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STABILITY AND BOUNDEDNESS OF SOLUTIONS OF CERTAIN VECTOR DELAY DIFFERENTIAL EQUATIONS

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ABSTRACT. In this paper, certain class of second-order vector delay differential equation of the form

$$\ddot{X} + A\dot{X} + H(X(t - r(t))) = P(t, X, \dot{X})$$

is considered where $X \in \mathbb{R}^n$, $0 \leq r(t) \leq \gamma$ and A is a real constant, symmetric positive definite $n \times n$ matrix. By using the second method of Lyapunov and Lyapunov-Krasovskii's funtion we established sufficient conditions for the asymptotic stability of the zero solution when $P(t, X, \dot{X}) = 0$ and boundedness of all solutions when $P(t, X, \dot{X}) \neq 0$. The results obtained here are generalizations of some of the results obtained for $\mathbb{R}^1$.

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1. INTRODUCTION

We consider the differential equation of the form:

$$\ddot{X} + A\dot{X} + H(X(t - r(t))) = P(t, X, \dot{X}) \tag{1.1}$$

or its equivalent system

$$\dot{X} = Y$$
$$\dot{Y} = -AY - H(X) + \int_{t-r(t)}^{t} J_h(X)Y\,ds + P(t, X, Y), \tag{1.2}$$

where $X, Y \in \mathbb{R}^n$, $H : \mathbb{R}^n \to \mathbb{R}^n$, $P : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$. A is a real constant, symmetric positive definite $n \times n$ matrix, $J_h(X)$ is a continuous, symmetric positive definite Jacobian matrix of H, $0 \leq r(t) \leq \gamma$, γ is a positive constant whose value will be determined later and the dots denote differentiation with respect to t. We assume that the nonlinear functions H and P be continuous and so constructed such that the uniqueness theorem is valid and

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solutions are continuously dependent on the initial conditions.

Equation (1.1) above represents a system of real second-order differential equations of the form

$$\ddot{x} + \sum_{k=1}^{n} a_{ik}\dot{x}_k + h_i(x_1(t-r(t)), \ldots, x_n(t-r(t)) = p_i(t, x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n)$$

$(i = 1, 2, \ldots, n)$. We shall assume as basic throughout what follows, that the partial derivatives $\frac{\partial h_i}{\partial x_j}$ exist and are continuous, $(i = 1, 2, \ldots, n)$.
The problem of interest here is to determine conditions under which all solutions of (1.1) are stable when $P(t, X, \dot{X}) = 0$ and bounded when $P(t, X, \dot{X}) \neq 0$. For over four decades, many authors have dealt with scalar, vector and matrix differential equations, and scalar delay differential equations and obtained many interesting results. For instance, Ogundare et.al [1] studied the boundedness and stability properties of solutions of

$$\ddot{x} + f(x)\dot{x} + g(x) = p(t, x, \dot{x}),$$

where f, g and p are continuous in their respective arguments t, x,and $\dot{x}$.

Ademola [2] considered the stability, boundedness and existence of unique periodic solutions to the following second order ordinary differential equation

$$[\phi(x)x']' + g(t, x, x')x' + \varphi(t)h(x) = p(t, x, x'),$$

where ϕ, g, φ, h and p are continuous functions in their respective argument. However, Ademola et.al [3] established stability, boundedness and existence of a unique periodic solution to certain second order delay differential equations of the form

$$\begin{aligned}[\phi(x(t)x'(t))]' &+ g(t, x(t-\tau(t)), x'(t-\tau(t)))x'(t)\\ &+h(x(t-\tau(t))) = p(t, x(t-\tau(t)), x'(t-\tau(t))),\end{aligned}$$

where ϕ, g, h, p and τ are continuous functions in their respective arguments.

Wiandt [4] in the article published in 1998, considered the vector Lienard differential equation

$$\ddot{X} + F(X)\dot{X} + G(X) = 0$$

and proved two theorems concerning the boundedness of all solutions of this equation. While Tunc [5] considered a class of second-order nonlinear differential equations of the form

$$\ddot{X} + G(X, \dot{X})\dot{X} + F(X) = P(t, X, \dot{X})$$

and established the result on ultimately boundedness of solutions. Beside, see also Afuwape and Omeike [6], Omeike [7], Tunc ([8],[9]), Zhu [10].

