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A HYBRID-EXTRAGRADIENT ITERATIVE METHOD FOR SPLIT MONOTONE VARIATIONAL INCLUSION, MIXED EQUILIBRIUM PROBLEM AND FIXED POINT PROBLEM FOR A NONEXPANSIVE MAPPING

K. R. KAZMI[1] S. H. RIZVI AND REHAN ALI

ABSTRACT. In this paper, we investigate a hybrid-extragradient iterative method to approximate a common element of the set of solutions of split monotone variational inclusion, mixed equilibrium problem and fixed-point problem for a nonexpansive mapping. Further, we establish a strong convergence theorem for the sequences generated by the proposed iterative algorithm. We also derive some consequences from our main result. A numerical example is given to support our main result. The method and results presented in this paper are the extension and generalization of the previously known iterative methods and results in this area.

Keywords and phrases: Split monotone variational inclusion; Mixed equilibrium problem; Fixed-point problem; Hybrid-extragradient method; Nonexpansive mapping.

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1. INTRODUCTION

Let H_1 and H_2 be two real Hilbert spaces with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. Let C and Q are nonempty, closed and convex subsets of H_1 and H_2 respectively, let $F : C \times C \to \mathbb{R}$ be a bifunction, where $\mathbb{R}$ is a set of real numbers, such that $F(x, x) = 0,\ \forall x \in C$ and let $A : C \to H_1$ be a nonlinear mapping. Then, we consider the following *mixed equilibrium problem* (in short, MEP): Find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x \rangle \geq 0,\ \forall y \in C. \tag{1}$$

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[1]Corresponding author

MEP(1) was introduced and studied by Moudafi and Théra [1]. The solution set of MEP(1) is denoted by Sol(MEP(1)). If $F = 0$, MEP(1) reduces to the classical *variational inequality problem* (in short, VIP): Find $x \in C$ such that

$$\langle Ax, y - x\rangle \geq 0, \ \forall y \in C, \tag{2}$$

which is introduced by Hartmann and Stampacchia [2]. If $A = 0$, MEP(1) reduces to the *equilibrium problem* (in short, EP): Find $x \in C$ such that

$$F(x, y) \geq 0, \ \forall y \in C, \tag{3}$$

which is introduced by Blum and Oettli [3]. The set of solutions of EP(3) is denoted by Sol(EP(3)).

Recall that a mapping $S : H_1 \to H_1$ is nonexpansive if $\|Sx - Sy\| \leq \|x-y\|$, $\forall x, y \in H_1$. Further, we consider the following *fixed point problem* (in short FPP) for a nonexpansive mapping S: Find $x \in H_1$ such that

$$Sx = x. \tag{4}$$

The solution set of FPP(4) is denoted by Fix(S). We note that if Fix(S) $\neq \emptyset$ then Fix(S) is closed and convex.

In 2007, Takahashi and Takahashi [4] proposed an iterative method based on viscosity approximation method for approximating a common solution of EP(3) and FPP for a nonexpansive mapping S in Hilbert space. Since then the common solution of these type of problems have been studied using different iterative methods, see for instance [5, 6, 7] and references therein.

Recently, Censor *et al.* [8] introduced and studied the following split variational inequality problem (in short, $\mathrm{S_P}$VIP): Let $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be nonlinear single-valued mappings and let $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*. Then $\mathrm{S_P}$VIP is to find $x^* \in C$ satisfying

$$\langle fx^*, x - x^*\rangle \geq 0, \ \forall x \in C, \tag{5}$$

and such that

$$y^* = Bx^* \in Q \text{ solves } \langle gy^*, x - y^*\rangle \geq 0, \ \forall y \in Q. \tag{6}$$

The solution set of $\mathrm{S_P}$VIP(5)-(6) is denoted by Sol($\mathrm{S_P}$VIP(5)-(6)) $= \{x^* \in C : x^* \in \text{Sol(VIP(5))} \text{ and } Bx^* \in \text{Sol(VIP(6))}\}$. They introduced and studied the following iterative method for solving

$\mathrm{S_P}$VIP(5)-(6): For a given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by the iterative algorithm:

$$x_{n+1} = U(x_n + \gamma B^*(T - I)Bx_n), \tag{7}$$

where $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator B^*B, $U := P_C(I - \lambda f)$ and $T := P_Q(I - \lambda g)$, for $\lambda > 0$, P_C is a metric projection onto C.

Further, Moudafi [9] introduced the following *split monotone variational inclusion problem* (in short, $\mathrm{S_P}$MVIP): Find $x^* \in H_1$ such that

$$0 \in f(x^*) + M_1(x^*), \tag{8}$$

and such that

$$y^* = Bx^* \in H_2 \text{ solves } 0 \in g(y^*) + M_2(y^*), \tag{9}$$

where $M_1 : H_1 \to 2^{H_1}$ and $M_2 : H_2 \to 2^{H_2}$ are multi-valued maximal monotone mappings.

Moudafi [9] introduced and studied the following iterative method for solving $\mathrm{S_P}$MVIP(8)-(9), which can be seen an important generalization of an iterative method (7) given in [8] for $\mathrm{S_P}$VIP(5)-(6): For a given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by the iterative algorithm:

$$x_{n+1} = U(x_n + \gamma B^*(T - I)Bx_n),$$

where $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator B^*B, $U := J_\lambda^{M_1}(I - \lambda f)$ and $T := J_\lambda^{M_2}(I - \lambda g)$, $J_\lambda^{M_1}$ is defined in Definition 2.4 below for $\lambda > 0$.

When looked separately, (8) is the monotone variational inclusion problem (in short, MVIP) and we denoted its solution set by Sol(MVIP(8)). The $\mathrm{S_P}$MVIP(8)-(9) constitutes a pair of monotone variational inclusion problems which have to be solved so that the image $y^* = Bx^*$ under a given bounded linear operator B, of the solution x^* of MVIP(8) in H_1 is the solution of another MVIP(9) in another space H_2. We denote the solution set of MVIP(9) by Sol(MVIP(9)).

The solution set of $\mathrm{S_P}$MVIP(8)-(9) is denoted by Sol($\mathrm{S_P}$MVIP(8)-(9)) $= \{x^* \in H_1 : x^* \in$ Sol(MVIP(8)) and $Bx^* \in$ Sol(MVIP(9))$\}$.

If $f_1 \equiv 0$ and $f_2 \equiv 0$ then $\mathrm{S_P}$MVIP(8)-(9) reduces to the following *split null point problem* (in short, $\mathrm{S_P}$NPP): Find $x^* \in H_1$ such that

$$0 \in M_1(x^*), \tag{10}$$

and such that

$$y^* = Bx^* \in H_2 \text{ solves } 0 \in M_2(y^*). \tag{11}$$

In 2012, Byrne *et al.* [10] introduced an iterative method and studied the weak and strong convergence theorems for $\mathrm{S_P}$NPP(10)-(11). For a given $x_0 \in H_1$, compute iterative sequence $\{x_n\}$ generated by the following scheme:

$$x_{n+1} = J_\lambda^{M_1}(x_n + \gamma B^*(J_\lambda^{M_2} - I)Bx_n), \text{ for} \lambda > 0.$$

Recently, Kazmi and Rizvi [11] introduced and studied an iterative method, based on viscosity approximation method to approximate a common solution of $\mathrm{S_P}$NPP(10)-(11) and fixed point problem of a nonexpansive mapping in the framework of real Hilbert spaces.

$$\begin{cases} u_n = J_\lambda^{M_1}(x_n + \gamma B^*(J_\lambda^{M_2} - I)Bx_n), \\ x_{n+1} = \alpha_n h(x_n) + (1-\alpha_n)Su_n, \end{cases}$$

where $h : H_1 \to H_1$ is a contraction mapping and $\lambda > 0$.

Very recently, Sitthithakerngkiet *et. al* [12] extended the work of Kazmi and Rizvi [11] and Byrne *et al.* [10] for $\mathrm{S_P}$VIP(10)-(11).

It is stressed in [12] that it is worth to study the strong convergence theorems for the sequences generated by iterative algorithms for the $\mathrm{S_P}$VIP(5)-(6) and $\mathrm{S_P}$MVIP(8)-(9). As Moudafi notes in [9] that $\mathrm{S_P}$MVIP(8)-(9) includes as special cases, $\mathrm{S_P}$VIP(5)-(6), $\mathrm{S_P}$NPP(10)-(11), the split common fixed point problem and split feasibility problem [8, 13, 14] which have already been studied and used in practice as a model in intensity-modulated radiation therapy treatment, see [13, 14]. This formulation is also at the core of modeling of many inverse problems arising from phase retrieval and other real-world problems; for instance, in sensor networks in computerized tomography and data compression; see e.g. [10, 15, 16, 17, 18, 19]. Therefore, it is worth to study the iterative methods for $\mathrm{S_P}$VIP(5)-(6) and $\mathrm{S_P}$MVIP(8)-(9).

