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ON BOUNDS OF RADIO NUMBER OF CERTAIN PRODUCT GRAPHS

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ABSTRACT. Given a graph G, whose vertex set is $V(G)$, the radio labelling of G is a variation of vertex labelling of G which satisfy the condition that given any $v_1, v_2 \in V(G)$, and some positive integer function $f(v)$ on $V(G)$, then $|f(v_1) - f(v_2)| \geq \text{diam}(G) + 1 - d(v_1, v_2)$. Radio labelling guarantees a better reduction in interference in signal-dependent networks since no two vertex have the same label. The radio number $rn(G)$ of G is the smallest possible value of $f(v)$ such that for any other $v_k \in V(G)$, $f(v_k) < f(v)$. In this work, we consider a Cartesian product graph obtained from a star and a path and determined upper and lower bounds of the radio number for the family of these graphs.

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1. INTRODUCTION

Let G represent a simple and undirected graph with vertex set $V(G)$ and edge set, $E(G)$, $e = uv \in E(G)$ if e connects two vertices $u, v \in G$. Furthermore, let $d(u, v)$ and $\text{diam}(G)$ be the distance between vertices u, v and the diameter of G respectively. Radio labelling, otherwise known as multilevel distance labeling is a channel assignment problem with the aim of reducing frequency interference. This was introduced by Hale in 1980 [3] and it involves the mapping $f : V(G) \to \mathbb{Z}_+$, such that the *radio* condition as follows is met:

$$|f(u) - f(v)| \geq \text{diam}G + 1 - d(u, v)$$

for any distinct pair $u, v \in V(G)$.

The least possible value of $f(v)$ in the range of f for which given any vertex $u \in V(G)$, $f(u) < f(v)$ is known as the radio number $rn(G)$ of G. Determining the radio number of many graphs

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is tedious, partly due to the diam$G + 1$ condition, which ensures that radio labelling is unique for every vertex in G. However, radio numbers for some graphs have been completely determined. Previously, Liu and Zhu [5] built on upper and lower bounds obtained by Chatrand, et. al. [1], [2], and determined the radio numbers for path and cycles. Marinescue-Ghemeci [6] obtained the numbers for a number of graphs including the thorn stars while Saha and Panigrahi [7] worked on the radio numbers of toroidal grid, which is the Cartesian product of two cycles.

In this paper, we determine upper and lower bounds of a Cartesian product graph $G = S_n \Box P_m$, where S_n and P_m are stars and paths of orders n and m respectively. The lower bound obtained is tight, as illustrated in an example where the exact radio number of $S_4 \Box P_2$ obtained by manual labeling coincides with the lower bound. Essentially, the radio number of $S_n \Box P_2$, for all $n \in \mathbb{Z}$ coincides with the lower bound. However, there is a considerable difference between the two bounds in this work, implying that the upper bound can be significantly improved. It should be noted also that $S_3 \Box P_m$ is a $G_{3,m}$ grid, a cartesian product of two paths. The complete radio numbers for grids have been obtained by Jiang [4]. Our lower bound for $S_3 \Box P_m$, compares favourably with the results.

2. PRELIMINARIES AND DEFINITIONS

We define $[1, k]$ as the set $\{1, 2, \cdots, k\}$ of positive integers from 1 to k. The star graph S_n in this work is a complete bipartite graph $K_{1,n-1}$ containing n vertices, one of which, say v_1, is the center vertex and for each v_r for the remaining $n-1$ vertices, $v_1 v_r$ is a leaf. The path P_m contains m vertices. Let $S_n(i)$, be a class of S_n stars, $i \in [1, m]$. A cartesian product graph $S_n \Box P_m$ primarily consists of $S_n(1), S_n(2), \cdots, S_n(m)$ such that for each $1 \leq i < m$, each of the n vertices on $S_n(i)$ is uniquely adjacent to and only to its corresponding vertex on $S_n(i+1)$.

Let $P_1, P_2, \cdots, P_s$ be the set of path between vertices v_a and v_b, let $\alpha_1, \alpha_2, \cdots \alpha_s$ be a set of positive integers, where α_i, $i \in [1, s]$, is the number of edges on P_i. The min $\{\alpha_1, \alpha_2, \cdots, \alpha_s\}$ is the distance $d(v_a, v_b)$ between v_a and v_b, the longest distance in G is the diameter diam(G) of a graph G.

