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HARMONIC INDEX AND RANDIĆ INDEX OF GENERALIZED TRANSFORMATION GRAPHS

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ABSTRACT. The harmonic index of a graph G is defined as the sum of weights $\frac{2}{d_G(u)+d_G(v)}$ of all edges uv of G and the Randić index of a graph G is defined as the sum of weights $\frac{1}{\sqrt{d_G(u)d_G(v)}}$ of all edges uv of G, where $d_G(u)$ is the degree of a vertex u in G. In this paper, the expressions for the harmonic index and Randić index of the generalized transformation graphs G^{xy} and for their complement graphs are obtained in terms of the parameters of underline graphs.

Keywords and phrases: Degree of a vertex, harmonic index, Randić index, generalized transformation graphs
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1. INTRODUCTION

Let G be a simple, undirected graph with n vertices and m edges. Let $V(G)$ and $E(G)$ be the vertex set and edge set of G respectively. If u and v are adjacent vertices of G, then the edge connecting them will be denoted by uv. The *degree* of a vertex u in G is the number of edges incident to it and is denoted by $d_G(u)$.

The *Randić index* [20] $R(G)$ is one of the most successful molecular descriptors in the studies of structure-property and structure-activity relationships [6, 10, 11, 19] and is defined as

$$R(G) = \sum_{uv\in E(G)} \frac{1}{\sqrt{d_G(u)d_G(v)}}.$$

For mathematical properties of this graph invariant, see [7, 12].

The *harmonic index* [4] is defined as

$$H(G) = \sum_{uv\in E(G)} \frac{2}{d_G(u)+d_G(v)}.$$

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The harmonic index and Randić index are well correlated [12]. Bounds for the harmonic index have been reported in [13, 23]. Favaron et al. [5] considered the relationship between the harmonic index and the eigenvalues of graphs. Deng at al. [3] studied the relationship between the harmonic index and chromatic number of a graph. Harmonic index of trees was considered in [2, 17]. Harmonic index of unicyclic graphs and bicyclic graphs was obtained in [8, 18, 26, 27, 28]. Bounds for the harmonic index of graph operations were obtained by Shwetha Shetty et al. [22]. For other results on harmonic index one can refer [2, 14, 15, 24, 25].

The *generalized transformation graph* G^{xy}, introduced recently by Basavanagoud et al. [1], is a graph whose vertex set is $V(G) \cup E(G)$, and $\alpha, \beta \in V(G^{xy})$. The vertices α and β are adjacent in G^{xy} if and only if (a) and (b) holds:

(a) $\alpha, \beta \in V(G)$, α, β are adjacent in G if $x = +$ and α, β are not adjacent in G if $x = -$.

(b) $\alpha \in V(G)$ and $\beta \in E(G)$, α, β are incident in G if $y = +$ and α, β are not incident in G if $y = -$.

One can obtain the four graphical transformations of graphs as G^{++}, G^{+-}, G^{-+} and G^{--}. An example of generalized transformation graphs and their complements are depicted in the Fig. 1. Note that G^{++} is just the *semitotal-point graph* of G, which was introduced by Sampathkumar and Chikkodimath [21]. The vertex v of G^{xy} corresponding to a vertex v of G is referred to as a *point vertex*. The vertex e of G^{xy} corresponding to an edge e of G is referred to as a *line vertex*.

Proposition 1.1: [1] Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then the degrees of point and line vertices in G^{xy} are
(i) $d_{G^{++}}(u) = 2d_G(u)$ and $d_{G^{++}}(e) = 2$.
(ii) $d_{G^{+-}}(u) = m$ and $d_{G^{+-}}(e) = n - 2$.
(iii) $d_{G^{-+}}(u) = n - 1$ and $d_{G^{-+}}(e) = 2$.
(iv) $d_{G^{--}}(u) = n + m - 1 - 2d_G(u)$ and $d_{G^{--}}(e) = n - 2$.

