Journal of the Nigerian Mathematical Society
Vol. 37, Issue 2, pp. 89-109, 2018

ON A QUASILINEAR WAVE EQUATION WITH MEMORY AND NONLINEAR SOURCE TERMS

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ABSTRACT. In this paper, we consider a quasilinear wave equation with memory and nonlinear source terms

$$u_{tt} - \Delta u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + \int_0^t m(t-s)\Delta u ds = g(u).$$

In the absence of the nonlinear damping term, and under certain polynomial growth conditions on the nonlinear functions $\sigma_i, (i = 1, 2, ..., n)$ and g, we obtain existence and uniqueness of solution, using Galerkin approach and monotonicity method. The finite time blow up result was obtained using the concavity method.

Keywords and phrases: Initial-boundary value problem, Quasilinear wave equation, Damped wave equation, weak solutions, Blow up, Concavity method.
2010 Mathematical Subject Classification: 35L05, 35L15, 35L70

1. Introduction

In this paper, we are concerned with existence and blow up of solutions to quasilinear wave equations of the form

$$\begin{cases} u_{tt} - \Delta u_t - \displaystyle\sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{x_i}) + \int_0^t m(t-s)\Delta u ds \\ \qquad = g(u) \qquad x \in \Omega, \qquad t > 0 \\ u(x,t)|_{\partial\Omega} = 0, \qquad t > 0 \\ u(x,0) = u_0, \qquad u_t(x,0) = u_1, \qquad x \in \Omega \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary $\delta\Omega$, $\Delta = \Sigma_{i=1}^n \frac{\partial^2}{\partial x_i^2}$ is a Laplace operator in $\mathbb{R}^n$ and $u = u(x,t)$ is an unknown real valued function on $\Omega \times [0, \infty)$.

Equation of the type (1.1) was first introduced in [9] noting the fact that viscoelastic materials exhibit natural damping due to the special property of the materials for retaining a memory of their

Received by the editors August 16, 2017; Revised: May 07, 2018; Accepted: May 07, 2018
www.nigerianmathematicalsociety.org; Journal available online at www.ojs.ictp.it

past. They studied the following one dimensional wave equation

$$u_{tt} - u_{xxt} - \sigma(u_x)_x = 0 \tag{1.2}$$

under certain monotonicity condition on the function $\sigma(s)$ and obtained global existence of classical solutions for the initial boundary value problem (1.2). Equations of the type (1.1) are wave equations describing the motion of a viscoelastic solid made up of materials of the rate type. In one dimension, they apply to an infinite slab of materials with faces normal to the x-axis, and are useful approximate model for purely longitudinal motion of homogeneous thin bar having uniform cross section and unit length. For results on the IBVP of the type (1.1) in one dimension see [1, 2, 3, 4, 5, 13].

In two and three dimensions, they describe antiplane shear motions of viscoelastic solids. Clement in [6] was the first to extend the study to the multidimensional case ($\Omega \subset \mathbb{R}^n$), He studied equations of the form

$$u_{tt} - \Delta u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i} a_i(x, t, u_{x_i}) = f(x, t) \tag{1.3}$$

and obtained global existence of weak solutions for the initial value problem (1.3), exploiting the monotone operator method under certain restriction on the function $\sigma_i (i = 1, 2, ..., n)$. For other results on the multidimensional case see [14, 16, 15, 17, 18]

Levin [10, 11] was the first to study the interaction between the damping and the source term, where he studied a nonlinear wave equation of the form

$$\begin{cases} u_{tt} - \Delta u + |u_t|^{\alpha-2} u_t = |u|^{p-2} u \qquad x \in \Omega, \quad t > 0 \\ u(x, 0) = u_0, \ u_t(x, 0) = u_1, \ x \in \Omega \quad u(x, t)|_{\partial\Omega} = 0, \quad t > 0 \end{cases} \tag{1.4}$$

He considered existence and asymptotic behaviour of solutions to (1.4) for the case $\alpha = 2$, and using the concavity method, He showed that the solution blow up when the energy is sufficiently negative,

His result was extended by Georgiev and Todorova [8], where they considered global existence and blow up results of (1.4) for $\alpha > 2$, using a different method (the method is based on the perturbation of the total energy). In considering the relationship between α and p, they showed that for $\alpha \geq p$ with negative energy, the solution is global in time and for $p > \alpha$ the solution cannot be global when the initial energy is sufficiently negative.