Lemma 1: [2] For path P_n and any positive integer n,

$$rn(P_n) \leq \begin{cases} 2k^2 + k & \text{if } n = 2k+1; \\ 2(k^2 - k) + 1 & \text{if } n = 2k \end{cases}$$

Lemma 2: [5] For path P_n and any integer $n \geq 4$,

$$rn(P_n) = \begin{cases} 2k^2 + 2 & \text{if } n = 2k+1; \\ 2(k^2 - 1) + 1 & \text{if } n = 2k \end{cases}$$

Remark 1: It should be noted that for $S_n \Box P_m$, $\text{diam}(G) = m + 1$.

Definition 1: Let for a star $S_t \subset S_n \Box P_m$, $t > n$, $V'(S_t) = \{v_3, v_4, \cdots, v_t\} = V(S_t) \setminus \{v_1, v_2\}$, where v_1 is the center of S_t and v_2 is some other vertex on S_t.

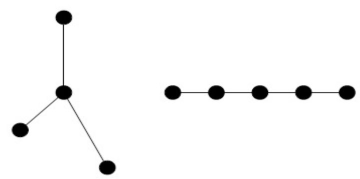

Fig. 1. Star S_4 and Path P_5

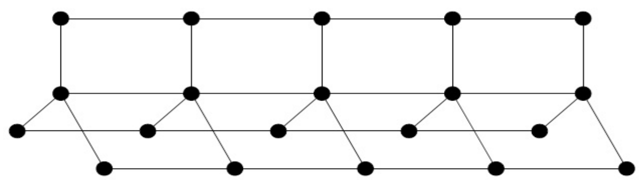

Fig. 2. A $S_4 \Box P_5$ Graph

3. BOUNDS OF THE RADIO NUMBER OF $G = S_n \Box P_m$

Here we present our results. We determine the lower bound and an upper bound of the radio number of $G = S_n \Box P_m$.

Theorem 1: Let $G = S_n \Box P_m$ and suppose that S_t is some star in G with $t \geq n+1$ and v_1 the center of S_t. Suppose that $f(v_1)$ is the smallest radio label on S_t. Then for any $v_k \in V'(S_t)$, $f(v_k) \geq f(v_1) + m(k-1) + 1$.

Proof: For v_1, the center of S_t, let $f(v_1) = q$ and let $v_2 \in V(S_t)$ such that $v_1 v_2 \in E(S_t)$. By the definition of radio labelling, let v_2 be the vertex such that

$$f(v_2) \geq f(v_1) + diam(G) + 1 - d(v_1, v_2).$$

Since $f(v_1)$ is the minimum label on S_t, then

$$\begin{aligned} f(v_2) &\geq q + m + 2 - 1 \\ &\geq q + m + 1. \end{aligned}$$

Now, let $v_3 \in V'(S_t)$. Clearly, $d(v_2, v_3) = 2$. Likewise, $d(v_2, v_k) = 2$ for all $k \in [3, t]$. By definition, suppose that $f(v_3) > f(v_2)$,

$$f(v_3) - f(v_2) \geq diam(G) - d(v_2, v_3) + 1.$$

Thus

$$\begin{aligned} f(v_3) &\geq f(v_2) + m \\ &\geq q + 2m + 1. \end{aligned}$$

Iteratively, for $k > 3$,

$$f(v_k) \geq q + (k-1)m + 1.$$

Therefore, for any v_k, $k \geq 3$, $f(v_k) \geq f(v_1) + (k-1)m + 1$, where $f(v_1)$ is the minimum radio label on S_t.

Remark 2: For $G = S_n \square P_m$, it should be observed that G contains two S_{n+1} stars (namely $S_{n+1}(1)$, $S_{n+1}(2)$) at its ends. It also contains $m-2$ number of S_{n+2} stars, namely $S_{n+2}(1), S_{n+2}(2), \cdots, S_{n+2}(m-2)$. It is clear therefore that for each S_{n+1} stars there exists some vertex v_k, say in $S_{n+1}(i)$, $i \in \{1, 2\}$ v_k not a center vertex of $S_{n+1}(i)$, but v_k is a center vertex of some S_{n+2} star. Likewise, there exist two vertices on each of the S_{n+2} stars, which are centers of two other S_t stars, $t \in \{n+1, n+2\}$.
Next, we give a result on the lower bound of the radio number of $G = S_n \square P_m$.

Corollary 1: Let $G = S_n \square P_m$, with $m \geq 3$. Then, $rn(S_n \square P_m) \geq m(n+1) + 2$, where $f(v_1) = 1$, for v_1, the center of some $S_t \in G$, $t = n + 2$.

Proof: Since $m \geq 3$, by definition of $S_n \square P_m$, there exists at least some star $S_{n+2}(i)$, $1 \leq i \leq m-2$, such that v_1 is the center vertex of $S_{n+2}(i)$, and $v_1 v_f, v_1 v_g \in E(S_{n+2}(i))$, where by earlier remark, v_f, v_g are center vertices for other stars S_t, $t \geq n+1$.

