ON BOUNDS OF RADIO NUMBER OF CERTAIN PRODUCT GRAPHS

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ABSTRACT. Given a graph $G$, whose vertex set is $V(G)$, the radio labelling of $G$ is a variation of vertex labelling of $G$ which satisfy the condition that given any $v_1, v_2 \in V(G)$, and some positive integer function $f$ on $V(G)$, then $|f(v_1) - f(v_2)| \geq \text{diam}(G) + 1 - d(v_1, v_2)$. Radio labelling guarantees a better reduction in interference in signal-dependent networks since no two vertex have the same label. The radio number $rn(G)$ of $G$ is the smallest possible value of $f(v)$ such that for any other $v_k \in V(G)$, $f(v_k) < f(v)$. In this work, we consider a Cartesian product graph obtained from a star and a path and determined upper and lower bounds of the radio number for the family of these graphs.

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1. INTRODUCTION

Let $G$ represent a simple and undirected graph with vertex set $V(G)$ and edge set, $E(G)$, $e = uv \in E(G)$ if $e$ connects two vertices $u, v \in G$. Furthermore, let $d(u, v)$ and diam$(G)$ be the distance between vertices $u, v$ and the diameter of $G$ respectively. Radio labelling, otherwise known as multilevel distance labeling is a channel assignment problem with the aim of reducing frequency interference. This was introduced by Hale in 1980 [3] and it involves the mapping $f : V(G) \rightarrow \mathbb{Z}_+$, such that the radio condition as follows is met:

$$|f(u) - f(v)| \geq \text{diam}(G) + 1 - d(u, v)$$

for any distinct pair $u, v \in V(G)$.

The least possible value of $f(v)$ in the range of $f$ for which given any vertex $u \in V(G)$, $f(u) < f(v)$ is known as the radio number $rn(G)$ of $G$. Determining the radio number of many graphs...

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is tedious, partly due to the \( \text{diam}G + 1 \) condition, which ensures that radio labelling is unique for every vertex in \( G \). However, radio numbers for some graphs have been completely determined. Previously, Liu and Zhu [5] built on upper and lower bounds obtained by Chatrand, et. al. [1], [2], and determined the radio numbers for path and cycles. Marinescue-Ghemeci [6] obtained the numbers for a number of graphs including the thorn stars while Saha and Panigrahi [7] worked on the radio numbers of toroidal grid, which is the Cartesian product of two cycles.

In this paper, we determine upper and lower bounds of a Cartesian product graph \( G = S_n \square P_m \), where \( S_n \) and \( P_m \) are stars and paths of orders \( n \) and \( m \) respectively. The lower bound obtained is tight, as illustrated in an example where the exact radio number of \( S_4 \square P_2 \) obtained by manual labeling coincides with the lower bound. Essentially, the radio number of \( S_n \square P_2 \), for all \( n \in \mathbb{Z} \) coincides with the lower bound. However, there is a considerable difference between the two bounds in this work, implying that the upper bound can be significantly improved. It should be noted also that \( S_3 \square P_m \) is a \( G_{3,m} \) grid, a cartesian product of two paths. The complete radio numbers for grids have been obtained by Jiang [4]. Our lower bound for \( S_3 \square P_m \), compares favourably with the results.

2. PRELIMINARIES AND DEFINITIONS

We define \([1,k]\) as the set \( \{1,2,\ldots,k\} \) of positive integers from 1 to \( k \). The star graph \( S_n \) in this work is a complete bipartite graph \( K_{1,n-1} \) containing \( n \) vertices, one of which, say \( v_1 \), is the center vertex and for each \( v_r \) for the remaining \( n-1 \) vertices, \( v_1v_r \) is a leaf. The path \( P_m \) contains \( m \) vertices. Let \( S_n(i) \), be a class of \( S_n \) stars, \( i \in [1,m] \). A cartesian product graph \( S_n \square P_m \) primarily consists of \( S_n(1), S_n(2), \ldots, S_n(m) \) such that for each \( 1 \leq i < m \), each of the \( n \) vertices on \( S_n(i) \) is uniquely adjacent to and only to its corresponding vertex on \( S_n(i+1) \).