The complement of G will be denoted by $\overline{G}$. If G has n vertices and m edges then the number of vertices of G^{xy} is $n+m$. By Proposition 1.1 and taking into account that $d_{\overline{G}}(u) = n - 1 - d_G(u)$, we have following proposition:

Proposition 1.2: Let G be a graph with n vertices and m edges. Let $u \in V(G)$ and $e \in E(G)$. Then the degrees of point and line vertices in $\overline{G^{xy}}$ are
(i) $d_{\overline{G^{++}}}(u) = n + m - 1 - 2d_G(u)$ and $d_{\overline{G^{++}}}(e) = n + m - 3$.
(ii) $d_{\overline{G^{+-}}}(u) = n - 1$ and $d_{\overline{G^{+-}}}(e) = m + 1$.
(iii) $d_{\overline{G^{-+}}}(u) = m$ and $d_{\overline{G^{-+}}}(e) = n + m - 3$.
(iv) $d_{\overline{G^{--}}}(u) = 2d_G(u)$ and $d_{\overline{G^{--}}}(e) = m + 1$.

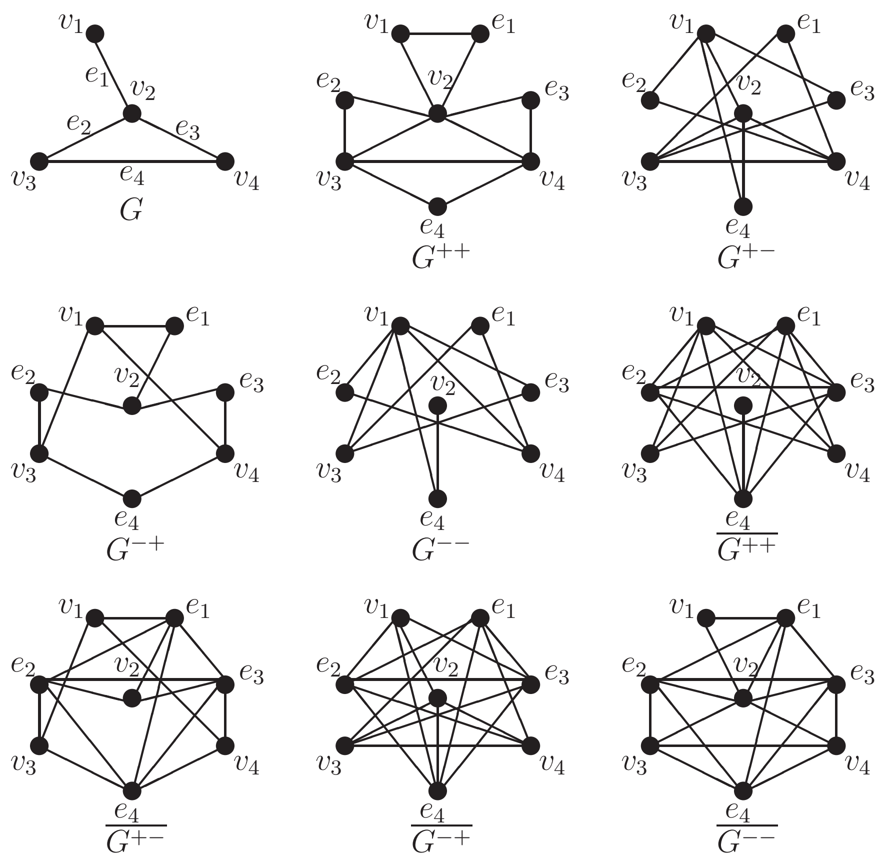

Figure 1: Graph G, its generalized transformations G^{xy} and their complements $\overline{G^{xy}}$

In this paper we obtain the expressions for the harmonic index and Randić index of generalized transformation graphs G^{xy} and of their complements $\overline{G^{xy}}$ in terms of the parameters of underline graphs.