More recently, there has been extensive literature on global existence, nonexistence and other properties of the IBVP

$$\begin{cases} u_{tt} - \Delta u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma(u_{x_i}) + f(u_t) = g(u) \qquad x \in \Omega, \quad t > 0 \\ u(x,t)|_{\partial\Omega} = 0, \ t > 0 \ u(x,0) = u_0, \ u_t(x,0) = u_1, \ x \in \Omega \end{cases} \tag{1.5}$$

having a nonlinear damping and source term, see [15, 17, 18].

In this paper, under mild assumption on the nonlinear function σ_i $(i = 1, ..., n)$ and g and in the absence of the nonlinear damping term, we prove existence and nonexistence of solutions to the IBVP (1.1), having a memory term m

2. PRELIMINARIES

In this section, we state some basic assumptions that will be used throughout the paper. For simplicity, we introduce the following notations

p' Hölder conjugate of p where $p' = \frac{p}{p-1}$.
$\|\cdot\|_p$ the usual $L^p(\Omega)$ norm for $1 \le p \le \infty$.
$W^{k,p}(\Omega)$ Banach space of functions in L^p with $k (k \in \mathbb{N})$ generalized derivatives .
$H^k(\Omega)$ Banach space $W^{k,2}(\Omega)$.
$C([a,b]; X)$ space of strongly continuous functions from $[a,b]$ to X.
$\langle \cdot, \cdot \rangle$ the inner product in $L^2(\Omega)$ or duality product between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$.

Consider the Hilbert space

$$H_0^1(\Omega) = \{u \in H^1(\Omega) : u|_{\partial\Omega} = 0\} \tag{2.1}$$

Lemma 2.1. *(Sobolev-Poincare inequality): Suppose that* $0 < p \le \frac{2}{n-2}$ *if* $n \ge 3$ *and* $p > 0$ *if* $n = 1, 2$. *Then there exist a constant* k *(optimal constant of Sobolev immersion) such that*

$$\|u\|_{2(p+1)} \le k \|\nabla u\|_2, \qquad \|u\|_{p+2} \le k \|u\|_{2(p+1)} \tag{2.2}$$

for all $u \in H_0^1(\Omega)$, *where we have the following embedding*

$$H_0^1(\Omega) \hookrightarrow L^{2(p+1)}(\Omega) \hookrightarrow L^{p+2}(\Omega) \tag{2.3}$$

Lemma 2.2. *Let* $\phi(t)$ *be a nonnegative function on* $[0, \infty]$ *satisfying*

$$\phi(t) \le B_1 + B_2 \int_0^t \phi^{\delta}(s) ds \tag{2.4}$$

where B_1, B_2 are positive constants, then $\phi(t)$ satisfy the inequality

$$\phi(t) \leq B_1[1-(\delta-1)B_2B_1^{\delta-1}t]^{\frac{-1}{\delta-1}} \qquad \textit{for} \quad \delta > 1. \tag{2.5}$$

We state the following assumptions on the nonlinear functions g and σ_i $(i = 1, ..., n)$. The nonlinear function g satisfies

(A_1) $g \in C(\mathbb{R})$ and $|g(s)| \leq \lambda |s|^{p+1}$, $s \in \mathbb{R}$

(A_2) $|g(u) - g(v)| \leq \lambda_*(|u|^p + |v|^p)|u - v|$ $\forall u, v \in \mathbb{R}$
and the function $\sigma_i(s)$ satisfies

(B_1) $\sigma_i \in C^1(\mathbb{R})$, $\sigma_i'(s) \geq \alpha_0$ and

(B_2) $\sum_{i=1}^n \int_\Omega \int_0^{u_{x_i}} \sigma_i(y)dydx \geq \alpha_1 \|u\|_{1,2}^{q+2}$, $\sum_{i=1}^n \|\sigma_i(u_{x_i})\|_2 \leq \alpha_2 \|u\|_{1,2}^{q+1}$, for $q \geq 0$

where λ, λ_*, α_0, α_1 and α_2 are positive constants.