The object of this paper is to obtain sufficient conditions for the stability and for the boundedness of solutions of equation (1.1) as $P(t, X, \dot{X}) = 0$ and $P(t, X, \dot{X}) \neq 0$ respectively. We make use of Lyapunov second method to establish our results.

Notations and definitions
Given any $X,\ Y$ in $\mathbb{R}^n$ the symbol $\langle X, Y\rangle$ will be used to denote the usual scalar product in $\mathbb{R}^n$, that is $\langle X, Y\rangle = \sum_{i=1}^{n} x_i y_i$; thus $\| X \|^2 = \langle X, X\rangle$. The matrix A is said to be positive definite when $\langle AX, X\rangle > 0$ for all non-zero X in $\mathbb{R}^n$.

The paper has four sections in all. The first section contains the introduction and literature review, section two contains some preliminary results, the two main results of the paper are in section three and last section, section four is the concluding remark.

2. SOME PRELIMINARY RESULTS

In this section, we state the algebraic results required in the proofs of our main results. The proofs of the results will not be given since they are found in [11,12,13,14,15].

Lemma 2.1 [11,12,13,14,15] *Let A be any real symmetric positive definite $n \times n$ matrix, then for any X in $\mathbb{R}^n$, we have*

$$\delta_a \| X \|^2 \leq \langle AX, X\rangle \leq \Delta_a \| X \|^2$$

where δ_a and Δ_a are the least and the greatest eigenvalues of A, respectively.

Lemma 2.2 [11,12,13,14,15] *Let $A,\ B$ be any two real symmetric positive definite $n \times n$ matrix. Then,*
(i) the eigenvalues $\lambda_i(AB),\ (i = 1, 2, \ldots, n)$, of the product matrix AB are real and satisfy

$$\min_{1\leq j,k\leq n} \lambda_j(A)\lambda_k(B) \leq \lambda_i(AB) \leq \max_{1\leq j,k\leq n} \lambda_j(A)\lambda_k(B);$$

(ii) the eigenvalues $\lambda_i(A+B)$, $(i=1,2,\ldots,n)$, of the sum of matrices A and B are real and satisfy

$$\{\min_{1\le j\le n}\lambda_j(A)+\min_{1\le k\le n}\lambda_k(B)\}\le\lambda_i(A+B)$$
$$\le\{\max_{1\le j\le n}\lambda_j(A)+\max_{1\le j\le n}\lambda_k(B)\}$$

Lemma 2.3 [9,11,12,13,14] *Let $H(X)$ be a continuous vector function and that $H(0)=0$, then*

$$\frac{d}{dt}\int_0^1\langle H(\sigma X),Y\rangle d\sigma=\langle H(X),Y\rangle.$$

Lemma 2.4 [9,11,12,13,14,15] *Let $H(X)$ be a continuous vector function and that $H(0)=0$, then*

$$\delta_h\parallel X\parallel^2\le\int_0^1\langle H(\sigma X),X\rangle d\sigma\le\Delta_h\parallel X\parallel^2,$$

where δ_h, Δ_h are the least and the greatest eigenvalues of $J_h(\sigma X)$, respectively.

Proof of Lemma 2.4: Let $H(X)$ be a continuous vector function and that $H(0)=0$ then,

$$H(X)=\int_0^1 J_h(\sigma X)Xd\sigma$$

for arbitrary vector X in $\mathbb{R}^n$. This follows from integrating

$$\frac{d}{d\sigma}H(\sigma X)=J_h(\sigma X)X$$

with respect to σ and then using the fact that $H(0)=0$.
Thus,

$$\int_0^1\langle H(\sigma X),X\rangle d\sigma=\int_0^1\int_0^1\sigma\langle J_h(\sigma\tau X)X,X\rangle d\sigma d\tau.$$

Then, following inequality of Lemma 2.1, we have

$$\delta_h\parallel X\parallel^2\le\int_0^1\int_0^1\sigma\langle J_h(\sigma\tau X)X,X\rangle d\sigma d\tau\le\Delta_h\parallel X\parallel^2.$$

3. MAIN RESULTS

3.1 Stability Result

First, we will give the stability criteria for the general autonomous delay differential system. We consider

$$x' = f(x_t),\ x_t = x(t+\theta),\ -r \leq \theta \leq 0,\ t \geq 0, \tag{3.1.1}$$

where $f : C_H \to \mathbb{R}^n$ is a continuous mapping, $f(0) = 0$, $C_H := \{\phi \in (C[-r, 0], \mathbb{R}^n) : \parallel \phi \parallel \leq H\}$ and for $H_1 < H$, there exists $L(H_1) > 0$, with $|f(\phi)| \leq L(H_1)$ when $\parallel \phi \parallel \leq H_1$.