2. HARMONIC INDEX OF G^{xy}

Theorem 2.1: Let G be a graph with n vertices and m edges. Then

$$H(G^{++}) = \frac{1}{2}H(G) + \sum_{u \in V(G)} \frac{d_G(u)}{1 + d_G(u)}.$$

Proof: Partition the edge set $E(G^{++})$ into subsets E_1 and E_2, where $E_1 = \{uv \mid uv \in E(G)\}$ and
$E_2 = \{ue \mid \text{the vertex } u \text{ is incident to the edge } e \text{ in } G\}$. It is easy to check that $|E_1| = m$ and $|E_2| = 2m$. By Proposition 1.1, if $u \in V(G)$ then $d_{G^{++}}(u) = 2d_G(u)$ and if $e \in E(G)$ then $d_{G^{++}}(e) = 2$. Therefore

$$\begin{aligned}
H(G^{++}) &= \sum_{uv \in E(G^{++})} \frac{2}{d_{G^{++}}(u) + d_{G^{++}}(v)} \\
&= \sum_{uv \in E_1} \frac{2}{d_{G^{++}}(u) + d_{G^{++}}(v)} + \sum_{ue \in E_2} \frac{2}{d_{G^{++}}(u) + d_{G^{++}}(e)} \\
&= \sum_{uv \in E(G)} \frac{2}{2d_G(u) + 2d_G(v)} + \sum_{ue \in E_2} \frac{2}{2d_G(u) + 2} \\
&= \frac{1}{2} \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)} + \sum_{ue \in E_2} \frac{1}{d_G(u) + 1} \\
&= \frac{1}{2}H(G) + \sum_{ue \in E_2} \frac{1}{1 + d_G(u)}.
\end{aligned}$$

In the second part of above equation, the quantity $\frac{1}{1+d_G(u)}$ appears $d_G(u)$ times. Hence above expression can be written as

$$H(G^{++}) = \frac{1}{2}H(G) + \sum_{u \in V(G)} \frac{d_G(u)}{1 + d_G(u)}.$$

Theorem 2.2: Let G be a graph with n vertices and m edges. Then

$$H(G^{+-}) = 1 + \frac{2m(n-2)}{m+n-2}.$$

Proof: Partition the edge set $E(G^{+-})$ into subsets E_1 and E_2, where $E_1 = \{uv \mid uv \in E(G)\}$ and
$E_2 = \{ue \mid \text{the vertex } u \text{ is not incident to the edge } e \text{ in } G\}$. It is easy to check that $|E_1| = m$ and $|E_2| = m(n-2)$. By Proposition 1.1, if $u \in V(G)$ then $d_{G^{+-}}(u) = m$ and if $e \in E(G)$ then $d_{G^{+-}}(e) =$

$n-2$. Therefore

$$\begin{aligned}
H(G^{+-}) &= \sum_{uv\in E(G^{+-})} \frac{2}{d_{G^{+-}}(u)+d_{G^{+-}}(v)} \\
&= \sum_{uv\in E_1} \frac{2}{d_{G^{+-}}(u)+d_{G^{+-}}(v)} + \sum_{ue\in E_2} \frac{2}{d_{G^{+-}}(u)+d_{G^{+-}}(e)} \\
&= \sum_{uv\in E(G)} \frac{2}{m+m} + \sum_{ue\in E_2} \frac{2}{m+n-2} \\
&= \frac{2m}{2m} + \frac{2m(n-2)}{m+n-2} = 1 + \frac{2m(n-2)}{m+n-2}.
\end{aligned}$$

Theorem 2.3: Let G be a graph with n vertices and m edges. Then

$$H(G^{-+}) = \frac{n}{2} + \frac{3mn-5m}{n^2-1}.$$

Proof: Partition the edge set $E(G^{-+})$ into subsets E_1 and E_2, where $E_1 = \{uv \mid uv \notin E(G)\}$ and
$E_2 = \{ue \mid \text{the vertex } u \text{ is incident to the edge } e \text{ in } G\}$. It is easy to check that $|E_1| = \binom{n}{2} - m$ and $|E_2| = 2m$. By Proposition 1.1, if $u \in V(G)$ then $d_{G^{-+}}(u) = n-1$ and if $e \in E(G)$ then $d_{G^{-+}}(e) = 2$. Therefore

$$\begin{aligned}
H(G^{-+}) &= \sum_{uv\in E(G^{-+})} \frac{2}{d_{G^{-+}}(u)+d_{G^{-+}}(v)} \\
&= \sum_{uv\in E_1} \frac{2}{d_{G^{-+}}(u)+d_{G^{-+}}(v)} + \sum_{ue\in E_2} \frac{2}{d_{G^{-+}}(u)+d_{G^{-+}}(e)} \\
&= \sum_{uv\notin E(G)} \frac{2}{n-1+n-1} + \sum_{ue\in E_2} \frac{2}{n-1+2} \\
&= \left[\binom{n}{2} - m\right]\left(\frac{2}{2n-2}\right) + 2m\left(\frac{2}{n+1}\right) \\
&= \frac{n}{2} + \frac{3mn-5m}{n^2-1}.
\end{aligned}$$