In 1976, Korpelevich [20] proposed an iterative method with iterative scheme for VIP in Euclidean space:

$$\begin{cases} x_1 = x \in C \\ y_n = P_C(x_n - \lambda Ax_n), \\ x_{n+1} = P_C(x_n - \lambda Ay_n), \end{cases} \tag{12}$$

where $\lambda > 0$ is a number. This iterative method is called extragradient iterative method. Since then the extragradient iterative method has been generalized and extended by many authors. In 2006, by combining a hybrid iterative method with an extragradient iterative method, Nadezhkina and Takahashi [21] introduced the following hybrid-extragradient iterative method in infinite dimensional Hilbert space H_1:

$$\begin{cases} x_1 = x \in C, \\ y_n = P_C(x_n - \lambda_n A x_n), \\ z_n = \beta_n x_n + (1-\beta_n) S P_C(x_n - \lambda_n A y_n), \\ C_n = \{z \in C : \|z_n - z\|^2 \leq \|x_n - z\|^2\}, \\ Q_n = \{z \in C : \langle x_n - z, x - x_n \rangle \geq 0\}, \\ x_{n+1} = P_{C_n \cap Q_n} x, \end{cases} \tag{13}$$

for every $n = 1, 2, \ldots$. They proved that under certain appropriate conditions imposed on $\{\beta_n\}$ and $\{\lambda_n\}$, the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ generated by (13) converge strongly to $z \in \mathrm{Fix}(S) \cap \mathrm{Sol}(\mathrm{VIP}(2))$. A lot of efficient generalizations and modifications exist at this moment, for instance, see [22, 23, 24, 25].

In this paper, we investigate an iterative method based on hybrid iterative method and extragradient iterative method to approximate a common element of the set of solutions of S$_\mathrm{P}$MVIP(8)-(9), MEP(1) and FPP for a nonexpansive mapping. Further, we establish a strong convergence theorem for the sequences generated by the proposed iterative algorithm. Furthermore, we derive some consequences from our main result. Finally, we justify our main result through a numerical example. The iterative method and result presented in this paper extend and unify the iterative methods and results due to Nadezhkina and Takahashi [21] and Djafari-Rouhani, Kazmi and Rizvi [7].

2. PRELIMINARY

We recall some concepts and results needed in the sequel. Let the symbols $\to$ and $\rightharpoonup$ denote strong and weak convergence, respectively.

It is well known that a real Hilbert space H_1 satisfies

(i) the identity

$$\|\lambda x + (1-\lambda) y\|^2 = \lambda \|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2, \tag{14}$$

for all $x, y \in H_1$ and $\lambda \in [0, 1]$.

(ii) the Opial's condition [26], i.e., for any sequence $\{x_n\}$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{15}$$

holds for every $y \in H_1$ with $y \neq x$;

(iii) the Kadec-Klee property [27], i.e., if $\{x_n\}$ be a sequence in H_1 which satisfies $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$ as $n \to \infty$ then $\|x_n - x\| \to 0$.

For every point $x \in H_1$, there exists a unique nearest point in C denoted by $P_C x$ such that

$$\|x - P_C x\| \leq \|x - y\|, \ \forall y \in C.$$

The mapping P_C is called the *metric projection* of H_1 onto C.

It is well known that P_C is nonexpansive and satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \ \forall x \in H_1. \tag{16}$$

Moreover, $P_C x$ is characterized by the fact $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \leq 0, \ \forall y \in C, \tag{17}$$

which implies that

$$\|x - y\|^2 \geq \|x - P_C x\|^2 + \|y - P_C x\|^2, \ \forall x \in H_1, \ y \in C. \tag{18}$$

It is well known that every nonexpansive operator $T : H_1 \to H_1$ satisfies, for all $(x, y) \in H_1 \times H_1$, the inequality

$$\langle (x - T(x)) - (y - T(y)), T(y) - T(x)\rangle \leq (1/2)\|(T(x) - x) - (T(y) - y\|^2 \tag{19}$$

and therefore, we get, for all $(x, y) \in H_1 \times \mathrm{Fix}(T)$,

$$\langle x - T(x), y - T(x)\rangle \leq (1/2)\|T(x) - x\|^2, \tag{20}$$

see e.g., [[28], Theorem 3.1].

Definition 2.1: A mapping $T : H_1 \to H_1$ is said to be

(i) *monotone*, if

$$\langle Tx - Ty, x - y\rangle \geq 0, \ \forall x, y \in H_1;$$

(ii) *α-inverse strongly monotone*, if there exists a constant $\alpha > 0$ such that

$$\langle Tx - Ty, x - y\rangle \ \geq \alpha\|Tx - Ty\|^2, \ \forall x, y \in H_1;$$

(iii) *β-Lipschitz continuous*, if there exists a constant $\beta > 0$ such that

$$\|Tx - Ty\| \leq \beta\|x - y\|, \ \forall x, y \in H_1.$$

We note that if T is α-inverse strongly monotone mapping, then T is monotone and $\frac{1}{\alpha}$-Lipschitz continuous.

Definition 2.2: A multi-valued mapping $M_1 : H_1 \to 2^{H_1}$ is called *monotone* if for all $x, y \in H_1$, $u \in M_1x$ and $v \in M_1y$ such that

$$\langle x - y, u - v\rangle \geq 0.$$

Definition 2.3: A monotone mapping $M_1 : H_1 \to 2^{H_1}$ is *maximal* if the Graph(M_1) is not properly contained in the graph of any other monotone mapping.

It is known that a monotone mapping M_1 is maximal if and only if for $(x, u) \in H_1 \times H_1$, $\langle x-y, u-v\rangle \geq 0$, for every $(y, v) \in$ Graph(M_1) implies that $u \in M_1x$.

Definition 2.4: Let $M_1 : H_1 \to 2^{H_1}$ be a multi-valued maximal monotone mapping. Then, the resolvent mapping $J_\lambda^{M_1} : H_1 \to H_1$ associated with M_1, is defined by

$$J_\lambda^{M_1}(x) := (I + \lambda M_1)^{-1}(x),\ \forall x \in H_1.$$

Remark 2.1:

(i) For all $\lambda > 0$, the resolvent operator $J_\lambda^{M_1}$ is single-valued, nonexpansive and firmly nonexpansive.

(ii) If we take $M_1 = \partial I_C$, the subdifferential of the indicator function I_C of C, where I_C is defined by

$$I_C(x) = \begin{cases} 0, & x \in C \\ +\infty, & x \notin C, \end{cases}$$

then

$$y = J_\lambda^{\partial I_C}(x) = (I + \lambda\partial I_C)^{-1}x \Leftrightarrow y = P_Cx.$$

Definition 2.5: A mapping $T : H_1 \to H_1$ is said to be *averaged* if and only if it can be written as the average of the identity mapping and a nonexpansive mapping, i.e.,

$$T := (1 - \alpha)I + \alpha S$$

where $\alpha \in (0, 1)$ and $S : H_1 \to H_1$ is nonexpansive and I is the identity operator on H_1.

We note that the firmly nonexpansive mappings (in particular, projection on nonempty closed and convex subset and resolvent operator of maximal monotone operator) are averaged.

The following are some key properties of averaged operators, see for instance [9, 10].

Proposition 2.1:

(i) If $T = (1-\alpha)S + \alpha V$, where $S : H_1 \to H_1$ is averaged, $V : H_1 \to H_1$ is nonexpansive and $\alpha \in (0,1)$, then T is averaged;
(ii) The composite of finitely many averaged mappings is averaged;
(iii) If the mappings $\{T_i\}_{i=1}^N$ are averaged and have a nonempty common fixed point set, then

$$\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(T_1 T_2 ... T_N);$$

(iv) If T is τ-ism, then for $\gamma > 0$, γT is $\frac{\tau}{\gamma}$-ism;
(v) T is averaged if and only if, its complement $I - T$ is τ-ism for some $\tau > \frac{1}{2}$.

Lemma 2.1: [27] Assume that T is nonexpansive self mapping of a closed convex subset C of a Hilbert space H_1. If T has a fixed point, then $I - T$ is demiclosed, i.e., whenever $\{x_n\}$ is a sequence in C converging weakly to some $x \in C$ and the sequence $\{(I-T)x_n\}$ converges strongly to some y, it follows that $(I-T)x = y$.