Definition 3.1.1. [8,16,17,18] An element $\psi \in C$ is in the ω-limit set of ϕ, if $x(t, 0, \phi)$ is defined on $[0, \infty)$ and there is a sequence $\{t_n\},\ t_n \to \infty$, as $n \to \infty$, with $\parallel x_{t_n}(\phi) - \psi \parallel \to 0$ as $n \to \infty$ where $x_{t_n} = x(t_n + \theta, 0, \phi)$ for $-r \leq \theta \leq 0$.

Definition 3.1.2.[8,16,17,18] A set $Q \subset C_H$ is an invariant set if for any $\phi \in Q$, the solutions of (3.1.1), $x(t, 0, \phi)$, is defined on $[0, \infty)$, and $x_t(\phi) \in Q$ for $t \in [0, \infty)$.

Lemma 3.1.1.[8,16,17,18] *If $\phi \in C_H$ is such that the solution $x_t(\theta)$ of (3.1.1) with $x_0(\phi) = \phi$ is defined on $[0, \infty)$ and $\parallel x_t(\phi) \parallel \leq H_1 < H$ for $t \in [0, \infty)$, then $\Omega(\phi)$ is a nonempty, compact, invariant set and*

$$dist(x_t(\phi), \Omega(\phi)) \to 0,\ as\ t \to \infty.$$

Lemma 3.1.2.[8,16,17,18] *Let $V(\phi) : C_H \to \mathbb{R}$ be a continuous functional satisfying a local Lipschitz condition. $V(0) = 0$ and such that*
(i) $W_1(\|\phi\|) \leq V(\phi) \leq W_2(\parallel \phi \parallel)$ where $W_1(r),\ W_2(r)$ are wedges.
(ii) $V'_{(3.1.1)}(\phi) \leq 0$, for $\phi \leq C_H$.
Then the zero solution of (3.1.1) is uniformly stable. If we define $Z = \{\phi \in C_H : V'_{(3.1.1)}(\phi) = 0\}$, then the zero solution of (3.1.1) is asymptotically stable, provided that the largest invariant set in Z is $Q = \{0\}$.

Before we state our main results in this section, we write (1.1) with $P(t, X, \dot{X}) = 0$ as

$$\dot{X} = Y$$

$$\dot{Y} = -AY - H(X) + \int_{t-r(t)}^{t} J_h(X)Y\,ds \tag{3.1.2}$$

We now state our stability result for (3.1.2) as follows.

Theorem 3.1.1. *Consider (3.1.2), let $H(0) = 0$ and suppose that:*
(i) $0 \leq r(t) \leq \gamma$ $(\gamma > 0)$, $r'(t) \leq \xi$ and $0 < \xi < 1$;
(ii) the matrices A and $J_h(X)$ (Jacobian matrix of $H(X)$) are symmetric and positive definite, and furthermore that the eigenvalues $\lambda_i(A)$ and $\lambda_i(J_h(X))(i = 1, 2, \ldots, n)$ of A and $J_h(X)$, respectively satisfy

$$0 < \delta_a \leq \lambda_i(A) \leq \Delta_a \tag{3.1.3}$$

$$0 < \delta_h \leq \lambda_i(J_h(X)) \leq \Delta_h, \; for \; X \in \mathbb{R}^n, \tag{3.1.4}$$

where $\delta_a, \delta_h, \Delta_a, \Delta_h$ are finite constants;
(iii) the matrices A and $J_h(X)$ commute. Then the zero solutions of (3.1.2) is asymptotically stable, provided

$$\gamma < \min\left(\frac{2\delta_a\delta_h}{\Delta_a\Delta_h}, \frac{\delta_a}{\mu + \Delta_h}\right). \tag{3.1.5}$$