Theorem 2.4: Let G be a graph with n vertices and m edges. Then

$$\begin{aligned}
H(G^{--}) &= \sum_{uv\notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))} \\
&\quad + \sum_{u\in V(G)} \frac{2(m-d_G(u))}{2n+m-3-2d_G(u)}.
\end{aligned}$$

Proof: Partition the edge set $E(G^{--})$ into subsets E_1 and E_2, where $E_1 = \{uv \mid uv \notin E(G)\}$ and
$E_2 = \{ue \mid$ the vertex u is not incident to the edge e in $G\}$. It is easy to check that $|E_1| = \binom{n}{2} - m$ and $|E_2| = m(n-2)$. By Proposition 1.1, if $u \in V(G)$ then $d_{G^{--}}(u) = n + m - 1 - 2d_G(u)$ and if $e \in E(G)$ then $d_{G^{--}}(e) = n - 2$. Therefore

$$\begin{aligned}
H(G^{--}) &= \sum_{uv\in E(G^{--})} \frac{2}{d_{G^{--}}(u) + d_{G^{--}}(v)} \\
&= \sum_{uv\in E_1} \frac{2}{d_{G^{--}}(u) + d_{G^{--}}(v)} + \sum_{ue\in E_2} \frac{2}{d_{G^{--}}(u) + d_{G^{--}}(e)} \\
&= \sum_{uv\notin E(G)} \frac{2}{n+m-1-2d_G(u)+n+m-1-2d_G(v)} \\
&\quad + \sum_{ue\in E_2} \frac{2}{n+m-1-2d_G(u)+n-2} \\
&= \sum_{uv\notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))} \\
&\quad + \sum_{ue\in E_2} \frac{2}{2n+m-3-2d_G(u)} \\
&= \sum_{uv\notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))} \\
&\quad + \sum_{u\in V(G)} \frac{2(m-d_G(u))}{2n+m-3-2d_G(u)}.
\end{aligned}$$

Remarks 2.5:

(1) By Theorem 2.2, for all graphs G having same number of vertices and same number of edges, $H(G^{+-})$ is same.
(2) By Theorem 2.3, for all graphs G having same number of vertices and same number of edges, $H(G^{-+})$ is same.
(3) Among all graphs with n vertices, the complete graph K_n has maximum $H(G^{-+})$.

3. HARMONIC INDEX OF $\overline{G^{xy}}$

Theorem 3.1: Let G be a graph with n vertices and m edges. Then

$$
\begin{aligned}
H\left(\overline{G^{++}}\right) &= \sum_{uv\notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))} \\
&\quad + \sum_{u\in V(G)} \frac{m-d_G(u)}{n+m-2-d_G(u)} + \frac{m(m-1)}{2(n+m-3)}.
\end{aligned}
$$

Proof: Partition the edge set $E(\overline{G^{++}})$ into subsets E_1, E_2 and E_3, where $E_1 = \{uv \mid uv \notin E(G)\}$,
$E_2 = \{ue \mid \text{the vertex } u \text{ is not incident to the edge } e \text{ in } G\}$ and $E_3 = \{ef \mid e, f \in E(G)\}$. It is easy to check that $|E_1| = \binom{n}{2} - m$, $|E_2| = m(n-2)$ and $E_3 = \binom{m}{2}$. By Proposition 1.2, if $u \in V(G)$ then $d_{\overline{G^{++}}}(u) = n+m-1-2d_G(u)$ and if $e \in E(G)$ then $d_{\overline{G^{++}}}(e) = n+m-3$. Therefore

