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I_2-STATISTICAL LIMIT SUPERIOR AND I_2-STATISTICAL LIMIT INFERIOR FOR DOUBLE SEQUENCES

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ABSTRACT. In this article, we extended the recently introduced concepts of I-statistical limit superior and I-statistical limit inferior to I_2-statistical limit superior and I_2-statistical limit inferior and examine some of their properties for double sequences of real numbers. Findings of the study revealed that all the properties of the single sequence are also preserved in the double sequences.

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1. INTRODUCTION

The concept of statistical convergence was formally introduced by Fast [11] and Schoenberg [23] independently. Although statistical convergence was introduced over fifty years ago, it has become an active area of research in recent years. This has been applied in various areas such as summability theory (Fridy [12] and Salat [21]), topological groups (Cakalli [3], [4]), topological spaces (Maio and Kocinac [15]), locally convex spaces (Maddox [16]), measure theory (Cheng et al [5]), (Connor and Swardson [7]) and (Miller [17]), Fuzzy Mathematics (Nuray and Savas [20] and Savas [22]). In recent years, generalization of statistical convergence has appeared in the study of strong summability and the structure of ideals of bounded functions, (Connor and Swardson [8]). Kostyrko et al [13] further extended the idea of statistical convergence to I-convergence using the notion of ideals of N with many interesting consequences. Das

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and Savas [9] introduced and studied I-statistical and I-lacunary statistical convergence of order α. Quite recently Brono et al. [1] introduced the concept of I_2-statistical and I_2-lacunary statistical convergence for double sequence of order α in line of Das and Savas [9]. Demirci [10] introduced the definition of I-limit superior and inferior of a real sequence and proved several properties. Later on it was further investigated by Lahiri and Das [14]. Mursaleen et al.[19] introduced the concept of I-statistical limit superior and I-statistical limit inferior. In this article, we in analogy to Mursaleen et al. [19], extend these concepts to I_2-statistical limit superior and I_2-statistical limit inferior for double sequences. Let $K \subseteq \mathbb{N} \times \mathbb{N}$ be a two-dimensional set of positive integers and let $K(n,m)$ be the numbers and (i,j) in K such that $i \leq n$ and $j \leq m$. Then the two dimensional analogue of natural density can be defined as follows:

The lower asymptotic density of a set $K \subseteq \mathbb{N} \times \mathbb{N}$ is defined as:

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{n,m}$$

In case the sequence $\left(\frac{K(n,m)}{n,m}\right)$ has a limit in Pringsheim's sense, then we say that K has a double natural density and is defined by

$$\liminf_{n,m} \frac{K(n,m)}{n,m} = \delta_2(K)$$

For example, let $K = \{(i^2, j^2); i, j \in \mathbb{N}\}$

$$\underline{\delta_2}(K) = \liminf_{n,m} \frac{K(n,m)}{n,m} = \lim_{n,m} \frac{\sqrt{n}\sqrt{m}}{nm} = 0$$

i.e, the set K has double natural density zero, while the set $K = \{(i^2, j^2); i, j \in \mathbb{N}\}$ has double natural density $\frac{1}{2}$. Note that if $n = m$, we have a two-dimensional natural density considered by Christopher [6].

Statistical convergence of double sequences $x = (x_{jk})$ is defined as follows:

Definition 1.1 [Mursaleen and Edely, (2003)]: A real double sequence $x = (x_{jk})$ is statistically convergent to a number l if for each $\varepsilon > 0$, the set

$$\{(j,k), j \leq n, k \leq m : |(x_{jk}) - L| \geq \varepsilon\}$$

has double natural density zero. In this case, we write $st_2 - \lim_{n,m} x_{jk} = L$ and the set of all statistically convergent double sequences were denote by st_2.

2. PRELIMINARY

In this section we study the concepts of I_2-statistical limit superior and I_2-statistical limit inferior for real double sequences. For a real double sequence $x = (x_{jk})$ let B_x denote the set

$$B_x = \left\{ b \in \mathbb{R} : \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |j \leq m, k \leq n : x_{jk} \geq b| \geq \delta \right\} \notin I_2 \right\}$$

$$A_x = \left\{ a \in \mathbb{R} : \left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |j \leq m, k \leq n : x_{jk} \leq a| \geq \delta \right\} \notin I_2 \right\}$$

Definition 3.1: Let $x = (x_{jk})$ be a real double sequence. Then I_2-statistical limit superior of x is given by:

$$I_2 - st \limsup x = \begin{cases} \sup B_x, & \text{if } B_x \neq \phi \\ -\infty & \text{if } B_x = \phi \end{cases}$$

Also, I_2-statistical limit inferior of x is given by,

$$I_2 - st \liminf x = \begin{cases} \sup B_x, & \text{if } A_x \neq \phi \\ -\infty & \text{if } A_x = \phi \end{cases}$$