Assumption 2.1: The bifunction $F : C \times C \longrightarrow \mathbb{R}$ satisfies the following assumptions:

(i) $F(x,x) = 0,\ \forall x \in C$;
(ii) F is monotone, i.e., $F(x,y) + F(y,x) \le 0,\ \forall x \in C$;
(iii) For each $x, y, z \in C$, $\limsup_{t \to 0} F(tz + (1-t)x, y) \le F(x,y)$;
(iv) For each $x \in C$, $y \to F(x,y)$ is convex and lower semicontinuous;

Assumption 2.2: The bifunction $F : C \times C \to \mathbb{R}$ satisfies

$$F(x,y) + F(y,z) + F(z,x) \le 0,\ \forall x, y, z \in C. \tag{21}$$

We easily observe that, for $y = z$, Assumption 2.1 (i) and Assumption 2.2 implies Assumption 2.1 (ii).

Lemma 2.2: [15] Let C be a nonempty closed convex subset of H_1. Assume that $F : C \times C \longrightarrow \mathbb{R}$ satisfying Assumption . For $r > 0$ and for all $x \in H_1$, define a mapping $T_r : H_1 \to C$ as follows:

$$T_r(x) = \{z \in C : F(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \ge 0,\ \forall y \in C\}. \tag{22}$$

Then the following results hold:

(i) For each $x \in H_1$, $T_r(x) \ne \emptyset$;
(ii) T_r is single-valued;

(iii) T_r is firmly nonexpansive, i.e.,

$$\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle, \ \forall x, y \in H_1; \tag{23}$$

(iv) $\mathrm{Fix}(T_r) = \mathrm{Sol}(\mathrm{EP}(3))$;
(v) $\mathrm{Sol}(\mathrm{EP}(3))$ is closed and convex.

Remark 2.2: It follows from Lemma 2.2 (i)-(ii) that

$$rF(T_rx, y) + \langle T_rx - x, y - T_rx\rangle \geq 0, \ \forall y \in C, \ x \in H_1. \tag{24}$$

Further, Lemma 2.2 (iii) implies the nonexpansivity of T_r, i.e.,

$$\|T_rx - T_ry \leq \|x - y\|, \ \forall x, y \in H_1. \tag{25}$$

Furthermore, (24) implies the following inequality

$$\|T_rx - y\|^2 \leq \|x - y\|^2 - \|T_rx - x\|^2 + 2rF(T_rx, y), \ \forall y \in C, x \in H_1. \tag{26}$$

3. Hybrid-extragradient iterative method

We establish a strong convergence theorem for the sequences generated by an iterative algorithm based on hybrid-extragradient iterative method which finds the approximate common element of the set of solution of split monotone variational inclusion problem ($\mathrm{S_PMVIP}$(8)-(9)), mixed equilibrium problem (MEP(1)) and FPP for a nonexpansive mapping S.

Theorem 3.1: Let H_1 and H_2 are real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1 ((i),(iii) and (iv)), and Assumption 2.2; let $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be the multi-valued maximal monotone mappings; let the mappings $A : C \to H_1$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be, respectively, $\sigma, \theta_1, \theta_2$-inverse strongly monotone and let $S : C \to C$ be a nonexpansive mapping such that $\Omega = \mathrm{Sol}(\mathrm{S_PMVIP}(8)\text{-}(9)) \cap \mathrm{Sol}(\mathrm{MEP}(1)) \cap \mathrm{Fix}(S) \neq \emptyset$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{u_n\}$ be generated by the following iterative algorithm:

$$
\begin{aligned}
x^0 &= x \in H_1, \\
y_n &= J_\lambda^{M_1}(I - \lambda f)x_n, && (27)\\
l_n &= J_\lambda^{M_2}(I - \lambda g)By_n, && (28)\\
z_n &= P_C[y_n + \gamma B^*(l_n - By_n)], && (29)\\
w_n &= T_{r_n}(I - r_n A)z_n, && (30)\\
u_n &= \alpha_n x_n + (1-\alpha_n)ST_{r_n}(z_n - r_n A w_n), && (31)\\
C_n &= \{z \in H_1 : \|u_n - z\|^2 \le \|x_n - z\|^2\}, && (32)\\
Q_n &= \{z \in H_1 : \langle x_n - z, x - x_n\rangle \ge 0\}, && (33)\\
x_{n+1} &= P_{C_n \cap Q_n} x, && (34)
\end{aligned}
$$

for $n = 1, 2, ...,$ where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \sigma)$, $\lambda \subset [a', b']$ for some $a', b' \in (0, \theta)$, where $\theta := \min\{\theta_1, \theta_2\}$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\gamma \in \left(0, \frac{1}{\|B^*\|^2}\right)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $z = P_\Omega x$.

Proof. We divide the proof of Theorem 3.1 into the following steps.

Step I. $P_\Omega(x)$ and $\{x_n\}$ are well defined. Further, the sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{w_n\}$, $\{t_n\}$ and $\{u_n\}$ are bounded, where $t_n := T_{r_n}(z_n - r_n A w_n)$.

Proof of Step I. First, we show that $P_\Omega(x)$ is well defined. Since f, g are inverse strongly monotone then $J_\lambda^{M_1}(I - \lambda f)$ and $J_\lambda^{M_2}(I - \lambda g)$ are nonexpansive and hence Sol(MVIP(8))=Fix($J_\lambda^{M_1}(I - \lambda f)$) and Sol(MVIP(9))=Fix($J_\lambda^{M_2}(I - \lambda g)$) are closed and convex sets. Further, it is easy to observe that Sol($\mathrm{S_P}$MVIP(8)-(9)) is closed and convex set. Since A is inverse strongly monotone then $T_{r_n}(I - r_n A)$ is nonexpansive and hence Sol(MEP(1))=Fix($J_{r_n}(I - r_n A)$) is closed and convex. Since $\Omega \neq \emptyset$, Ω is closed and convex set in H_1 and thus $P_\Omega(x)$ is well defined.

Next, we show that $\{x_n\}$ is well defined. Indeed, let $\bar{x} \in \Omega$ then $\bar{x} \in$ Sol($\mathrm{S_P}$MVIP(8)-(9)) and hence $\bar{x} = J_\lambda^{M_1}(I - \lambda f)x$ and

$B\bar{x} = J_\lambda^{M_2}(I-\lambda g)B\bar{x}$. We estimates

$$\begin{aligned}
\|y_n-\bar{x}\|^2 &= \|J_\lambda^{M_1}(x_n-\lambda f x_n) - J_\lambda^{M_1}(\bar{x}-\lambda f\bar{x})\|^2 \\
&\le \|(x_n-\bar{x})-\lambda(f x_n - f\bar{x})\|^2 \\
&= \|x_n-\bar{x}\|^2 + \lambda^2\|f x_n - f\bar{x}\|^2 + 2\lambda\langle x_n-\bar{x}, f x_n - f\bar{x}\rangle \\
&\le \|x_n-\bar{x}\|^2 - \lambda(2\theta_1-\lambda)\|f x_n - f\bar{x}\|^2 \qquad (35)\\
&\le \|x_n-\bar{x}\|^2; \qquad (36)
\end{aligned}$$

$$\begin{aligned}
\|l_n-B\bar{x}\|^2 &= \|J_\lambda^{M_2}(I-\lambda g)By_n - J_\lambda^{M_2}(I-\lambda g)B\bar{x}\|^2 \\
&\le \|By_n-B\bar{x}\|^2 - \lambda(2\theta_2-\lambda)\|gBy_n - gB\bar{x}\|^2 \qquad (37)\\
&\le \|By_n-B\bar{x}\|^2; \qquad (38)
\end{aligned}$$

$$\begin{aligned}
\|z_n-\bar{x}\|^2 &= \|P_C[y_n+\gamma B^*(l_n-By_n)]-\bar{x}\|^2 \\
&\le \|y_n+\gamma B^*(l_n-By_n)-\bar{x})\|^2 \\
&= \|y_n-\bar{x}\|^2 + \|\gamma B^*(l_n-By_n)\|^2 \\
&\quad +2\gamma\langle y_n-\bar{x}, B^*(l_n-By_n)\rangle
\end{aligned}$$

$$\begin{aligned}
&\le \|y_n-\bar{x}\|^2 + \gamma^2\|B^*\|^2\|l_n-By_n\|^2 \\
&\quad +2\gamma\langle B(y_n-\bar{x})+(l_n-By_n)-(l_n-By_n), l_n-By_n\rangle \\
&= \|y_n-\bar{x}\|^2 + \gamma^2\|B^*\|^2\|l_n-By_n\|^2 + 2\gamma\big[\frac{1}{2}\|l_n-B\bar{x}\|^2 \\
&\quad +\frac{1}{2}\|l_n-By_n\|^2 - \frac{1}{2}\|By_n-B\bar{x}\|^2 - \frac{1}{2}\|l_n-By_n\|^2\big] \\
&= \|y_n-\bar{x}\|^2 - \gamma(1-\gamma\|B^*\|^2)\|l_n-By_n\|^2 \qquad (39)\\
&\le \|y_n-\bar{x}\|^2 \le \|x_n-\bar{x}\|^2; \qquad (40)
\end{aligned}$$

and

$$\begin{aligned}
\|w_n-\bar{x}\|^2 &= \|T_{r_n}(z_n-r_nAz_n) - T_{r_n}(\bar{x}-r_nA\bar{x})\|^2 \\
&\le \|(z_n-\bar{x})-r_n(Az_n-A\bar{x})\|^2 \\
&= \|z_n-\bar{x}\|^2 + r_n^2\|Az_n-A\bar{x}\|^2 \\
&\quad -2r_n\langle z_n-\bar{x}, Az_n-A\bar{x}\rangle \\
&\le \|z_n-\bar{x}\|^2 - r_n(2\sigma-r_n)\|Az_n-A\bar{x}\|^2 \qquad (41)\\
&\le \|z_n-\bar{x}\|^2 \le \|x_n-\bar{x}\|^2. \qquad (42)
\end{aligned}$$

