let (X, d, K) be a complete metric type, (X, D, K) the d-induced D-complete G-metric type space and $\{T_n\}$ be a sequence of self mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$\begin{aligned} D(T_i(x), T_j(y), T_j(z)) \leq \delta_{i,j}[D(x, T_i(x), T_i(x)) + D(y, T_j(y), T_j(z))] \\ + \gamma_{i,j} D(x, y, z) \end{aligned} \tag{4}$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \cdots$, where $\delta_{i,j} = D(a_i, a_j, a_j)$ and $\gamma_{i,j} = D(b_i, b_j, b_j)$.

If the sequence (s_n) where $s_i = \frac{\delta_{i,i+1} + \gamma_{i,i+1}}{1 - \delta_{i,i+1}}$ is a λ-sequence of $\mathbb{R}^+$ endowed with the max metric max, then $\{T_n\}$ has a unique common fixed point in X.

Proof: Apply Theorem 1 by putting $F = I_{[0,\infty)}$, the identity map.

Corollary 2: (Compare [10, Corollary 2.1.]) Let (X,d,K) be a complete metric type, (X,D,K) the d-induced D-complete G-metric type space and $\{T_n\}$ be a sequence of self mappings on X. Assume that there exists a sequences (a_n) of elements of X such

$$F(D(T_i(x),T_j(y),T_j(z))) \leq F(\delta_{i,j}[D(x,T_i(x),T_i(x)) + D(y,T_j(y),T_j(z))]) \qquad (5)$$

for $x,y \in X$ with $x \neq y$, $0 \leq \delta_{i,j} < 1,\ i,j = 1,2,\cdots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i,a_j,a_j)$. If the sequence (s_n) where $s_i = \frac{\delta^s_{i,i+1}}{1-\delta^s_{i,i+1}}$ is a non-decreasing λ-sequence of $\mathbb{R}^+$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

Proof. Apply Theorem 1 by putting[4] $\gamma_{i,i+1} = 0$.

Example 1: (Compare [2]) Let $X = [0,1]$ and $d(x,y) = \max\{x,y\}$ whenever $x,y \in [0,1]$. Clearly, (X,d) is a complete metric space. The d-induced D-complete G-metric is therefore $D(x,y,z) = \max\{x,y,z\}$ whenever $x,y,z \in [0,1]$ Following the notation in the definition, we set $a_i = \left(\frac{1}{1+2^i}\right)^2$ so that $\delta_{i,j} = \left(\frac{1}{1+2^\eta}\right)^2$ where $\eta = \min\{i,j\}$. We also define $T_i(x) = \frac{x}{16^i}$ for all $x \in X$ and $i = 1,2,\cdots$ and $F : [0,\infty) \to [0,\infty),\ x \mapsto \sqrt{x}$. Then F is continuous, non-decreasing, sub-additive and homogeneous of degree $s = \frac{1}{2}$ and $F^{-1}(0) = \{0\}$. Assume $i < j$ and $x > y \geq z$. Hence we have

$$F(D(T_i(x),T_j(y),T_j(z))) = \sqrt{\frac{x}{16^i}}$$

and

$$F(\delta_{i,j}[D(x,T_i(x),T_i(x))+D(y,T_j(y),T_j(z)]) = \sqrt{\left(\frac{1}{1+2^i}\right)^2 (x+y)}.$$

Therefore condition (5) is satisfied for all $x,y \in X$ with $x \neq y$. Moreover, since F is homogeneous of degree $s = \frac{1}{2}$, the sequence

$$s_i = \frac{\delta^s_{i,i+1}}{1-\delta^s_{i,i+1}} = \frac{1}{2^i}$$

satisfies the conditions of Theorem . Then by Corollary 2, $\{T_n\}$ has a common fixed point, which is this case $x^* = 0$.

[4] In this case, we can choose (b_n) to be any constant sequence of elements of X.

Using the same idea as in the proof of Theorem 1, one can establish the following result.

Theorem 2: (Copmare [10, Theorem 2.2.]) Let (X, D, K) be a complete metric type space and $\{T_n\}$ be a sequence of self mappings on X. Assume that there exist two sequences (a_n) and (b_n) of elements of X such

$$
\begin{aligned}
F(D(T_i^p(x), T_j^p(y), T_j^p(z))) \leq \ & F(\delta_{i,j}[D(x, T_i^p(x), T_i^p(x)) \\
& + D(y, T_j^p(y), T_j^p(y))]) \\
& + F(\gamma_{i,j} D(x, y, z))
\end{aligned}
\tag{6}
$$

for $x, y \in X$ with $x \neq y$, $0 \leq \delta_{i,j}, \gamma_{i,j} < 1$, $i, j = 1, 2, \cdots$, and for some $F \in \Phi$ homogeneous with degree s, where $\delta_{i,j} = D(a_i, a_j, a_j)$ and $\gamma_{i,j} = D(b_i, b_j, b_j)$.

If the sequence (s_n) where $s_i = \frac{\delta^s_{i,i+1} + \gamma^s_{i,i+1}}{1 - \delta^s_{i,i+1}}$ is a non-decreasing λ-sequence of $\mathbb{R}^+$ endowed with the max metric, then $\{T_n\}$ has a unique common fixed point in X.

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