Without loss of generality, set v_g as the last vertex v_{n+2}. Then, by Theorem 1,

$$\begin{aligned} f(v_{n+2}) &\geq f(v_1) + m(n+2-1) + 1 \\ &\geq 1 + m(n+1) + 1 \\ &\geq m(n+1) + 2. \end{aligned}$$

Thus $rn(S_n \square P_m) \geq m(n+1) + 2$.

Corollary 2: For $m = 2$, $rn(S_n \square P_m) \geq 2n + 2$.

Proof: Let $f(v_1) = 1$, v_1 being a central vertex of one of the major stars on $S_n \square P_2$ and let $\Delta(G)$ be the highest degree of a graph G.

For $S_n \Box P_2$, $\Delta(S_n \Box P_2) = n+1$. By applying the result in Theorem 1, $f(v_{n+1}) \geq f(v_1) + (n+1-1)2 + 1 = 2n+2$.

Definitions 2:

I. We describe $S_n(i)$, $i \in [1, m]$ as primary star, obtained by deleting vertices which are centers of neighbouring S_{n+1} and S_{n+2} stars. Obviously, the number of $S_n(i)$ stars in any $S_n \Box P_m$ graph is m.

II. For a $S_n(i)$ star, $\{v_1(s_n(i)), v_2(s_n(i)), \cdots, v_n(s_n(i))\}$ are the members of $V(S_n(i))$, where $v_1(s_n(i))$ is the center vertex.

Theorem 2: For $m \geq 2$ and $G = S_n \Box P_m$, $rn(G) \leq nm^2 + 1$, where the least label on G is $f(v_k) = 1$, for some $v_k \in V(G)$.

Proof: Let $v_k = v_1(s_n(i))$ and set $f(v_1(s_n(i))) = 1$. By the result in Theorem 1 let $f(v_n(s_n(1))) = 1 + (n-1)m + 1 = (n-1)m + 2$. There exists $v_1(s_n(2)) \in S_{n+1}(1)$ such that $v_1(s_n(2))$ is the center of $S_n(2)$. Therefore, without loss of generality, set

$$\begin{aligned} f(v_1(s_n(2))) &= f(v_n(s_n(1))) + m \\ &= (n-1)m + 2 + m \\ &= mn + 2. \end{aligned}$$

Still by Theorem 1,

$$\begin{aligned} f(v_n(s_n(2))) &= mn + 2 + (n-1)m + 1 \\ &= 2mn - m + 3. \end{aligned}$$

Using similar technique as employed earlier, we have that

$$\begin{aligned} f(v_1(s_n(3))) &= f(v_n(s_n(2))) + m \\ &= 2mn + 3, \end{aligned}$$

while

$$\begin{aligned} f(v_n)(s_n(3)) &= f(v_1(s_n(3))) + (n-1)m + 1 \\ &= 3mn - m + 4. \end{aligned}$$

By continuing the iteration, it will be seen that for $v_1(s_n(m))$,

$$f(v_1(s_n(m))) = m^2 n - mn + m.$$

And thus,

$$\begin{aligned} f(v_n(s_n(m))) &= m^2 n - mn + m + (n-1)m + 1 \\ &= nm^2 + 1. \end{aligned}$$

Thus we conclude that $rn(G) \leq m^2 n + 1$.

Remark 3: From the result in Corollary 2, the lower bound of $rn(S_4 \Box P_2)$ is 10. In Figure 3, we see that the the highest value of f on $V(S_4 \Box P_2)$ is also 10 after manual radio labelling, thereby confirming the radio number of that graph as 10. Applying our

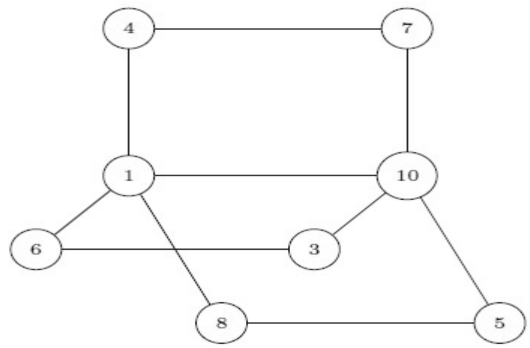

Fig. 3. Radio labeling for $S_4\Box P_2$

result on upper bound for the same graph, the highest is at most 17. Our upper bound can be improved significantly.

Note that from our results, trivially $rn(S_n\Box P_2) = 2n + 2$.

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