Now, evidently Q_n is closed and convex for every $n = 0, 1, 2, \ldots$. Further, since

$$C_n := \{z \in H_1 : \|u_n-x_n\|^2 + 2\langle u_n-x_n, x_n-z\rangle \le 0\}, \qquad (43)$$

then we observe that C_n is closed and convex for every $n = 0, 1, 2, \ldots$. Hence $C_n \cap Q_n$ are closed and convex for all n. Further, we claim that $C_n \cap Q_n$ is nonempty for all n. For this, it is enough to show that $\Omega \subset C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$. Let $\bar{x} \in \Omega$ then $\bar{x}$ is a solution of MEP(1) and hence

$$F(\bar{x}, w_n) + \langle A\bar{x}, w_n - \bar{x}\rangle \geq 0, \ \forall w_n \in C. \tag{44}$$

Applying (26) with $z_n - r_nAw_n$ and $\bar{x}$, we have

$$\begin{aligned}
\|t_n \quad - \quad & \bar{x}\|^2 \\
\leq \quad & \|z_n - r_nAw_n - \bar{x}\|^2 - \|t_n - (z_n - r_nAw_n)\|^2 + 2r_nF(t_n, \bar{x}) \\
= \quad & \|z_n - \bar{x}\|^2 - \|t_n - z_n\|^2 + 2r_n\langle Aw_n, \bar{x} - t_n\rangle + 2r_nF(t_n, \bar{x}) \\
= \quad & \|z_n - \bar{x}\|^2 - \|t_n - z_n\|^2 + 2r_n\big[\langle Aw_n - A\bar{x}, \bar{x} - w_n\rangle \\
& + \langle A\bar{x}, \bar{x} - w_n\rangle - \langle Aw_n, t_n - w_n\rangle\big] + 2r_nF(t_n, \bar{x}). \qquad (45)
\end{aligned}$$

Since A is σ-inverse strongly monotone, then A is monotone and $\frac{1}{\sigma}$-Lipschitz continuous. Using (24), (44) and monotonicity of A in (45), we have

$$\begin{aligned}
\|t_n - \bar{x}\|^2 \quad \leq \quad & \|z_n - \bar{x}\|^2 - \|t_n - z_n\|^2 + 2r_n\langle Aw_n, w_n - t_n\rangle \\
& + 2r_n\big[F(\bar{x}, w_n) + F(t_n, \bar{x})\big] \\
\leq \quad & \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 \\
& - 2\langle z_n - w_n, w_n - t_n\rangle + 2r_n\langle Aw_n, w_n - t_n\rangle \\
& + 2r_n\big[F(\bar{x}, w_n) + F(t_n, \bar{x})\big] \\
= \quad & \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 \\
& - 2\langle w_n - (z_n - r_nAz_n), t_n - w_n\rangle \\
& + 2r_n\langle Az_n - Aw_n, t_n - w_n\rangle + 2r_n\big[F(\bar{x}, w_n) + F(t_n, \bar{x})\big] \\
= \quad & \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 \\
& + 2r_n\langle Az_n - Aw_n, t_n - w_n\rangle \\
& + 2r_n\big[F(\bar{x}, w_n) + F(w_n, t_n) + F(t_n, \bar{x})\big].
\end{aligned}$$

Now, using Assumption 2.2 in the above inequality, we have

$$\begin{aligned}
\|t_n - \bar{x}\|^2 \quad \leq \quad & \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 \\
& + 2r_n\frac{1}{\sigma}\|z_n - w_n\|\|t_n - w_n\| \qquad (46) \\
\leq \quad & \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 \\
& + \|w_n - t_n\|^2 + \left(\frac{r_n}{\sigma}\right)^2 \|z_n - w_n\|^2 \\
\leq \quad & \|z_n - \bar{x}\|^2 - \left(1 - \left(\frac{r_n}{\sigma}\right)^2\right) \|z_n - w_n\|^2. \qquad (47)
\end{aligned}$$

Since $r_n \in [a, b]$, we obtain

$$\|t_n - \bar{x}\|^2 \leq \|z_n - \bar{x}\|^2 \leq \|y_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2. \tag{48}$$

Since $\bar{x} \in \Omega$ then $\bar{x} = S\bar{x}$ and we have the following

$$\begin{aligned}
\|u_n - \bar{x}\|^2 &= \|\alpha_n x_n + (1-\alpha_n)St_n - \bar{x}\|^2 \\
&= \|\alpha_n(x_n - \bar{x}) + (1-\alpha_n)(St_n - \bar{x})\|^2 \\
&= \alpha_n\|x_n - \bar{x}\|^2 + (1-\alpha_n)\|St_n - \bar{x}\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|St_n - x_n\|^2 \\
&\leq \alpha_n\|x_n - \bar{x}\|^2 + (1-\alpha_n)\|St_n - \bar{x}\|^2 \\
&\leq \alpha_n\|x_n - \bar{x}\|^2 + (1-\alpha_n)\|t_n - \bar{x}\|^2 \qquad (49) \\
&\leq \alpha_n\|x_n - \bar{x}\|^2 + (1-\alpha_n)\|x_n - \bar{x}\|^2 \\
&= \|x_n - \bar{x}\|^2. \qquad (50)
\end{aligned}$$

This implies that $\bar{x} \in C_n$ and hence $\Omega \subseteq C_n$ for every $n = 0, 1, 2, \ldots$. Further, since $\Omega \subseteq C_0$ and $\Omega \subseteq Q_0 = H_1$. It follows that $\Omega \subset C_0 \cap Q_0$ and hence $C_0 \cap Q_0$ is nonempty closed and convex set. Therefore $x_1 = P_{C_0 \cap Q_0}x$ is well defined. Now suppose that $\Omega \subseteq C_{n-1} \cap Q_{n-1}$ for some $n > 1$. Let $x_n = P_{C_{n-1} \cap Q_{n-1}}x$. Again, since $\Omega \subseteq C_n$ and for any $\bar{x} \in \Omega$, it follows from (17) that $\langle x - x_n, x_n - \bar{x}\rangle = \langle x - P_{C_{n-1} \cap Q_{n-1}}x, P_{C_{n-1} \cap Q_{n-1}}x - \bar{x}\rangle \geq 0$, and hence $\bar{x} \in Q_n$. Therefore $\Omega \subseteq C_n \cap Q_n$ for every $n = 0, 1, 2, \ldots$ and hence $x_{n+1} = P_{C_n \cap Q_n}x$ is well defined for every $n = 0, 1, 2, \ldots$. Thus the sequence $\{x_n\}$ is well defined.

Let $l = P_\Omega x$. From $x_{n+1} = P_{C_n \cap Q_n}x$ and $l \in \Omega \subset C_n \cap Q_n$, we have

$$\|x_{n+1} - x\| \leq \|l - x\|, \tag{51}$$

for every $n = 0, 1, 2, \ldots$. Therefore $\{x_n\}$ is bounded. Further, it follows from (36), (38), (40), (42), (48) and (50) that the sequences $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{w_n\}$, $\{t_n\}$ and $\{u_n\}$ are bounded.

Step II. $\lim\limits_{n\to\infty}\|x_{n+1} - x_n\| = \lim\limits_{n\to\infty}\|z_n - x_n\| = \lim\limits_{n\to\infty}\|u_n - x_n\| = \lim\limits_{n\to\infty}\|x_n - y_n\| = \lim\limits_{n\to\infty}\|x_n - t_n\| = \lim\limits_{n\to\infty}\|St_n - t_n\| = 0.$

Proof of Step II. It follows from (33) and (34) that $x_n = P_{Q_n}x$, and $x_{n+1} \in C_n \cap Q_n$. Hence, we have

$$\|x_n - x\| \leq \|x_{n+1} - x\|, \tag{52}$$

for every $n = 0, 1, 2, \ldots$. Further, it follows from (51) and (52) that the sequence $\{\|x_n - x\|\}$ is monotonically increasing and bounded, and hence convergent. Therefore $\lim\limits_{n\to\infty}\|x_n - x\|$ exists.