We also assume here that $m \in C^1[0, \infty)$ is a nonnegative and non increasing function satisfying

$$m(0) > 0, \qquad m'(s) \leq 0, \qquad \int_0^\infty m(s)ds \leq \gamma < 1. \tag{2.6}$$

and through out this paper, we will make use of Young's inequality of the form

$$XY \leq \epsilon X^{m_0} + c(\epsilon)Y^{n_0}$$

where X, Y, m_0 , n_0, ϵ, $c(\epsilon)$ are positive constants and $\frac{1}{m_0} + \frac{1}{n_0} = 1$

2.1. **Local existence.** In this section, we consider local existence of weak solutions to (1.1) in the maximal interval $(0, T]$, $0 < t_n < T$. We use the Faedo- Galerkin approximation procedure [12], see also [7].

Definition 2.1. *By a weak solution of (1.1) over [0 , T], we mean a function*

$u \in C^0\big([0, T], H_0^1(\Omega) \cap H^2(\Omega)\big) \cap C^1\big([0, T]; L^2(\Omega)\big) \cap C^2\big([0, T], H^{-1}(\Omega)\big)$ *with* $u_t \in L^2\big([0, T]; H_0^1(\Omega)\big)$ *such that* $u(0) = u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_t(0) = u_1 \in L^2(\Omega)$ *and for a.e* $t \in [0, T]$

$$\big\langle u_n''(t), w\big\rangle + \sum_{i=1}^n \big\langle \sigma_i(u_{n_{x_i}}(t)), \nabla w_{x_i}\big\rangle + \big\langle \nabla u_n'(t), \nabla w\big\rangle$$
$$+\Big\langle \int_0^t m(t-s)\Delta u_n(s)ds, w\Big\rangle = \big\langle g(u_n(t)), w\big\rangle, \textit{for } w \in H_0^1(\Omega) \tag{2.7}$$

Lemma 2.3. *Suppose that the assumptions (A_1), (A_2), (B_1) and (B_2) hold for $0 < p \leq \frac{2}{n-2}$ if $n \geq 3$ and $p > 0$, if $n = 1, 2$. Then the problem (1.1) with $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$, admits a unique solution u on $[0, T)$ such that*

$u \in L^\infty\big([0,T]; H_0^1(\Omega) \cap H^2(\Omega)\big)$
$u_t \in L^\infty\big([0,T]; L^2(\Omega) \cap L^2\big([0,T], H_0^1(\Omega)\big)\big)$
$u_{tt} \in L^2\big([0,T]; H^{-1}(\Omega)\big),$

Proof: Let the sequence of functions $(w_n)_{n\in\mathbb{N}}$ be a basis in $H_0^1(\Omega)\cap H^2(\Omega)$ which is orthogonal in L^2. We look for a weak solution of the form

$$u_n(t) = \sum_{i=1}^{n} b_{in}(t) w_i \tag{2.8}$$

satisfying the approximate problem corresponding to (1.1)

$$\begin{aligned} &\big\langle u_n''(t), w_i\big\rangle + \sum_{i=1}^{n} \big\langle \sigma_i(u_{n_{x_i}}(t)), \nabla w_{x_i}\big\rangle + \big\langle \nabla u_n'(t), \nabla w_i\big\rangle \\ &\qquad + \big\langle \int_0^t m(t-s)\nabla u_n(s) ds, \nabla w_i\big\rangle = \big\langle g(u_n(t)), w_i\big\rangle \end{aligned} \tag{2.9}$$