Theorem 3.1: If $\gamma = I_2 - st \limsup x$ is finite, then, for every posiive number ε,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |j \leq m, k \leq n : x_{jk} \geq \gamma - \varepsilon| \geq \delta \right\} \notin I_2$$

and

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |j \leq m, k \leq n : x_{jk} \geq \gamma + \varepsilon| \geq \delta \right\} \in I_2$$

Similarly, if $\mu = I_2 - stliminf x$ is finite, then for every positive number ε,

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |j \leq m, k \leq n : x_{jk} \geq \gamma - \varepsilon| \geq \mu \right\} \notin I_2$$

and

$$\left\{ (m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn} |j \leq m, k \leq n : x_{jk} \geq \gamma + \varepsilon| \geq \mu \right\} \in I_2$$

Proof: It follows from the definition.

Theorem 3.2: For any real double sequence $x = (x_{jk})), I_2 - st \liminf x \leq I_2 - st \limsup x$

Proof:
Case-I: If $I_2 - \limsup x = -\infty$, then, we have $B_x = 0$. So for every $b \in \mathbb{R}$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq b| \geq \delta\right\} \in I_2$$

which implies,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq b| \leq \delta\right\} \in F(I)$$

i.e.,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \leq b| \geq \delta\right\} \in F(I)$$

so for every

$$a \in \mathbb{R} \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \leq a| \geq \delta\right\} \notin F(I)$$

Hence, $I_2 - st \inf x = -\infty (since A_x = \mathbb{N})$
Case-II: If $I_2 - st \limsup x = \infty$, then, we need no proof.
Case-III: If $\gamma = I_2 - st \limsup x$ is finite and $\mu = I_2 - st \liminf x$ so for
$\epsilon \geq 0, \delta \geq 0, \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq \gamma + \varepsilon| \geq \delta\right\} \in I_2$
this implies,
$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \leq \gamma + \varepsilon| \geq \delta\right\} \in F(I)$
i.e. $\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \leq \gamma + \varepsilon| \geq \delta\right\} \in I_2$
so, $\gamma + \varepsilon \in A_x$. Since ϵ is arbitrary and by definition $\mu = \inf A_x$. Therefore,

$$\gamma \leq \mu + \varepsilon.$$

This proves that

$$\gamma \leq \mu$$

Definition 3.2: A real double sequence $x = (x_{jk})$ is said to be $I_2 - st$ bound if there is number H such that:
$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : |x_{jk}| \geq H| \geq \delta\right\} \in I_2$

Remark 3.1: If a double sequence is $I_2 - st$ bounded then $I_2 - st \limsup$ and $I_2 - st \liminf$ of that sequence are finite.

Definition 3.3: An element l is said to be an I_2-statistical cluster point of a double sequences $x = (x_{jk}) if for each$ $\varepsilon \geq 0$ and $\delta \geq 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \le m, k \le n : |x_{jk} - l| \ge \varepsilon| \le \delta\right\} \notin I_2$$

Theorem 3.3: If a I_2-statistically bounded sequence has one cluster point, then, it is I_2-statistically convergent.

Proof: Let $x = (x_{jk})$ be a I_2-statistically bounded double sequence which has one cluster point. Then,

$$T = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \le m, k \le n : |x_{jk}| \ge H| \ge \delta\right\} \in I_2$$

so, there exists a set $T' = m_1, n_1 \le m_2, n_2 \le ... \supset \mathbb{N} \times \mathbb{N}$ such that $T' \notin I_2$ and $(x_{m_j n_k})$ is statistically bounded double sequence.

Now, since (x_{mn}) has only one cluster point and $(x_{m_j n_k})$ is a statistically bounded sub-sequence of (x_{mn}), so $(x_{m_j n_k})$ also has only one cluster point. Hence $(x_{m_j n_k})$ is statistically convergent double sequence.

Let

$$St_2 - \lim x_{m_j n_k} = l$$

then for any $\varepsilon \ge 0$ and $\delta \ge 0$, we have the inclusion

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \le m, k \le n : |x_{jk} - l| \ge \varepsilon| \ge \delta\right\} \subseteq T \cup A \in I_2$$

where A is a finite set i.e., (x_{mn}) is I_2-statistically convergent to l.

Theorem 3.4: A double sequence $x = (x_{jk})$ is $I_2 - st$ convergent if and only if $I_2 - st \limsup x$, provided x is $I_2 - st$ bounded.