Now, applying (18) with $x_n = P_{Q_n}x$ and $x_{n+1} \in Q_n$, we have

$$\|x_{n+1} - x_n\|^2 \leq \|x_{n+1} - x\|^2 - \|x_n - x\|^2,$$

for every $n = 0, 1, 2, \ldots$. This implies that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{53}$$

Since $x_{n+1} \in C_n$, it follows from (43) that

$$\begin{aligned}\|u_n - x_n\|^2 &\leq 2\langle u_n - x_n, x_{n+1} - x_n\rangle \\ &\leq 2\|u_n - x_n\|\|x_{n+1} - x_n\|.\end{aligned}$$

Therefore

$$\|u_n - x_n\| \leq 2\|x_{n+1} - x_n\|,$$

and hence, using (53), we have

$$\lim_{n\to\infty} \|u_n - x_n\| = 0. \tag{54}$$

It follows from (47) and (49) that

$$\begin{aligned}\|z_n &- w_n\|^2 \\ &\leq \left[(1-\alpha_n)(1-(\frac{r_n}{\sigma})^2)\right]^{-1}(\|x_n - \bar{x}\|^2 - \|u_n - \bar{x}\|^2) \\ &= \left[(1-\alpha_n)(1-(\frac{r_n}{\sigma})^2)\right]^{-1}(\|x_n - \bar{x}\| - \|u_n - \bar{x}\|) \\ &\quad \times(\|x_n - \bar{x}\| + \|u_n - \bar{x}\|) \\ &\leq \left[(1-\alpha_n)(1-(\frac{r_n}{\sigma})^2)\right]^{-1}\|x_n - u_n\|(\|x_n - \bar{x}\| + \|u_n - \bar{x}\|).\end{aligned}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and $\lim_{n\to\infty} \|u_n - x_n\| = 0$, therefore above inequality implies that

$$\lim_{n\to\infty} \|z_n - w_n\| = 0. \tag{55}$$

By the same arguments used as in (46), we have

$$\begin{aligned}\|t_n - \bar{x}\|^2 &\leq \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 \\ &\quad + \frac{2r_n}{\sigma}\|z_n - w_n\|\|t_n - w_n\| \\ &\leq \|z_n - \bar{x}\|^2 - \|z_n - w_n\|^2 - \|w_n - t_n\|^2 + \|z_n - w_n\|^2 \\ &\quad + \left(\frac{r_n}{\sigma}\right)^2\|t_n - w_n\|^2 \\ &= \|z_n - \bar{x}\|^2 - \left[1 - \left(\frac{r_n}{\sigma}\right)^2\right]\|w_n - t_n\|^2. \\ &\leq \|x_n - \bar{x}\|^2 - \left[1 - \left(\frac{r_n}{\sigma}\right)^2\right]\|w_n - t_n\|^2. \end{aligned} \tag{56}$$

Further, using (56) in (49), we have

$$\|u_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - (1-\alpha_n)\left(1 - \left(\frac{r_n}{\sigma}\right)^2\right)\|w_n - t_n\|^2,$$

which implies that

$$\begin{aligned}\|t_n \quad - \quad & w_n\|^2 \\ \leq \quad & \left[(1-\alpha_n)(1-(\frac{r_n}{\sigma})^2)\right]^{-1}(\|x_n - \bar{x}\|^2 - \|u_n - \bar{x}\|^2) \\ = \quad & \left[(1-\alpha_n)(1-(\frac{r_n}{\sigma})^2)\right]^{-1}(\|x_n - \bar{x}\| - \|u_n - \bar{x}\|) \\ & \times(\|x_n - \bar{x}\| + \|u_n - \bar{x}\|) \\ \leq \quad & [(1-\alpha_n)(1-(\frac{r_n}{\sigma})^2)]^{-1}(\|x_n - \bar{x}\| + \|u_n - \bar{x}\|) \\ & \times\|x_n - u_n\|. \end{aligned} \tag{57}$$

Again, since $\{x_n\}$ and $\{u_n\}$ are bounded and $\lim_{n\to\infty}\|u_n - x_n\| = 0$, therefore (57) implies that

$$\lim_{n\to\infty}\|t_n - w_n\| = 0. \tag{58}$$

Next, it follows from (35), (48) and (49) that

$$\|u_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - (1-\alpha_n)\lambda(2\theta_1 - \lambda)\|fx_n - f\bar{x}\|^2,$$

which implies that

$$\begin{aligned}\|fx_n \quad - \quad & f\bar{x}\|^2 \\ \leq \quad & [(1-\alpha_n)\lambda(2\theta_1 - \lambda)]^{-1}(\|x_n - \bar{x}\|^2 - \|u_n - \bar{x}\|^2) \\ = \quad & [(1-\alpha_n)\lambda(2\theta_1 - \lambda)]^{-1}(\|x_n - \bar{x}\| - \|u_n - \bar{x}\|) \\ & \times(\|x_n - \bar{x}\| + \|u_n - \bar{x}\|) \\ \leq \quad & [(1-\alpha_n)\lambda(2\theta_1 - \lambda)]^{-1}(\|x_n - \bar{x}\| \\ & +\|u_n - \bar{x}\|)\|x_n - u_n\|. \end{aligned} \tag{59}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and $\lim_{n\to\infty}\|u_n - x_n\| = 0$, therefore (59) implies that

$$\lim_{n\to\infty}\|fx_n - f\bar{x}\| = 0. \tag{60}$$

Further, it follows from (39), (48) and (49) that

$$\|u_n - \bar{x}\|^2 \leq \|x_n - \bar{x}\|^2 - (1-\alpha_n)\gamma(1-\gamma\|B^*\|^2)\|l_n - By_n\|^2,$$

which implies that

$$\begin{aligned} \|l_n &- By_n\|^2 \\ &\leq \left[(1-\alpha_n)\gamma(1-\gamma\|B^*\|^2)\right]^{-1}\left(\|x_n-\bar{x}\|^2-\|u_n-\bar{x}\|^2\right) \\ &= \left[(1-\alpha_n)\gamma(1-\gamma\|B^*\|^2)\right]^{-1}\left(\|x_n-\bar{x}\|-\|u_n-\bar{x}\|\right) \\ &\quad \times\left(\|x_n-\bar{x}\|+\|u_n-\bar{x}\|\right) \\ &\leq \left[(1-\alpha_n)\gamma(1-\gamma\|B^*\|^2)\right]^{-1}\left(\|x_n-\bar{x}\|\right. \\ &\quad \left.+\|u_n-\bar{x}\|\right)\|x_n-u_n\|. \end{aligned} \tag{61}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and $\lim_{n\to\infty}\|u_n-x_n\|=0$, therefore (61) implies that

$$\lim_{n\to\infty}\|l_n-By_n\|=0. \tag{62}$$

Next, the inequality (37), i.e.,

$$\|l_n-B\bar{x}\|^2\leq\|By_n-B\bar{x}\|^2-\lambda(2\theta_2-\lambda)\|gBy_n-gB\bar{x}\|^2,$$

implies that

$$\begin{aligned} \|gBy_n &- gB\bar{x}\|^2 \\ &\leq [\lambda(2\theta_2-\lambda)]^{-1}\left(\|By_n-B\bar{x}\|^2-\|l_n-B\bar{x}\|^2\right) \\ &= [\lambda(2\theta_2-\lambda)]^{-1}\left(\|By_n-B\bar{x}\|-\|l_n-B\bar{x}\|\right) \\ &\quad \times\left(\|By_n-B\bar{x}\|^2+\|l_n-B\bar{x}\|\right) \\ &\leq [\lambda(2\theta_2-\lambda)]^{-1}\left(\|By_n-\bar{x}\|+\|l_n-B\bar{x}\|\right) \\ &\quad \times\|By_n-l_n\|. \end{aligned} \tag{63}$$