Proof of Theorem 3.1.1. Using the equivalent system form (3.1.2), our main tool is the following Lyapunov functional, $V(X_t, Y_t)$ defined as

$$2V(X_t, Y_t) = \langle AX + Y, AX + Y\rangle + \langle Y, Y\rangle \; + \; 4\int_0^1 \langle H(\sigma X), X\rangle d\sigma$$

$$+\mu \int_{-r(t)}^0 \int_{t+s}^t \langle Y(\theta), Y(\theta)\rangle d\theta ds. \tag{3.1.6}$$

But since

$$\mu \int_{-r(t)}^0 \int_{t+s}^t \langle Y(\theta), Y(\theta)\rangle d\theta ds$$

is non-negative and by Lemma 2.4, it follows that

$$\begin{aligned}
2V(X_t, Y_t) &\geq \langle AX + Y, AX + Y\rangle + \langle Y, Y\rangle \; + \; 4\delta_h \parallel X \parallel^2 \\
V(X_t, Y_t) &\geq \frac{1}{2} \parallel AX + Y \parallel^2 + \frac{1}{2} \parallel Y \parallel^2 + 2\delta_h \parallel X \parallel^2 \\
&\geq \frac{1}{2} \parallel Y \parallel^2 + 2\delta_h \parallel X \parallel^2 .
\end{aligned}$$

Hence, we can find a constant $K = \min \frac{1}{2}\{1, 4\delta_h\} > 0$ (small enough) such that

$$2V(X_t, Y_t) \geq K(\parallel Y \parallel^2 + \parallel X \parallel^2). \tag{3.1.7}$$

Next, we show that $V(X_t, Y_t)$ satisfy the second condition of Lemma

3.1.2. First, by the system (3.1.2), equation (3.1.6) and Lemma 2.3, we get

$$\frac{d}{dt}V_{(3.1.2)}(X_t,Y_t)=-\langle AX(t),H(X(t))\rangle-\langle AY(t),Y(t)\rangle$$

$$-\mu(1-r'(t))\int_{t-r(t)}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta+2\int_{t-r(t)}^{t}\langle Y(t),J_h(X(s))Y(s)\rangle ds$$

$$+\mu r(t)\langle Y(t),Y(t)\rangle+\int_{t-r(t)}^{t}\langle AX(t),J_h(X(s))Y(s)\rangle ds. \qquad (3.1.8)$$

Following Lemma 2.1, Lemma 2.2, inequalities (3.1.3), (3.1.4) and also the identity $2|\langle X,Y\rangle|\le\parallel X\parallel^2+\parallel Y\parallel^2$, we obtain

$$\frac{d}{dt}V_{(3.1.2)}(X_t,Y_t)\le-\delta_a\delta_h\parallel X(t)\parallel^2-\delta_a\parallel Y(t)\parallel^2$$

$$+\frac{1}{2}\Delta_a\Delta_h\gamma\parallel X(t)\parallel^2+\frac{1}{2}\Delta_a\Delta_h\int_{t-r(t)}^{t}\parallel Y(s)\parallel^2 ds$$

$$-\mu(1-\xi)\int_{t-r(t)}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta+\mu\gamma\parallel Y(t)\parallel^2$$

$$+\Delta_h\gamma\parallel Y(t)\parallel^2+\Delta_h\int_{t-r(t)}^{t}\parallel Y(s)\parallel^2 ds \qquad (3.1.9)$$

By simplifying further, we obtain

$$\frac{d}{dt}V_{(3.1.2)}(X_t,Y_t)\le-\Big(\delta_a\delta_h-\frac{1}{2}\Delta_a\Delta_h\gamma\Big)\parallel X(t)\parallel^2$$

$$-\Big(\delta_a-\mu\gamma-\Delta_h\gamma\Big)\parallel Y(t)\parallel^2$$

$$+\Big(\frac{1}{2}\Delta_a\Delta_h+\Delta_h-\mu(1-\xi)\Big)\int_{t-r(t)}^{t}\langle Y(\theta),Y(\theta)\rangle d\theta. \qquad (3.1.10)$$