for $w_i \in H_0^1(\Omega)$ with initial conditions

$$u_n(0) = u_{0n} \equiv \sum_{i=1}^{n} c_{in} w_i \to u_0 \ \text{ strongly in } H_0^1(\Omega) \cap H^2(\Omega) \tag{2.10}$$

as $n \to \infty$ and

$$u_n'(0) = u_{1n} \equiv \sum_{i=1}^{n} d_{in} w_i \to u_1 \text{ strongly in } L^2(\Omega) \text{ as } n \to \infty \tag{2.11}$$

where $b_{in}(t) = \big\langle u_n(t), w_i\big\rangle$, $c_{in} = \big\langle u_0, w_i\big\rangle$, $d_{in} = \big\langle u_1, w_i\big\rangle$ and $u_n' = \frac{du_n}{dt}$, $u_n'' = \frac{d^2u_n}{dt^2}$. By the continuity assumption on $\sigma_i(i = 1,2,...,n)$ and g, there exist a solution $u_n(t)$ to the system (2.9) - (2.11) on the interval $(0, t_n)$. Hence, using standard methods in differential equations, we prove the existence of solutions to (1.1) on some interval $[0, t_n)$, for $0 < t_n < T$. We will need the a-priori estimates below to show that the local solution is uniformly bounded on the whole interval [0 , T] for all n.

A-priori energy estimates. Setting $w_i = u_n'(t)$ in (2.9), we obtain

$$\begin{aligned} &\Big\langle u_n'(t), u_n''(t)\Big\rangle + \sum_{i=1}^{n} \Big\langle \sigma_i(u_{nx_i}(t)), u_{nx_i}'(t)\Big\rangle + \Big\langle \nabla u_n'(t), \nabla u_n'(t)\Big\rangle \\ &= \Big\langle g(u_n(t)), u_n'(t)\Big\rangle + \Big\langle \int_0^t m(t-s)\nabla u_n(s), \nabla u_n'(t)\Big\rangle ds \end{aligned} \tag{2.12}$$

which gives

$$\begin{aligned}&\frac{d}{dt}\Big[\frac{1}{2}\|u_n'(t)\|_2^2+\sum_{i=1}^{n}\int_\Omega\int_0^{u_{nx_i}(t)}\sigma_i(y)dydx\Big]+\|u_n'(t)\|_{1,2}^2\\&\le\int_\Omega|g(u_n(t))||u_n'(t)|\,dx+\int_0^t m(t-s)\int_\Omega\nabla u_n(s)\nabla u_n'(t)dxds\end{aligned}\tag{2.13}$$

We estimate the terms on the right hand side using Hölder and Young's inequality

$$\begin{aligned}\int_\Omega|g(u_n(t))||u_n'(t)|dx&\le\lambda\|u_n(t)\|_{2(p+1)}^{p+1}\|u_n'(t)\|_2\\&\le\lambda c(\epsilon_1)\|u_n(t)\|_{2(p+1)}^{2(p+1)}+\lambda\epsilon_1\|u_n'(t)\|_2^2\\&\le\lambda c(\epsilon_1)k^{2(p+1)}\big(\|u_n(t)\|_{1,2}^2\big)^{(p+1)}+\lambda\epsilon_1\|u_n'(t)\|_2^2\end{aligned}\tag{2.14}$$

and for $\int_0^t m(t-s)\int_\Omega\nabla u_n(s)\nabla u_n'(t)dxds$, we have

$$\begin{aligned}&\int_0^t m(t-s)\int_\Omega\nabla u_n(s)\nabla u_n'(t)dxdt\\&\le c(\epsilon_1)\int_0^t m(t-s)\|\nabla u_n(s)\|_2^2ds+\epsilon_1\int_0^t m(s)ds\|\nabla u_n'(t)\|_2^2\end{aligned}\tag{2.15}$$