Since $\{y_n\}$ and $\{l_n\}$ are bounded and $\lim_{n\to\infty}\|l_n-By_n\|=0$, therefore (63) implies that

$$\lim_{n\to\infty}\|gBy_n-gB\bar{x}\|=0. \tag{64}$$

Next, by using the firmly nonexpansivity of $J_\lambda^{M_1}$ and arguments used in (36), we estimate

$$\begin{aligned}
\|y_n-\bar{x}\|^2 &= \|J_\lambda^{M_1}(I-\lambda f)x_n - J_\lambda^{M_1}(I-\lambda f)\bar{x})\|^2\\
&\le \langle (I-\lambda f)x_n-(I-\lambda f)\bar{x}, y_n-\bar{x}\rangle\\
&= \frac{1}{2}\Bigg[\|(I-\lambda f)x_n-(I-\lambda f)\bar{x}\|^2+\|y_n-\bar{x}\|^2\\
&\quad -\|x_n-y_n-\lambda(fx_n-f\bar{x})\|^2\Bigg]\\
&\le \frac{1}{2}\Bigg[\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2-\|x_n-y_n\|^2\\
&\quad +2\lambda\langle x_n-y_n, fx_n-f\bar{x}\rangle-\lambda^2\|fx_n-f\bar{x}\|^2\Bigg]\\
&\le \frac{1}{2}\big[\|x_n-\bar{x}\|^2+\|y_n-\bar{x}\|^2-\|x_n-y_n\|^2\\
&\quad +2\lambda\|x_n-y_n\|\|fx_n-f\bar{x}\|\big],
\end{aligned}$$

which in turns yields

$$\|y_n-\bar{x}\|^2\le\|x_n-\bar{x}\|^2-\|x_n-y_n\|^2+2\lambda\|x_n-y_n\|\|fx_n-f\bar{x}\|. \quad (65)$$

It follows from (48), (49) and (65) that

$$\begin{aligned}
\|u_n-\bar{x}\|^2 &\le \|x_n-\bar{x}\|^2-(1-\alpha_n)\|x_n-y_n\|^2\\
&\quad +2\lambda(1-\alpha_n)\|x_n-y_n\|\|fx_n-f\bar{x}\|,
\end{aligned}$$

which implies that

$$\begin{aligned}
\|x_n &- y_n\|^2\\
&\le (1-\alpha_n)^{-1}\Bigg[\|x_n-\bar{x}\|^2-\|u_n-\bar{x}\|^2\\
&\quad +2\lambda(1-\alpha_n)\|x_n-y_n\|\|fx_n-f\bar{x}\|\Bigg]\\
&= (1-\alpha_n)^{-1}\big[(\|x_n-\bar{x}\|-\|u_n-\bar{x}\|)(\|x_n-\bar{x}\|+\|u_n-\bar{x}\|)\\
&\quad +2\lambda(1-\alpha_n)\|x_n-y_n\|\|fx_n-f\bar{x}\|\big]\\
&\le (1-\alpha_n)^{-1}\big[(\|x_n-\bar{x}\|+\|u_n-\bar{x}\|)\|x_n-u_n\|\\
&\quad +2\lambda(1-\alpha_n)\|x_n-y_n\|\|fx_n-f\bar{x}\|\big]. \qquad (66)
\end{aligned}$$

Since $\{x_n\}$ and $\{u_n\}$ are bounded and $\lim\limits_{n\to\infty}\|u_n-x_n\|=0$ and $\lim\limits_{n\to\infty}\|fx_n-f\bar{x}\|=0$, therefore (66) implies that

$$\lim_{n\to\infty}\|x_n-y_n\|=0. \quad (67)$$

Further, using the firmly nonexpansivity of P_C, we estimate

$$\begin{aligned}
\|z_n-\bar{x}\|^2 &= \|P_C[y_n+\gamma B^*(l_n-By_n)]-\bar{x}\|^2\\
&\le \langle y_n+\gamma B^*(l_n-By_n)-\bar{x}, z_n-\bar{x}\rangle\\
&= \frac{1}{2}\Big[\|(y_n-\bar{x})+\gamma B^*(l_n-By_n)\|^2+\|z_n-\bar{x}\|^2\\
&\quad -\|y_n+\gamma B^*(l_n-By_n)-\bar{x}-z_n+\bar{x}\|^2\Big]\\
&= \frac{1}{2}\Big[\|y_n-\bar{x}\|^2+\|z_n-\bar{x}\|^2+\|\gamma B^*(l_n-By_n)\|^2\\
&\quad +2\gamma\langle By_n-B\bar{x}, l_n-By_n\rangle\\
&\quad -\|(y_n-z_n)+\gamma B^*(l_n-By_n)\|^2\Big]\\
&\le \frac{1}{2}\Big[\|y_n-\bar{x}\|^2+\|z_n-\bar{x}\|^2+\|\gamma B^*(l_n-By_n)\|^2\\
&\quad +2\gamma\|By_n-B\bar{x}\|\|l_n-By_n\|-\|y_n-z_n\|^2\\
&\quad -\|\gamma B^*(l_n-By_n)\|^2-2\gamma\langle y_n-z_n, B^*(l_n-By_n)\rangle\Big],
\end{aligned}$$

which in turns yields

$$\begin{aligned}
\|z_n-\bar{x}\|^2 &\le \|y_n-\bar{x}\|^2-\|y_n-z_n\|^2+2\gamma\|By_n-B\bar{x}\|\|l_n-By_n\|\\
&\quad +2\gamma\|y_n-z_n\|\|B^*\|\|l_n-By_n\|\\
&\le \|y_n-\bar{x}\|^2-\|y_n-z_n\|^2\\
&\quad +2\gamma\|l_n-By_n\|(\|By_n-B\bar{x}\|+\|B^*\|\|y_n-z_n\|) \qquad (68)
\end{aligned}$$

It follows from (48), (49) and (68) that

$$\begin{aligned}
\|u_n-\bar{x}\|^2 &\le \|x_n-\bar{x}\|^2-(1-\alpha_n)\|y_n-z_n\|^2\\
&\quad +2\gamma(1-\alpha_n)[\|l_n-By_n\|(\|By_n-B\bar{x}\|\\
&\quad +\|B^*\|\|y_n-z_n\|)],
\end{aligned}$$

which implies that

$$\begin{aligned}
\|y_n &- z_n\|^2\\
&\le (1-\alpha_n)^{-1}\big[\|x_n-\bar{x}\|^2-\|u_n-\bar{x}\|^2\\
&\quad +2\gamma(1-\alpha_n)\|l_n-By_n\|(\|By_n-B\bar{x}\|+\|B^*\|\|y_n-z_n\|)\big]\\
&= (1-\alpha_n)^{-1}\big[\,(\|x_n-\bar{x}\|-\|u_n-\bar{x}\|)\,(\|x_n-\bar{x}\|+\|u_n-\bar{x}\|)\\
&\quad +2\gamma(1-\alpha_n)\|l_n-By_n\|(\|By_n-B\bar{x}\|+\|B^*\|\|y_n-z_n\|)\big]\\
&\le (1-\alpha_n)^{-1}\big[\,(\|x_n-\bar{x}\|+\|u_n-\bar{x}\|)\,\|x_n-u_n\|\\
&\quad +2\gamma(1-\alpha_n)\|l_n-By_n\|(\|By_n-B\bar{x}\|\\
&\quad +\|B^*\|\|y_n-z_n\|)\big]. \qquad (69)
\end{aligned}$$

Since $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$ are bounded and $\lim_{n\to\infty} \|u_n - x_n\| = 0$ and $\lim_{n\to\infty} \|l_n - By_n\| = 0$, therefore (69) implies that

$$\lim_{n\to\infty} \|y_n - z_n\| = 0. \tag{70}$$

Since

$$\|x_n - z_n\| \leq \|x_n - y_n\| + \|y_n - z_n\|,$$

then using (67) and (70), we have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{71}$$

Since

$$\|w_n - x_n\| \leq \|w_n - z_n\| + \|z_n - x_n\|,$$

then using (55) and (71), we have

$$\lim_{n\to\infty} \|w_n - x_n\| = 0. \tag{72}$$

Since

$$\|t_n - x_n\| \leq \|t_n - w_n\| + \|w_n - x_n\|,$$

then using (58) and (72), we have

$$\lim_{n\to\infty} \|t_n - x_n\| = 0. \tag{73}$$

Next, we show that $\lim_{n\to\infty} \|St_n - t_n\| = 0$. Since

$$u_n = \alpha_n x_n + (1 - \alpha_n) St_n,$$

therefore

$$\begin{aligned} u_n - x_n &= \alpha_n x_n + (1 - \alpha_n) St_n - x_n \\ &= (1 - \alpha_n)(St_n - x_n), \end{aligned}$$

which implies that

$$(1 - \alpha_n)\|St_n - x_n\| = \|u_n - x_n\|.$$

Since $\alpha_n \in [0, c]$ and $c \in [0, 1)$, it follows from above equality that

$$\begin{aligned} (1 - c)\|St_n - x_n\| &\leq (1 - \alpha_n)\|St_n - x_n\| \\ &= \|u_n - x_n\|. \end{aligned}$$

Since $\lim_{n\to\infty} \|u_n - x_n\| = 0$, we have

$$\lim_{n\to\infty} \|St_n - x_n\| = 0.$$

Further, it follows from

$$\|St_n - t_n\| \leq \|St_n - x_n\| + \|x_n - t_n\|,$$

$\lim_{n\to\infty} \|St_n - x_n\| = 0$ and $\lim_{n\to\infty} \|x_n - t_n\| = 0$ that

$$\lim_{n\to\infty} \|St_n - t_n\| = 0. \tag{74}$$

Step III. The weak limit of weakly convergent sequence of $\{x_n\}$ belongs to Ω.