If we now choose $\mu=\frac{(\Delta_a+2)\Delta_h}{2(1-\xi)}$,

$$\begin{aligned}\frac{d}{dt}V_{(3.1.2)}(X_t,Y_t) &\le -\Big(\delta_a\delta_h-\frac{1}{2}\Delta_a\Delta_h\gamma\Big)\parallel X(t)\parallel^2\\ &- \Big(\delta_a-\mu\gamma-\Delta_h\gamma\Big)\parallel Y(t)\parallel^2,\end{aligned}$$

and now choosing

$$\gamma<\min\Big(\frac{2\delta_a\delta_h}{\Delta_a\Delta_h},\frac{2\delta_a(1-\xi)}{\Delta_h[\Delta_a+2(2-\xi)]}\Big),$$

then there is a constant $K_1>0$ such that

$$\frac{d}{dt}V_{(3.1.2)}(X_t,Y_t)\le-K_1(\parallel X\parallel^2+\parallel Y\parallel^2). \qquad (3.1.11)$$

Hence the result follows from inequalities (3.1.7), (3.1.11) and Lemmas 3.1.1 and 3.1.2.

3.2 Boundedness Result

First, consider a system of delay differential equations

$$x' = F(t, x_t),\ x_t = x(t+\theta),\ -r \le \theta \le 0, \tag{3.2.1}$$

where $F : \mathbb{R} \times C \to \mathbb{R}^n$ is a continuous mapping and takes bounded set into bounded sets.
The following Lemma is a well-known results obtained by Burton[6].
Lemma 3.2.1 [8,16,17,18] *Let $V(t, \phi) : \mathbb{R} \times C \to \mathbb{R}$ be continuous and locally Lipschitz in ϕ. If:*
(i) $W(|x(t)|) \le V(t, x_t) \le W_1(|x(t)|) + W_2\big(\int_{t-r(t)}^{t} W_3(|x(s)|)ds\big)$
and
(ii)$\dot{V}_{(3.2.1)} \le -W_3(|x(s)|) + M,\ for\ some\ M > 0,$
where W_i $(i = 1, 2, 3)$ are wedges, then the solutions of (3.2.1) are uniformly bounded and uniformly ultimately bounded for bound **B**.

Theorem 3.2.1 *If the conditions of Theorem 3.1.1 hold, and*

$$\| P(t, X, Y) \| \le m + \delta(\| X \| + \| Y \|) \tag{3.2.2}$$

$(m,\ \delta$ are positive constants) then the solutions of Equation (1.2) (for which $\| P(t, X, \dot{X}) \| \ne 0$) are uniformly bounded and uniformly ultimately bounded provided γ satisfies

$$\gamma < \min\Big(\frac{2\delta_a\delta_h}{\Delta_a\Delta_h}, \frac{2\delta_a(1-\xi)}{\Delta_h[\Delta_a + 2(2-\xi)]}\Big).$$

Proof of Theorem 3.2.1 Consider the function V defined in (3.1.6). We only concentrate on the hypothesis (ii) of Lemma 3.2.1 since hypothesis (i) of Lemma 3.2.1 was taken care of in the preceding section.
Thus,

$$\frac{d}{dt}V_{(1.2)}(X_t, Y_t) = -\langle AX(t), H(X(t))\rangle - \langle AY(t), Y(t)\rangle$$

$$-\mu(1-r'(t))\int_{t-r(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta + 2\int_{t-r(t)}^{t}\langle Y(t), J_h(X(s))Y(s)\rangle ds$$

$$+\mu r(t)\langle Y(t), Y(t)\rangle + \int_{t-r(t)}^{t}\langle AX(t), J_h(X(s))Y(s)\rangle ds$$

$$+\langle P(t, X, Y), AX + 2Y\rangle$$

$$= -\int_0^1 \langle AX(t), J_h((t))X(t)\rangle d\sigma - \langle AY(t), Y(t)\rangle$$

$$-\mu(1-r'(t))\int_{t-r(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta + 2\int_{t-r(t)}^{t}\langle Y(t), J_h(X(s))Y(s)\rangle ds$$

$$+\mu r(t)\langle Y(t), Y(t)\rangle + \int_{t-r(t)}^{t}\langle AX(t), J_h(X(s))Y(s)\rangle ds$$

$$+\langle P(t,X,Y), AX + 2Y\rangle$$

Following Lemma 2.1 -2.3, inequalities (3.1.3), (3.1.4) and identity

$$2|\langle X, Y\rangle| \leq \parallel X \parallel^2 + \parallel Y \parallel^2,$$