Using (2.6), (2.14) and (2.15), we obtain

$$\begin{aligned}\frac{d}{dt}\Big[\frac{1}{2}\|u_n'(t)\|_2^2+\sum_{i=1}^{n}\int_\Omega\int_0^{u_{nx_i}(t)}\sigma_i(y)dydx\Big]&+(1-\gamma\epsilon_1)\|u_n'(t)\|_{1,2}^2\\&\le\lambda\epsilon_1\|u_n'(t)\|_2^2+\lambda c(\epsilon_1)k^{2(p+1)}\big(\|u_n(t)\|_{1,2}^2\big)^{p+1}\\&\quad+c(\epsilon_1)\int_0^t m(t-s)\|u_n(s)\|_{1,2}^2ds\end{aligned}\tag{2.16}$$

Integrating (2.16) over t for $t\in[0,T]$, we get

$$\begin{aligned}&\frac{1}{2}\|u_n'(t)\|_2^2+\sum_{i=1}^{n}\int_\Omega\int_0^{u_{nx_i}(t)}\sigma_i(y)dydx+\int_0^t\big(1-\gamma\epsilon_1\big)\|u_n'(s)\|_{1,2}^2ds\\&\le C_0+\lambda\epsilon_1\int_0^t\|u_n'(s)\|_2^2ds+\lambda c(\epsilon_1)k^{2(p+1)}\int_0^t\big(\|u_n(s)\|_{1,2}^2\big)^{p+1}ds\\&\quad+\gamma c(\epsilon_1)\int_0^t\|u_n(s)\|_{1,2}^2ds\end{aligned}\tag{2.17}$$

where

$$C_0=\frac{1}{2}\|u_n'(0)\|_2^2+\sum_{i=1}^{n}\int_\Omega\int_0^{u_{nx_i}(0)}\sigma_i(y)dydx$$

and we can choose ϵ_1 such that $0 < \epsilon_1 < 1$. From assumption (B_2), we have

$$\sum_{i=1}^{n} \int_\Omega \int_0^{u_{nx_i}(t)} \sigma_i(y)dydx \geq \alpha_1 \|u_n(t)\|_{1,2}^2 \tag{2.18}$$

hence, we can choose a positive constant C_0^* such that (2.17) becomes

$$\begin{aligned} &\|u_n'(t)\|_2^2 + \|u_n(t)\|_{1,2}^2 + \int_0^t \|u_n'(s)\|_{1,2}^2 ds \\ &\leq C_0 + C_0^* \int_0^t \left[\|u_n'(s)\|_2^2 + \|u_n(s)\|_{1,2}^2 + \int_0^s \|u_n'(\tau)\|_{1,2}^2 d\tau\right]^{p+1} ds \end{aligned} \tag{2.19}$$

Therefore, setting

$$h_n(t) = \|u_n'(t)\|_2^2 + \|u_n(t)\|_{1,2}^2 + \int_0^t \|u_n'(s)\|_{1,2}^2 ds \tag{2.20}$$

for $t \in [0, T]$, hence from (2.19) and (2.20), we obtain

$$h_n(t) \leq C_0 + C_0^* \int_0^t h_n^{p+1}(s)ds \tag{2.21}$$

where C_0^* is independent of $n \in \mathbb{N}$. Hence from Lemma 2.2, since $p > 0$, we have

$$h_n(t) \leq C_0\left[1 - pC_0^* C_0^p t\right]^{\frac{-1}{p}} \tag{2.22}$$

thus from (2.22), there exist a small time $T_0 \in [0, T]$ satisfying

$$T_0 < \left[pC_0^* C_0^p\right]^{-1}$$

such that the right hand side exist. Hence we can choose a constant $K_0 > 0$ independent of $n \in \mathbb{N}$ such that

$$\|u_n'(t)\|_2^2 + \|u_n(t)\|_{1,2}^2 + \int_0^t \|\nabla u_n'(s)\|_2^2 ds \leq K_0 \tag{2.23}$$

for $t \in [0, T]$. Now (2.23) implies that

$$\|u_n'(t)\|_2^2 \leq K_0 \tag{2.24}$$

$$\|u_n(t)\|_{1,2}^2 \leq K_0 \tag{2.25}$$

and

$$\int_0^t \|\nabla u_n'(s)\|_2^2 ds \leq K_0 \tag{2.26}$$

furthermore, from (2.25) and (2.26), it follows that

$$\|u_n(t)\|_2^2 \leq K_0 \tag{2.27}$$

and

$$\int_0^t \|u_n'(s)\|_2^2 ds \leq K_0 \tag{2.28}$$

.