Let \( P_1, P_2, \ldots, P_s \) be the set of path between vertices \( v_a \) and \( v_b \), let \( \alpha_1, \alpha_2, \ldots, \alpha_s \) be a set of positive integers, where \( \alpha_i, i \in [1,s] \), is the number of edges on \( P_i \). The min \( \{\alpha_1, \alpha_2, \ldots, \alpha_s\} \) is the distance \( d(v_a, v_b) \) between \( v_a \) and \( v_b \), the longest distance in \( G \) is the diameter \( \text{diam}(G) \) of a graph \( G \).

**Lemma 1:** [2] For path \( P_n \) and any positive integer \( n \),

\[
    r_n(P_n) \leq \begin{cases} 
    2k^2 + k & \text{if } n = 2k + 1; \\
    2(k^2 - k) + 1 & \text{if } n = 2k 
    \end{cases}
\]
Lemma 2: [5] For path $P_n$ and any integer $n \geq 4$,

$$rn(P_n) = \begin{cases} 2k^2 + 2 & \text{if } n = 2k + 1; \\ 2(k^2 - 1) + 1 & \text{if } n = 2k \end{cases}$$

Remark 1: It should be noted that for $S_n \Box P_m$, $\text{diam}(G) = m + 1$.

Definition 1: Let for a star $S_t \subset S_n \Box P_m$, $t > n$, $V'(S_t) = \{v_3, v_4, \ldots, v_t\} = V(S_t) \setminus \{v_1, v_2\}$, where $v_1$ is the center of $S_t$ and $v_2$ is some other vertex on $S_t$.

3. BOUNDS OF THE RADIO NUMBER OF $G = S_n \Box P_m$

Here we present our results. We determine the lower bound and an upper bound of the radio number of $G = S_n \Box P_m$.

Theorem 1: Let $G = S_n \Box P_m$ and suppose that $S_t$ is some star in $G$ with $t \geq n + 1$ and $v_1$ the center of $S_t$. Suppose that $f(v_1)$ is the smallest radio label on $S_t$. Then for any $v_k \in V'(S_t)$, $f(v_k) \geq f(v_1) + \text{diam}(G) + m(k - 1) + 1$.

Proof: For $v_1$, the center of $S_t$, let $f(v_1) = q$ and let $v_2 \in V(S_t)$ such that $v_1v_2 \in E(S_t)$. By the definition of radio labelling, let $v_2$ be the vertex such that

$$f(v_2) \geq f(v_1) + \text{diam}(G) + 1 - d(v_1, v_2).$$

Since $f(v_1)$ is the minimum label on $S_t$, then

$$f(v_2) \geq q + m + 2 - 1 \geq q + m + 1.$$
Now, let \( v_3 \in V'(S_t) \). Clearly, \( d(v_2, v_3) = 2 \). Likewise, \( d(v_2, v_k) = 2 \) for all \( k \in [3, t] \). By definition, suppose that \( f(v_3) > f(v_2) \),
\[
f(v_3) - f(v_2) \geq \text{diam}(G) - d(v_2, v_3) + 1.
\]
Thus
\[
f(v_3) \geq f(v_2) + m \\
\geq q + 2m + 1.
\]
Iteratively, for \( k > 3 \),
\[
f(v_k) \geq q + (k - 1)m + 1.
\]
Therefore, for any \( v_k, k \geq 3 \), \( f(v_k) \geq f(v_1) + (k - 1)m + 1 \), where \( f(v_1) \) is the minimum radio label on \( S_t \).

**Remark 2:** For \( G = S_n \square P_m \), it should be observed that \( G \) contains two \( S_{n+1} \) stars (namely \( S_{n+1}(1), S_{n+1}(2) \)) at its ends. It also contains \( m-2 \) number of \( S_{n+2} \) stars, namely \( S_{n+2}(1), S_{n+2}(2), \ldots, S_{n+2}(m-2) \). It is clear therefore that for each \( S_{n+1} \) stars there exists some vertex \( v_k \), say in \( S_{n+1}(i), i \in \{1, 2\} \) \( v_k \) not a center vertex of \( S_{n+1}(i) \), but \( v_k \) is a center vertex of some \( S_{n+2} \) star. Likewise, there exist two vertices on each of the \( S_{n+2} \) stars, which are centers of two other \( S_t \) stars, \( t \in \{n + 1, n + 2\} \).

Next, we give a result on the lower bound of the radio number of \( G = S_n \square P_m \).