Proof of Step III. Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$, say. It follows from (67) and (73) that the sequences $\{x_n\}$, $\{y_n\}$ and $\{t_n\}$ have the same asymptotic behavior, therefore, there exist subsequences $\{y_{n_k}\}$ of $\{y_n\}$ and $\{t_{n_k}\}$ of $\{t_n\}$ such that $y_{n_k} \rightharpoonup \hat{x}$ and $t_{n_k} \rightharpoonup \hat{x}$.

Now, we show that $\hat{x} \in \mathrm{Fix}(S)$. On contrary, we assume that $\hat{x} \notin \mathrm{Fix}(S)$. Since $S\hat{x} \neq \hat{x}$, then from Opial's condition (15) and (50), we have

$$\begin{aligned} \liminf_{k\to\infty} \|t_{n_k} - \hat{x}\| &< \liminf_{k\to\infty} \|t_{n_k} - S\hat{x}\| \\ &\leq \liminf_{k\to\infty} \left\{\|t_{n_k} - St_{n_k}\| + \|St_{n_k} - S\hat{x}\|\right\} \\ &\leq \liminf_{k\to\infty} \|t_{n_k} - \hat{x}\|, \end{aligned}$$

which is a contradiction. Thus, $\hat{x} \in \mathrm{Fix}(S)$. On the other hand $y_{n_k} = J_\lambda^{M_1}(x_{n_k} - \lambda f(x_{n_k}))$ can be rewritten as

$$\frac{(x_{n_k} - y_{n_k}) - \lambda f(x_{n_k})}{\lambda} \in M_1 y_{n_k}. \tag{75}$$

By passing to the limit $k \to \infty$ in (75) and by taking account (67) and the fact that f is $\frac{1}{\theta_1}$-Lipschitz continuous and the graph of maximal monotone operator is weakly-strongly closed, we obtain $0 \in M_1(\hat{x}) + f(\hat{x})$, i.e., $\hat{x} \in \mathrm{Sol}(\mathrm{MVIP}(8))$. Further, again since $\{x_n\}$ and $\{y_n\}$ have the same asymptotical behavior, $\{By_n\}$ weakly converges to $B\hat{x}$. By (62) and the fact that the mapping $J_\lambda^{M_2}(I-\lambda g)$ is nonexpansive and Lemma 2.1 that $0 \in M_2(B\hat{x}) + g(B\hat{x})$, i.e., $B\hat{x} \in \mathrm{Sol}(\mathrm{MVIP}(9))$.

Next, we show $\hat{x} \in \mathrm{Sol}(\mathrm{MEP}(1))$. The relation $w_n = T_{r_n}(z_n - r_n A z_n)$ implies

$$F(w_n, y) + \langle Az_n, y - w_n\rangle + \frac{1}{r_n}\langle y - w_n, w_n - z_n\rangle \geq 0,\ \forall y \in C.$$

Since F is monotone, the above inequality implies

$$\langle Az_n, y - w_n\rangle + \frac{1}{r_n}\langle y - w_n, w_n - z_n\rangle \geq F(y, w_n),\ \forall y \in C.$$

Hence,

$$\langle Az_{n_k}, y-w_{n_k}\rangle+\left\langle y-w_{n_k}, \frac{w_{n_k}-z_{n_k}}{r_{n_k}}\right\rangle \geq F(y,w_{n_k}),\ \forall y\in C. \quad (76)$$

For t with $0<t\leq 1$, let $y_t = ty+(1-t)\hat{x}\in C$. So, from (76), we have

$$\begin{aligned}\langle y_t \ &- \ w_{n_k}, Ay_t\rangle \\ &\geq \langle y_t-w_{n_k}, Ay_t\rangle - \langle y_t-w_{n_k}, Az_{n_k}\rangle \\ &\quad -\left\langle y_t-w_{n_k}, \frac{w_{n_k}-z_{n_k}}{r_{n_k}}\right\rangle + F(y_t,w_{n_k}) \\ &= \langle y_t-w_{n_k}, Ay_t-Aw_{n_k}\rangle + \langle y_t-w_{n_k}, Aw_{n_k}-Az_{n_k}\rangle \\ &\quad -\left\langle y_t-w_{n_k}, \frac{w_{n_k}-z_{n_k}}{r_{n_k}}\right\rangle + F(y_t,w_{n_k}).\end{aligned}$$

Since $\lim_{k\to\infty}\|w_{n_k}-z_{n_k}\|=0$ and A is Lipschitz continuous, we have $\lim_{k\to\infty}\|Aw_{n_k}-Az_{n_k}\|=0$. Further, from the monotonicity of A and the convexity and lower semicontinuity of F, $\frac{w_{n_k}-z_{n_k}}{r_{n_k}}\to 0$ and $w_{n_k}\rightharpoonup \hat{x}$, we have

$$\langle y_t-\hat{x}, Ay_t\rangle \geq F(y_t,\hat{x}), \quad (77)$$

as $k\to\infty$. Furthermore, we have

$$\begin{aligned} 0 &\leq F(y_t,y_t) \\ &\leq tF(y_t,y)+(1-t)F(y_t,\hat{x}) \\ &\leq tF(y_t,y)+(1-t)\langle y_t-\hat{x}, Ay_t\rangle \\ &= tF(y_t,y)+(1-t)t\langle y-\hat{x}, Ay_t\rangle \end{aligned}$$

and hence

$$0\leq F(y_t,y)+(1-t)\langle y-\hat{x}, Ay_t\rangle.$$

Letting $t\to 0_+$ then, for each $y\in C$, we have

$$F(\hat{x},y)+\langle y-\hat{x}, A\hat{x}\rangle \geq 0.$$

This implies that $\hat{x}\in \mathrm{Sol(MEP(1))}$. Hence $\hat{x}\in\Omega$.

Step IV. $\{x_n\}$ strongly converges to $\hat{x}=P_\Omega x$.

Proof of Step IV: It follows from $l = P_\Omega x$, $\hat{x} \in \Omega$, (51) and (52) we have

$$\|l-x\| \leq \|\hat{x}-x\| \leq \lim\inf_{k\to\infty} \|x_{n_k}-x\| \leq \lim\sup_{k\to\infty} \|x_{n_k}-x\| \leq \|l-x\|.$$

Thus, we have

$$\lim_{k\to\infty} \|x_{n_k} - x\| = \|\hat{x} - x\|.$$

Since $x_{n_k} - x \rightharpoonup \hat{x} - x$ and from Kadec-Klee property of Hilbert space, we have $x_{n_k}-x \to \hat{x}-x$ and hence $x_{n_k} \to \hat{x}$. Since $x_n = P_{Q_n}x$ and $l \in \Omega \subset C_n \cap Q_n \subset Q_n$, on using (33), we have

$$-\|l - x_{n_k}\|^2 = \langle l - x_{n_k}, x_{n_k} - x\rangle + \langle l - x_{n_k}, x - l\rangle \geq \langle l - x_{n_k}, x - l\rangle.$$

As $k \to \infty$, we obtain $-\|l-\hat{x}\|^2 \geq \langle l-\hat{x}, x-l\rangle \geq 0$ by $l = P_\Omega x$ and $\hat{x} \in \Omega$. Hence we have $\hat{x} = l$. This implies that $x_n \to l$. Further, it is easy to see $u_n \to l$, $y_n \to l$, $z_n \to l$ and $t_n \to l$.

This completes the proof Theorem 3.1.

Now, we derive some consequences from Theorem 3.1. First, we derive the following strong convergence theorem for the sequences generated by an iterative algorithm which finds the approximate common element of the set of solution of split variational inequality problem ($\mathrm{S_P}$VIP(5)-(6)), mixed equilibrium problem (MEP(1)) and FPP for a nonexpansive mapping S.