Setting $w_i = -\Delta u_n(t)$ in (2.9), we have

$$
\begin{aligned}
&-\Big\langle u_n''(t), \Delta u_n(t)\Big\rangle - \Big\langle \int_0^t m(t-s)\Delta u_n(s)ds, \Delta u_n(t)\Big\rangle \\
&= -\Big\langle \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\sigma_i(u_{nx_i}(t)), \Delta u_n(t)\Big\rangle - \Big\langle \Delta u_n'(t), \Delta u_n(t)\Big\rangle \\
&\quad - \Big\langle g(u_n(t)), \Delta u_n(t)\Big\rangle,
\end{aligned} \tag{2.29}
$$

which yields

$$
\begin{aligned}
&\frac{d}{dt}\big(\|\Delta u_n(t)\|_2^2 - 2\int_\Omega u_n'(t)\Delta u_n(t)dx\big) + 2\sum_{i=1}^{n}\int_\Omega \sigma_i'(u_{nx_i}(t))|u_{nx_ix_i}(t)|^2dx \\
&\le 2\|\nabla u_n'(t)\|_2^2 + 2\int_0^t m(t-s)\int_\Omega \Delta u_n(s)\Delta u_n(t)dxds \\
&\quad +2\int_\Omega |g(u_n(t))||\Delta u_n(t)|dx
\end{aligned} \tag{2.30}
$$

for the second term on the left hand side, using Hölder inequality and Young's inequality, we obtain the following estimate

$$
\begin{aligned}
2\int_\Omega |\Delta u_n(t)||u_n'(t)| &\le 2\|\Delta u_n(t)\|_2\|u_n'(t)\|_2 \\
&\le 2\epsilon_2\|\Delta u_n(t)\|_2^2 + 2c(\epsilon_2)\|u_n'(t)\|_2^2
\end{aligned} \tag{2.31}
$$

and for the second and third term on the right hand side, by using assumption (A_1) together with Hölder and Young's inequality, we obtain

$$
\begin{aligned}
2\int_\Omega |g(u_n(t))||\Delta u_n(t)|dx &\le 2\lambda\|u_n(t)\|_{2(p+1)}^{p+1}\|\Delta u_n(t)\|_2 \\
&\le 2\lambda c(\epsilon_2)\|u_n(t)\|_{2(p+1)}^{2(p+1)} + 2\lambda\epsilon\|\Delta u_n(t)\|_2^2 \\
&\le 2\lambda c(\epsilon_2)k^{4(p+1)}\big(\|\Delta u_n(t)\|_2^2\big)^{p+1} \\
&\quad +2\lambda\epsilon_2\|\Delta u_n(t)\|_2^2
\end{aligned} \tag{2.32}
$$

and the term $2\int_0^t m(t-s)\Delta u_n(s)\Delta u_n(t)ds$ yields

$$
\begin{aligned}
&2\int_0^t m(t-s)\int_\Omega \Delta u_n(s)\Delta u_n(t)dxds \\
&\le 2c(\epsilon_2)\int_0^t m(t-s)\|\Delta u_n(s)\|_2^2ds + 2\epsilon_2\int_0^t m(s)ds\|\Delta u_n(t)\|_2^2
\end{aligned} \tag{2.33}
$$

using assumption (B_1) and substituting (2.6) and (2.31) - (2.33) into (2.30), we obtain