$$
\begin{aligned}
H\left(\overline{G^{++}}\right) &= \sum_{uv\in E(\overline{G^{++}})} \frac{2}{d_{\overline{G^{++}}}(u)+d_{\overline{G^{++}}}(v)} \\
&= \sum_{uv\in E_1} \frac{2}{d_{\overline{G^{++}}}(u)+d_{\overline{G^{++}}}(v)} + \sum_{ue\in E_2} \frac{2}{d_{\overline{G^{++}}}(u)+d_{\overline{G^{++}}}(e)} \\
&\quad + \sum_{ef\in E_3} \frac{2}{d_{\overline{G^{++}}}(e)+d_{\overline{G^{++}}}(f)} \\
&= \sum_{uv\notin E(G)} \frac{2}{n+m-1-2d_G(u)+n+m-1-2d_G(v)} \\
&\quad + \sum_{ue\in E_2} \frac{2}{n+m-1-2d_G(u)+n+m-3} \\
&\quad + \sum_{ef\in E_3} \frac{2}{n+m-3+n+m-3} \\
&= \sum_{uv\notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))} \\
&\quad + \sum_{ue\in E_2} \frac{1}{n+m-2-d_G(u)} + \sum_{ef\in E_3} \frac{1}{n+m-3}
\end{aligned}
$$

$$= \sum_{uv \notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))}$$
$$+ \sum_{u \in V(G)} \frac{m-d_G(u)}{n+m-2-d_G(u)} + \binom{m}{2}\left(\frac{1}{n+m-3}\right)$$
$$= \sum_{uv \notin E(G)} \frac{1}{n+m-1-(d_G(u)+d_G(v))}$$
$$+ \sum_{u \in V(G)} \frac{m-d_G(u)}{n+m-2-d_G(u)} + \frac{m(m-1)}{2(n+m-3)}.$$

Theorem 3.2: Let G be a graph with n vertices and m edges. Then

$$H\left(\overline{G^{+-}}\right) = \frac{n}{2} - \frac{m}{n-1} + \frac{4m}{m+n} + \frac{m(m-1)}{2(m+1)}.$$

Proof: Partition the edge set $E(\overline{G^{+-}})$ into subsets E_1, E_2 and E_3, where $E_1 = \{uv \mid uv \notin E(G)\}$,
$E_2 = \{ue \mid \text{the vertex } u \text{ is incident to the edge } e \text{ in } G\}$ and $E_3 = \{ef \mid e, f \in E(G)\}$. It is easy to check that $|E_1| = \binom{n}{2} - m$, $|E_2| = 2m$ and $E_3 = \binom{m}{2}$. By Proposition 1.2, if $u \in V(G)$ then $d_{\overline{G^{+-}}}(u) = n-1$ and if $e \in E(G)$ then $d_{\overline{G^{+-}}}(e) = m+1$. Therefore

$$H\left(\overline{G^{+-}}\right) = \sum_{uv \in E(\overline{G^{+-}})} \frac{2}{d_{\overline{G^{+-}}}(u) + d_{\overline{G^{+-}}}(v)}$$
$$= \sum_{uv \in E_1} \frac{2}{d_{\overline{G^{+-}}}(u) + d_{\overline{G^{+-}}}(v)} + \sum_{ue \in E_2} \frac{2}{d_{\overline{G^{+-}}}(u) + d_{\overline{G^{+-}}}(e)}$$
$$+ \sum_{ef \in E_3} \frac{2}{d_{\overline{G^{+-}}}(e) + d_{\overline{G^{+-}}}(f)}$$
$$= \sum_{uv \notin E(G)} \frac{2}{n-1+n-1} + \sum_{ue \in E_2} \frac{2}{n-1+m+1}$$
$$+ \sum_{ef \in E_3} \frac{2}{m+1+m+1}$$
$$= \sum_{uv \notin E(G)} \frac{1}{n-1} + \sum_{ue \in E_2} \frac{2}{n+m} + \sum_{ef \in E_3} \frac{1}{m+1}$$

$$
\begin{aligned}
&= \left[\binom{n}{2} - m\right]\left(\frac{1}{n-1}\right) + 2m\left(\frac{2}{n+m}\right) \\
&\quad + \binom{m}{2}\left(\frac{1}{m+1}\right) \\
&= \frac{n}{2} - \frac{m}{n-1} + \frac{4m}{m+n} + \frac{m(m-1)}{2(m+1)}.
\end{aligned}
$$