**Corollary 1:** Let \( G = S_n \square P_m \), with \( m \geq 3 \). Then, \( \text{rn}(S_n \square P_m) \geq m(n + 1) + 2 \), where \( f(v_1) = 1 \), for \( v_1 \), the center of some \( S_t \in G \), \( t = n + 2 \).

**Proof:** Since \( m \geq 3 \), by definition of \( S_n \square P_m \), there exists at least some star \( S_{n+2}(i), 1 \leq i \leq m - 2 \), such that \( v_1 \) is the center vertex of \( S_{n+2}(i) \), and \( v_1v_f, v_1v_g \in E(S_{n+2}(i)) \), where by earlier remark, \( v_f, v_g \) are center vertices for other stars \( S_t, t \geq n + 1 \).

Without loss of generality, set \( v_g \) as the last vertex \( v_{n+2} \). Then, by Theorem 1,
\[
f(v_{n+2}) \geq f(v_1) + m(n + 2 - 1) + 1 \\
\geq 1 + m(n + 1) + 1 \\
\geq m(n + 1) + 2.
\]
Thus \( \text{rn}(S_n \square P_m) \geq m(n + 1) + 2 \).

**Corollary 2:** For \( m = 2 \), \( \text{rn}(S_n \square P_m) \geq 2n + 2 \).

**Proof:** Let \( f(v_1) = 1 \), \( v_1 \) being a central vertex of one of the major stars on \( S_n \square P_2 \) and let \( \Delta(G) \) be the highest degree of a graph \( G \).
For $S_n \Box P_2$, $\Delta(S_n \Box P_2) = n + 1$. By applying the result in Theorem 1, $f(v_{n+1}) \geq f(v_1) + (n + 1 - 1)2 + 1 = 2n + 2$.

**Definitions 2:**

I. We describe $S_n(i), i \in [1, m]$ as primary star, obtained by deleting vertices which are centers of neighbouring $S_{n+1}$ and $S_{n+2}$ stars. Obviously, the number of $S_n(i)$ stars in any $S_n \Box P_m$ graph is $m$.

II. For a $S_n(i)$ star, \{v_1(s_n(i)), v_2(s_n(i)), \ldots, v_n(s_n(i))\} are the members of $V(S_n(i))$, where $v_1(s_n(i))$ is the center vertex.

**Theorem 2:** For $m \geq 2$ and $G = S_n \Box P_m$, $rn(G) \leq nm^2 + 1$, where the least label on $G$ is $f(v_k) = 1$, for some $v_k \in V(G)$.

**Proof:** Let $v_k = v_1(s_n(i))$ and set $f(v_1(s_n(i))) = 1$. By the result in Theorem 1 let $f(v_n(s_n(1))) = 1 + (n - 1)m + 1 = (n - 1)m + 2$. There exists $v_1(s_n(2)) \in S_{n+1}(1)$ such that $v_1(s_n(2))$ is the center of $S_n(2)$. Therefore, without loss of generality, set $f(v_1(s_n(2))) = f(v_n(s_n(1))) + m = (n - 1)m + 2 + m = mn + 2$.

Still by Theorem 1, $f(v_n(s_n(2))) = mn + 2 + (n - 1)m + 1 = 2mn - m + 3$.

Using similar technique as employed earlier, we have that $f(v_1(s_n(3))) = f(v_n(s_n(2))) + m = 2mn + 3$,

while $f(v_n(s_n(3))) = f(v_1(s_n(3))) + (n - 1)m + 1 = 3mn - m + 4$.

By continuing the iteration, it will be seen that for $v_1(s_n(m))$, $f(v_1(s_n(m))) = m^2n - mn + m$.

And thus, $f(v_n(s_n(m))) = m^2n - mn + m + (n - 1)m + 1 = nm^2 + 1$.

Thus we conclude that $rn(G) \leq m^2n + 1$.

**Remark 3:** From the result in Corollary 2, the lower bound of $rn(S_4 \Box P_2)$ is 10. In Figure 3, we see that the the highest value of $f$ on $V(S_4 \Box P_2)$ is also 10 after manual radio labelling, thereby confirming the radio number of that graph as 10. Applying our
result on upper bound for the same graph, the highest is at most 17. Our upper bound can be improved significantly.

Note that from our results, trivially $rn(S_n \Box P_2) = 2n + 2$.

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