we obtain

$$\begin{aligned} &\frac{d}{dt}V_{(1.2)}(X_t, Y_t) \leq -\Big(\delta_a\delta_h - \frac{1}{2}\Delta_a\Delta_h\gamma\Big) \parallel X(t) \parallel^2 \\ - &\Big(\delta_a - \mu\gamma - \Delta_h\gamma\Big) \parallel Y(t) \parallel^2 \\ + &\Big(\frac{1}{2}\Delta_a\Delta_h + \Delta_h - \mu(1-\xi)\Big)\int_{t-r(t)}^{t}\langle Y(\theta), Y(\theta)\rangle d\theta \\ + &\parallel P(t,X,Y) \parallel \Big(\Delta_a \parallel X \parallel + 2 \parallel Y \parallel\Big). \end{aligned}$$

Now, using $\parallel P(t,X,Y) \parallel \leq m + \delta(\parallel X \parallel + \parallel Y \parallel)$, and choosing $\mu = \frac{(\Delta_a+2)\Delta_h}{2(1-\xi)} > 0$ and $\gamma < \min\Big(\frac{2\delta_a\delta_h}{\Delta_a\Delta_h}, \frac{2\delta_a(1-\xi)}{\Delta_h[\Delta_a+2(2-\xi)]}\Big)$, we have

$$\begin{aligned} &\frac{d}{dt}V_{(1.2)}(X_t, Y_t) \leq -K_1(\parallel X \parallel^2 + \parallel Y \parallel^2) \\ + &(m + \delta(\parallel X \parallel + \parallel Y \parallel))(\Delta_a \parallel X \parallel + 2 \parallel Y \parallel) \\ = &-K_1(\parallel X \parallel^2 + \parallel Y \parallel^2) + m(\Delta_a \parallel X \parallel + 2 \parallel Y \parallel) \\ + &\delta\Delta_a \parallel X \parallel^2 + 2\delta \parallel X \parallel\parallel Y \parallel + \delta\Delta_a \parallel X \parallel\parallel Y \parallel + 2\delta \parallel Y \parallel^2 . \end{aligned}$$

Using the identity $2|\langle X, Y\rangle| \leq \parallel X \parallel^2 + \parallel Y \parallel^2$ and simplify, we obtain

$$\begin{aligned} &\frac{d}{dt}V_{(1.2)}(X_t, Y_t) \leq -K_1(\parallel X \parallel^2 + \parallel Y \parallel^2) \\ + &m(\Delta_a \parallel X \parallel + 2 \parallel Y \parallel) + \delta(\frac{3}{2}\Delta_a + 1) \parallel X \parallel^2 \\ + &\delta(\frac{1}{2}\Delta_a + 3) \parallel Y \parallel^2 \\ = &-(K_1 - \delta K_2)(\parallel X \parallel^2 + \parallel Y \parallel^2) \\ + &m(\Delta_a \parallel X \parallel + 2 \parallel Y \parallel). \end{aligned}$$

where $K_2 = \max(\frac{3}{2}\Delta_a + 1, \frac{1}{2}\Delta_a + 3)$. If we take $\delta < \frac{K_1}{K_2}$ then there exist some constant $\theta > 0$ such that

$$\frac{d}{dt}V(X_t, Y_t) \leq -\theta(\parallel X \parallel^2 + \parallel Y \parallel^2) + k\theta(\Delta_a \parallel X \parallel + 2 \parallel Y \parallel)$$
$$= -\frac{\theta}{2}(\parallel X \parallel^2 + \parallel Y \parallel^2) - \frac{\theta}{2}\{(\parallel X \parallel - k)^2 + (\parallel Y \parallel - k)^2\} + \theta k^2$$
$$\leq -\frac{\theta}{2}(\parallel X \parallel^2 + \parallel Y \parallel^2) + \theta k^2, \ for\ some\ k,\ \theta > 0.$$

This completes the proof.

4. CONCLUDING REMARKS

In the study of qualitative properties of solutions of both linear and non-linear differential equations, Lyapunov direct method remains one of the most powerful methods. Therefore, we have employed this method to prove the stability and boundedness of solutions of second order delay differential equations.

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