Theorem 3.3: Let G be a graph with n vertices and m edges. Then

$$
H\left(\overline{G^{-+}}\right) = 1 + \frac{2m(n-2)}{n+2m-3} + \frac{m(m-1)}{2(n+m-3)}.
$$

Proof: Partition the edge set $E(\overline{G^{-+}})$ into subsets E_1, E_2 and E_3, where $E_1 = \{uv \mid uv \in E(G)\}$,
$E_2 = \{ue \mid \text{the vertex } u \text{ is not incident to the edge } e \text{ in } G\}$ and $E_3 = \{ef \mid e, f \in E(G)\}$. It is easy to check that $|E_1| = m$, $|E_2| = m(n-2)$ and $E_3 = \binom{m}{2}$. By Proposition 1.2, if $u \in V(G)$ then $d_{\overline{G^{-+}}}(u) = m$ and if $e \in E(G)$ then $d_{\overline{G^{-+}}}(e) = n+m-3$. Therefore

$$
\begin{aligned}
H\left(\overline{G^{-+}}\right) &= \sum_{uv \in E(\overline{G^{-+}})} \frac{2}{d_{\overline{G^{-+}}}(u) + d_{\overline{G^{-+}}}(v)} \\
&= \sum_{uv \in E_1} \frac{2}{d_{\overline{G^{-+}}}(u) + d_{\overline{G^{-+}}}(v)} + \sum_{ue \in E_2} \frac{2}{d_{\overline{G^{-+}}}(u) + d_{\overline{G^{-+}}}(e)} \\
&\quad + \sum_{ef \in E_3} \frac{2}{d_{\overline{G^{-+}}}(e) + d_{\overline{G^{-+}}}(f)} \\
&= \sum_{uv \in E(G)} \frac{2}{m+m} + \sum_{ue \in E_2} \frac{2}{m+n+m-3} \\
&\quad + \sum_{ef \in E_3} \frac{2}{n+m-3+n+m-3} \\
&= \sum_{uv \in E(G)} \frac{1}{m} + \sum_{ue \in E_2} \frac{2}{n+2m-3} + \sum_{ef \in E_3} \frac{1}{n+m-3} \\
&= \frac{m}{m} + m(n-2)\left(\frac{2}{n+2m-3}\right) + \binom{m}{2}\left(\frac{1}{n+m-3}\right) \\
&= 1 + \frac{2m(n-2)}{n+2m-3} + \frac{m(m-1)}{2(n+m-3)}.
\end{aligned}
$$

Theorem 3.4: Let G be a graph with n vertices and m edges. Then

$$H\left(\overline{G^{--}}\right) = \frac{1}{2}H(G) + \frac{m(m-1)}{2(m+1)} + \sum_{u \in V(G)} \frac{2d_G(u)}{m+1+2d_G(u)}.$$

Proof: Partition the edge set $E(\overline{G^{--}})$ into subsets E_1, E_2 and E_3, where $E_1 = \{uv \mid uv \in E(G)\}$,
$E_2 = \{ue \mid \text{the vertex } u \text{ is incident to the edge } e \text{ in } G\}$ and $E_3 = \{ef \mid e, f \in E(G)\}$. It is easy to check that $|E_1| = m$, $|E_2| = 2m$ and $E_3 = \binom{m}{2}$. By Proposition 1.2, if $u \in V(G)$ then $d_{\overline{G^{--}}}(u) = 2d_G(u)$ and if $e \in E(G)$ then $d_{\overline{G^{--}}}(e) = m+1$. Therefore