Proof: Let $\gamma = I_2 - st \liminf x$ and $\mu = I_2 - st \limsup x$ Let $I_2 - st \lim x = L$, so,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \le m, k \le n : |x_{jk} - L| \ge \varepsilon| \ge \delta\right\} \in I_2$$

that is

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \le m, k \le n : x_{jk} \ge L + \varepsilon| \ge \delta\right\} \in I_2$$

we also have

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \le m, k \le n : x_{jk} \le L - \varepsilon| \ge \delta\right\} \in I_2$$

which implies $L \leq \gamma$. Therefore, $\mu \leq \gamma$. But we know that, $\gamma \leq \mu$. i.e. $\gamma = \mu$.

Now we let $\gamma = \mu$ and define Ł $= \gamma$. For each $\varepsilon \geq 0, \delta \geq 0$

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq L + \frac{\varepsilon}{2}| \geq \delta\right\} \in I_2$$

and

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \leq L - \frac{\varepsilon}{2}| \geq \delta\right\} \in I_2$$

that is

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : |x_{jk} - L| \geq \varepsilon| \geq \delta\right\} \in I_2$$

so x is I_2-statistically convergent.

Theorem 3.5: If x, y are two $I_2 - st$ bounded sequence, then,

(i) $I_2 - st \limsup(x+y) \leq I_2 - st \limsup x + l_2 - st \limsup y$

(ii) $I_2 - st \limsup(x+y) \leq I_2 - st \liminf x + l_2 - st \liminf y$

Proof:

(i) Let, $l_1 = l_2 - st \limsup x$ and $l_2 = l_2 - st \limsup y$ So,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l_1 + \frac{\varepsilon}{2}| \geq \delta\right\} \in I_2$$

and

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : y_{jk} \geq l_1 + \frac{\varepsilon}{2}| \geq \delta\right\} \in I_2$$

Now,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} + y_{jk} \geq l_1 + l_2 + \varepsilon| \geq \delta\right\}$$
$$\subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l_1 \frac{\varepsilon}{2}| \geq \delta\right\}$$
$$\cup \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : y_{jk} \geq l_2 \frac{\varepsilon}{2}| \geq \delta\right\}$$

so,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} + y_{jk} \geq l_1 + l_2 + \varepsilon| \geq \delta\right\} \in I_2$$

if

$$r \in B_{x+y}$$

then, by definition

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} + y_{jk} \geq r| \geq \delta\right\} \notin I_2$$

We show that

$$r \leq l_1 + l_2 + \varepsilon$$

. If

$$r \geq l_1 + l_2 + \varepsilon$$

then,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} + y_{jk} \geq r| \geq \delta\right\}$$

$$\subseteq \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} + y_{jk} \geq l_1 + l_2 + \varepsilon| \geq \delta\right\}$$

Therefore,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} + y_{jk} \geq r| \geq \delta\right\} \in I_2$$

which is a contradiction

Hence, $r \leq l_1 + l_2 + \varepsilon$. As this is true for all $r \in B_{(x+y)}$, so $I_2 - st \limsup(x+y) = \sup B_{(x+y)} \leq l_1 + l_2 + \varepsilon$

Since, $\varepsilon \geq 0$ is arbitrary, so, $I_2 - st \limsup(x+y) \leq I_2 - st \limsup y$. Definition 3.4: A double sequence $x = (x_{jk})$ is said to be $I_2 - st$ convergent to $+\infty (or - \infty)$ if for every real number $H \geq 0$,

Theorem 3.6: If $I_2 - st \limsup x = l$, then, there exist a subsequence of x that is $I_2 - st$ convergent to l.

Proof:

Case I: If $l = -\infty$ then, $B_x = \varnothing$.
so for any real $H \geq 0$,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq -H| \geq \delta\right\} \in I_2$$

i.e. $I_2 - st \lim x = -\infty$
Case II: If $l = +\infty$, then, $B_x = \mathbb{R}$.

So for any $b \in \mathbb{R}$, $\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq b| \geq \delta\right\} \notin I_2$

$$A_{m_1n_1} = \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq x_{m_1n_1}| \geq \delta\right\} \notin I_2$$

Since, I_2 is an admissible ideal, so, $A_{m_1n_1}$ must be an infinite set.

i.e., $d(j \geq m, k \geq n : x_{jk} \geq x_{m_1n_1} + 1) \neq 0$.

We claim that there are at least
$j, k \in j \leq m, k \leq n : x_{jk} \geq xm_1n_1 + 1 \subseteq 1, 2, ..., m_1, n_1, m_1 + 1, n_1 + 1$
i.e.
$d(j \leq m, k \leq n : x_{jk} \geq x_{m_1n_1} + 1) \leq d(1, 2, ..., m_1, n_1, m_1 + 1, n_1 + 1) = 0$,
which is a contradiction.

We call these j, k as m_2, n_2 thus $x_{m_2n_2} \geq x_{m_1n_1} + 1$. Proceeding in this way we obtain a sub-sequence $x_{m_jn_k}$ of x with $x_{m_jk} + x_{m_{j-1}n_{k-1}} + 1$. Since for any $H \geq 0$,

$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \leq H| \geq \delta\right\} \in I_2 so, I_2 - st \lim x_{m_jn_j} = +\infty$.