Corollary 3.1: Let H_1 and H_2 are real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1((i),(iii) and (iv)), and Assumption 2.2; and let the mappings $A : C \to H_1$, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ be respectively, $\sigma, \theta_1, \theta_2$-inverse strongly monotone and let $S : C \to C$ be a nonexpansive mapping such that $\Omega = \mathrm{Sol}(\mathrm{S_P}\mathrm{VIP}(5)\text{-}(6)) \cap \mathrm{Sol}(\mathrm{MEP}(1)) \cap \mathrm{Fix}(S) \neq \emptyset$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{u_n\}$ be generated by the following iterative algorithm:

$$\begin{aligned} x^0 &= x \in H_1, \\ y_n &= P_C(I - \lambda f)x_n, \\ l_n &= P_C(I - \lambda g)By_n, \\ z_n &= P_C[y_n + \gamma B^*(l_n - By_n)], \\ w_n &= T_{r_n}(I - r_nA)z_n, \end{aligned} \tag{78}$$

$$\begin{aligned} u_n &= \alpha_n x_n + (1-\alpha_n) S T_{r_n}(z_n - r_n A w_n), \\ C_n &= \{z \in H_1 : \|u_n - z\|^2 \le \|x_n - z\|^2\}, \\ Q_n &= \{z \in H_1 : \langle x_n - z, x - x_n \rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n} x, \end{aligned}$$

for $n = 1, 2, ...$, where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \sigma)$, $\lambda \subset [a', b']$ for some $a', b' \in (0, \theta)$, where $\theta := \min\{\theta_1, \theta_2\}$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\gamma \in \left(0, \frac{1}{\|B^*\|^2}\right)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $z = P_\Omega x$.

Proof: Take $M_1 = \partial I_C$ and $M_2 = \partial I_Q$ in Theorem 3.1.

Finally, we derive the following strong convergence theorem for the sequences generated by an iterative algorithm which finds the approximate common element of the set of solution of split null point problem ($\mathrm{S_P}$NPP(10)-(11)), mixed equilibrium problem (MEP(1)) and FPP for a nonexpansive mapping S.

Corollary 3.2: Let H_1 and H_2 are real Hilbert spaces and $B : H_1 \to H_2$ be a bounded linear operator with its adjoint operator B^*. Let $F : C \times C \to \mathbb{R}$ be a bifunction satisfying Assumption 2.1((i),(iii) and (iv)), and Assumption 2.2; let $M_1 : H_1 \to 2^{H_1}$, $M_2 : H_2 \to 2^{H_2}$ be the multi-valued maximal monotone mappings; and let the mapping $A : C \to H_1$ be σ-inverse strongly monotone and let $S : C \to C$ be a nonexpansive mapping such that $\Omega = \mathrm{Sol}(\mathrm{S_P}\mathrm{NPP}(10)\text{-}(11)) \cap \mathrm{Sol}(\mathrm{MEP}(1)) \cap \mathrm{Fix}(S) \neq \emptyset$. Let the iterative sequences $\{x_n\}$, $\{y_n\}$, $\{l_n\}$, $\{z_n\}$, $\{w_n\}$ and $\{u_n\}$ be generated by the following iterative algorithm:

$$\begin{aligned} x^0 &= x \in H_1, \\ y_n &= J_\lambda^{M_1} x_n, \\ l_n &= J_\lambda^{M_2} B y_n, \\ z_n &= P_C[y_n + \gamma B^*(l_n - B y_n)], \\ w_n &= T_{r_n}(I - r_n A) z_n, \\ u_n &= \alpha_n x_n + (1-\alpha_n) S T_{r_n}(z_n - r_n A w_n), \end{aligned} \tag{79}$$

$$
\begin{aligned}
C_n &= \{z \in H_1 : \|u_n - z\|^2 \leq \|x_n - z\|^2\}, \\
Q_n &= \{z \in H_1 : \langle x_n - z, x - x_n \rangle \geq 0\}, \\
x_{n+1} &= P_{C_n \cap Q_n} x,
\end{aligned}
$$

for $n = 1, 2, ...$, where $\{r_n\} \subset [a, b]$ for some $a, b \in (0, \sigma)$, $\lambda \subset [a', b']$ for some $a', b' \in (0, \theta)$, where $\theta := \min\{\theta_1, \theta_2\}$ and $\{\alpha_n\} \subset [0, c]$ for some $c \in [0, 1)$ and $\gamma \in \left(0, \dfrac{1}{\|B^*\|^2}\right)$. Then the sequences $\{x_n\}$, $\{y_n\}$ and $\{z_n\}$ converge strongly to $z = P_\Omega x$.

Proof: Take $f = 0$ and $g = 0$ in Theorem 3.1.

4. Numerical Example

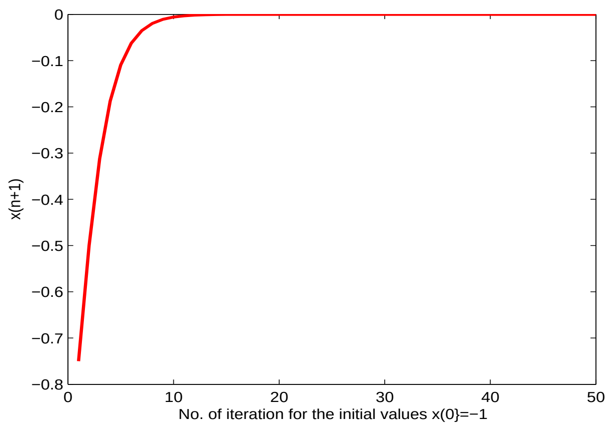
Now, we give a numerical example which justify Theorem 3.1.

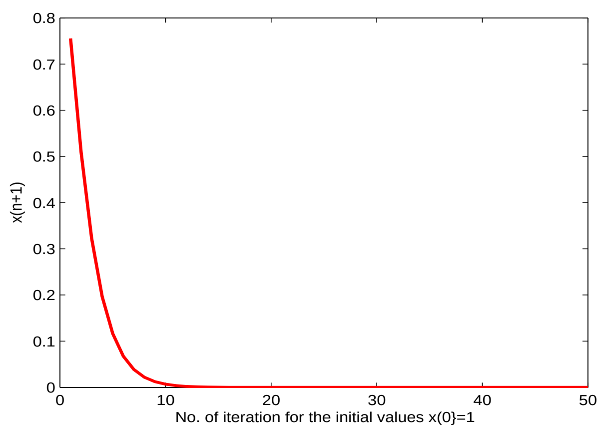
Example 4.1: Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x, y \rangle = xy$, $\forall x, y \in \mathbb{R}$, and induced norm $|.|$. Let $C = [0, 1]$ and $Q = (-\infty, 0]$; let $F : C \times C \to \mathbb{R}$ be a bifunction defined by $F(x, y) = x(y - x)$, $\forall x, y \in C$; let $M_1, M_2 : \mathbb{R} \to \mathbb{R}$ be defined by $M_1(x) = 2x$ and $M_2(x) = 4x$, $\forall x \in \mathbb{R}$; let the mappings $A : C \to \mathbb{R}$, $B : \mathbb{R} \to \mathbb{R}$ and $S : C \to C$ be defined by $A(x) = 2x$, $B(x) = -2x$ and $S(x) = x$, $\forall x \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = 0$, $\forall x \in \mathbb{R}$ and $g(y) = 0$, $\forall y \in H_2$. Then it is easy to prove that F is a bifunction and satisfying Assumption 2.1 and Assumption 2.2; M_1, M_2 are maximal monotone; A is $\frac{1}{2}$-inverse strongly monotone; S is nonexpansive and B is a bounded linear operator with its adjoint B^* such that $\|B\| = \|B^*\| = 2$. The iterative sequences $\{x_n\}$, $\{y_n\}$ $\{l_n\}$, $\{z_n\}$, $\{w_n\}$, $\{u_n\}$ generated by (27)-(34) are then reduced to the following iterative schemes:

$$
\begin{aligned}
y_n &= \left(\frac{1}{3}\right) x_n;\ l_n = \left(\frac{1}{4}\right) y_n; \\
z_n &= \begin{cases} 0, \text{ if } x < 0, \\ 1, \text{ if } x > 1, \\ [y_n - 0.4(l_n + 2y_n)] \text{ otherwise}; \end{cases} \\
w_n &= \left(\frac{-1}{2}\right) z_n;\ u_n = \left(\frac{1}{n+1}\right) x_n - \frac{4}{3}\left(1 - \frac{1}{n+1}\right) w_n \\
C_n &= \left[\frac{u_n + x_n}{2}, \infty\right),\ Q_n = [x_n, \infty); \\
x_{n+1} &= P_{C_n \cap Q_n} x,
\end{aligned}
$$

where $\alpha_n = \frac{1}{n+1}$ and $r_n = 1$. Then $\{x_n\}$ converges strongly to $0 \in \Omega = \{0\}$.

Next, using the software Matlab 7.0, we have following figures which shows that $\{x_n\}$ converges strongly to 0.





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DEPARTMENT OF MATHEMATICS, ALIGARH MUSLIM UNIVERSITY, ALIGARH 202002, INDIA

E-mail addresses: `krkazmi@gmail.com`

DEPARTMENT OF MATHEMATICS, BABU BANARASI DAS UNIVERSITY, LUCKNOW 226028, INDIA

E-mail address: `shujarizvi07@gmail.com`

DEPARTMENT OF MATHEMATICS, ALIGARH MUSLIM UNIVERSITY, ALIGARH 202002, INDIA

E-mail addresses: `rehan08amu@gmail.com`