$$\begin{aligned}
H\left(\overline{G^{--}}\right) &= \sum_{uv \in E(\overline{G^{--}})} \frac{2}{d_{\overline{G^{--}}}(u) + d_{\overline{G^{--}}}(v)} \\
&= \sum_{uv \in E_1} \frac{2}{d_{\overline{G^{--}}}(u) + d_{\overline{G^{--}}}(v)} + \sum_{ue \in E_2} \frac{2}{d_{\overline{G^{--}}}(u) + d_{\overline{G^{--}}}(e)} \\
&\quad + \sum_{ef \in E_3} \frac{2}{d_{\overline{G^{--}}}(e) + d_{\overline{G^{--}}}(f)} \\
&= \sum_{uv \in E(G)} \frac{2}{2d_G(u) + 2d_G(v)} + \sum_{ue \in E_2} \frac{2}{2d_G(u) + m + 1} \\
&\quad + \sum_{ef \in E_3} \frac{2}{m+1+m+1} \\
&= \frac{1}{2} \sum_{uv \in E(G)} \frac{2}{d_G(u) + d_G(v)} + \sum_{ue \in E_2} \frac{2}{m+1+2d_G(u)} \\
&\quad + \sum_{ef \in E_3} \frac{1}{m+1} \\
&= \frac{1}{2}H(G) + \sum_{u \in V(G)} \frac{2d_G(u)}{m+1+2d_G(u)} + \binom{m}{2}\left(\frac{1}{m+1}\right) \\
&= \frac{1}{2}H(G) + \frac{m(m-1)}{2(m+1)} + \sum_{u \in V(G)} \frac{2d_G(u)}{m+1+2d_G(u)}.
\end{aligned}$$

Remarks 3.5:

(1) By Theorem 3.2, for all graphs G having same number of vertices and same number of edges, $H\left(\overline{G^{+-}}\right)$ is same.
(2) By Theorem 3.3, for all graphs G having same number of vertices and same number of edges, $H\left(\overline{G^{-+}}\right)$ is same.

4. RANDIĆ INDEX OF G^{xy} AND $\overline{G^{xy}}$

In fully analogous manner, applying the proof techniques of the Sections 2 and 3 and by the definition of the Randić index, we arrive at:

Theorem 4.1: Let G be a graph with n vertices and m edges. Then

$$\begin{aligned}
R(G^{++}) &= \frac{1}{2}\left[R(G) + \sum_{u\in V(G)} \sqrt{d_G(u)}\right];\\
R(G^{+-}) &= 1 + \sqrt{m(n-2)};\\
R(G^{-+}) &= \frac{n}{2} - m\left[\frac{1}{n-1} - \sqrt{\frac{2}{n-1}}\right];\\
R(G^{--}) &= \sum_{uv\notin E(G)} \frac{1}{\sqrt{(n+m-1-2d_G(u))(n+m-1-2d_G(v))}}\\
&\quad + \sum_{u\in V(G)} \frac{m-d_G(u)}{\sqrt{(n-2)(n+m-1-2d_G(u))}};\\
R\left(\overline{G^{++}}\right) &= \sum_{uv\notin E(G)} \frac{1}{\sqrt{(n+m-1-2d_G(u))(n+m-1-2d_G(v))}}\\
&\quad + \sum_{u\in V(G)} \frac{m-d_G(u)}{\sqrt{(n+m-3)(n+m-1-2d_G(u))}}\\
&\quad + \frac{m(m-1)}{2(n+m-3)};\\
R\left(\overline{G^{+-}}\right) &= \frac{n}{2} - \frac{m}{n-1} + \frac{2m}{\sqrt{(n-1)(m+1)}} + \frac{m(m-1)}{2(m+1)};\\
R\left(\overline{G^{-+}}\right) &= 1 + \frac{m(n-2)}{\sqrt{m(n+m-3)}} + \frac{m(m-1)}{2(n+m-3)};\\
R\left(\overline{G^{--}}\right) &= \frac{1}{2}R(G) + \frac{m(m-1)}{2(m+1)} + \sum_{u\in V(G)} \sqrt{\frac{d_G(u)}{2(m+1)}}.
\end{aligned}$$

Remarks 4.2:

(1) If G_1 and G_2 are two different graphs having same number of vertices and same number of edges, then

$$\begin{aligned} R(G_1^{+-}) &= R(G_2^{+-}) \\ R(G_1^{-+}) &= R(G_2^{-+}) \\ R\left(\overline{G_1^{+-}}\right) &= R\left(\overline{G_2^{+-}}\right) \\ R\left(\overline{G_1^{-+}}\right) &= R\left(\overline{G_2^{-+}}\right). \end{aligned}$$

(2) Among all graphs with n vertices, the complete graph K_n has maximum $R(G^{+-})$.

(3) Among all graphs with n vertices, the complete graph K_n has minimum $R(G^{-+})$.

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