Case III: $-\infty \leq l \leq +\infty$. So,

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l + \frac{1}{2}| \geq \delta\right\} \in I_2$$

and

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l - 1| \geq \delta\right\} \notin I_2$$

So there must be integers p and q in this set for which
$\frac{p}{q}|j \leq p, k \leq q : x_{jk} \geq l - 1| \geq \delta$ and $\frac{p}{q}|j \leq p, k \leq q : x_{jk} \geq l + \frac{1}{2}| \geq \delta$

For otherwise

$$\left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l - 1| \geq \delta\right\}$$
$$\subset \left\{(m,n) \in \mathbb{N} \times \mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l + \frac{1}{2}| \geq \delta\right\} \in I_2$$

which is a contradiction. Now for the maximum $j \leq p, k \leq q$ will satisfy $x_{jk} \geq l - 1$ and $x_{jk} \leq l + \frac{1}{2}$ sp we ,must have m_1, n_1 for whcih $l - 1 \leq x_{m_1,n_1} \leq l + \frac{1}{2} \leq l + 1$

Next, we choose an element x_{m_2,n_2} from $xm_2, n_2 \geq m_1n_1$ such that $l - \frac{1}{2} \leq x_{m_2,n_2} \leq l + \frac{1}{2}$
Now

$$\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{mn}|j \leq m, k \leq n : x_{jk} \geq l - \frac{1}{2}| \geq \delta\right\}$$

is an infinite set. So

$$d\left(\left\{j \leq m, k \leq n : x_{jk} \geq l - \frac{1}{2}\right\}\right) \neq 0$$

We observed that there are at least $j, k \geq m_1, n_1$for which $x_{jk} \geq l - \frac{1}{2}$, or otherwise.

$d\left(\left\{j \leq m, k \leq n : x_{jk} \geq l - \frac{1}{2}\right\}\right) \leq d(1, 2, ..., m_1, n_1) = 0,$
which is a contradiction.

Let $E_{m_1n_1} = \left\{j \leq m, k \leq n : j \geq m_1, k \geq n_1, x_{jk} \geq l - \frac{1}{2}\right\} \neq \varnothing$ if $j, k \in E_{m_1n_1}$ always implies $x_{jk} \geq l + \frac{1}{2}$
then,
$E_{m_1n_1} \subset \left\{j \leq m, k \leq n : x_{jk} \geq l + \frac{1}{2}\right\}$
i.e.
$d\left(E_{m_1n_1}\right) \leq d\left(\left\{j \leq m, k \leq n : x_{jk} \geq l + \frac{1}{2}\right\}\right) = 0.$

Since $\left\{(m,n) \in \mathbb{N}\times\mathbb{N} : \frac{1}{2}\left|\left\{j \leq m, k \leq n : x_{jk} \geq l + \frac{1}{2}\right\}\right| \leq \delta\right\} \in F(I)$

Thus, $\left\{j \leq m, k \leq n : x_{jk} \geq l - \frac{1}{2}\right\} \subseteq 1, 2, ..., m_1, n_1 \cup E_{m_1n_1}$ So $d\left(\left\{j \leq m, k \leq n : x_{jk} \geq l - \frac{1}{2}\right\}\right) \leq d1, 2, ..., m_1, n_1 + (E_{m_1n_1}) \leq 0,$
which is a contradiction.

This shows that there are $m_2n_2 \geq m_1n_1$ such that $l - \frac{1}{2} \leq x_{m_2n_2} \leq l + \frac{1}{2}$. Proceeding in this way, we obtain a sub-sequence $x_{m_2n_2}$ of $x, m_j, n_k \geq m_{j-1}n_{k-1}$ such that $l - \frac{1}{jk} \leq x_{m_jn_k} \leq l + \frac{1}{jk}$ for each j, k. This sequence $x_{m_jn_k}$ ordinarily converges to l and thus $I_2 - st$ convergent to l.

Theorem 3.7: If $I_2 - st \liminf x = l$, then, there exists a sub-sequence of x that is $I_2 - st$ convergent to l.
Proof: The proof is analogous to Theorem 3.6.

Theorem 3.8: Every $I_2 - st$ bounded double sequence x has a sub-sequence which is $I_2 - st$ convergent to a finite real number.
Proof: The proof follows from Remark 3.1 and Theorem 3.6.

3. CONCLUDING REMARKS

The concept of I-statistical Limit superior and I-statistical Limit inferior of single sequence has been extended to I_2-statistical Limit superior and I_2-statistical Limit inferior of double sequence and all the properties of the single sequence were preserved in the double sequence.

In Fig. 1, the critical value of the temperature is plotted against Newtonian parameter δ.

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