$$\begin{aligned}&\frac{d}{dt}\big((1-2\epsilon_2)\|u_n(t)\|_{2,2}^2-2c(\epsilon_2)\|u_n'(t)\|_2^2\big)+2\alpha_0\|u_n(t)\|_{2,2}^2\\&\leq 2\|u_n'(t)\|_{1,2}^2+2\epsilon_2\big(\lambda+\gamma\big)\|u_n(t)\|_{2,2}^2+2\lambda c(\epsilon_2)k^{4(p+1)}\big(\|u_n(t)\|_{2,2}^2\big)^{p+1}\\&\quad+2c(\epsilon_2)\int_0^t m(t-s)\|u_n(s)\|_{2,2}^2ds\end{aligned}\tag{2.34}$$

integrating over t for $t\in[0,T]$, we have

$$\begin{aligned}&(1-2\epsilon_2)\|u_n(t)\|_{2,2}^2+2\alpha_0\int_0^t\|u_n(s)\|_{2,2}^2ds\\&\leq C_1+2c(\epsilon_2)\|u_n'(t)\|_2^2+2\int_0^t\|u_n'(s)\|_{1,2}^2ds\\&\quad+2\lambda c(\epsilon_2)k^{4(p+1)}\int_0^t\big(\|u_n(s)\|_{2,2}^2\big)^{p+1}ds\\&\quad+2\epsilon_2\big(\lambda+\gamma\big)\int_0^t\|u_n(s)\|_{2,2}^2ds+2\gamma c(\epsilon_2)\int_0^t\|u_n(s)\|_{2,2}^2ds\end{aligned}\tag{2.35}$$

where

$$C_1=(1-2\epsilon_2)\|u_n(0)\|_{2,2}^2-2c(\epsilon_2)\|u_n'(0)\|_2^2$$

Using (2.24) and (2.26), we can choose positive constants C_2 and C_2^* independent of $n\in\mathbb{N}$ and $\epsilon_2<\frac{1}{2}$ such that (2.35) yields

$$\begin{aligned}&\|u_n(t)\|_{2,2}^2+\int_0^t\|u_n(s)\|_{2,2}^2ds\\&\leq C_2+C_2^*\int_0^t\big(\|u_n(t)\|_{2,2}^2+\int_0^s\|u_n(\tau)\|_{2,2}^2d\tau\big)^{p+1}ds\end{aligned}\tag{2.36}$$

using the same approach as before, we can show using Lemma 2.2 that there exist a time $t\in[0,T]$ and a positive constant K_1 such that

$$\|u_n(t)\|_{2,2}^2+\int_0^t\|u_n(s)\|_{2,2}^2ds\leq K_1\tag{2.37}$$

Hence, we have that

$$\|u_n(t)\|_{2,2}^2\leq K_1\tag{2.38}$$

and

$$\int_0^t\|u_n(s)\|_{2,2}^2ds\leq K_1\tag{2.39}$$

Now for the nonlinear terms, we adopt the ideas used in [14, 16, 18]. Here, we define the operator $A : H_0^1 \to H^{-1}$ by

$$\big\langle Au_n, v\big\rangle = \sum_{i=1}^{n}\big\langle \sigma_i(u_{nx_i}(t)), v_{x_i}\big\rangle \quad \text{for any } u, v \in H_0^1 \tag{2.40}$$

hence from (2.25), assumption (B_2) and the Hölder inequality we obtain

$$\begin{aligned}\big\langle Au_n(t), v\big\rangle &\le \sum_{i=1}^{n}\|\sigma_i(u_{nx_i}(t))\|_2\|v_{x_i}\|_2 \\ &\le C_3\|u_n(t)\|_{1,2}^{q+1}\|v\|_{1,2} \le K_2\|v\|_{1,2}\end{aligned} \tag{2.41}$$

thus we have

$$\|Au_n(t)\|_{-1,2} \le K_2 \quad \text{for } t \in [0, T]. \tag{2.42}$$

Likewise for the nonlinear function g, we have from (2.27) and assumption (A_1)

$$\|g(u_n(t))\|_2 \le C_4\|u_n(t)\|_2^{p+1} \le K_3 \quad \text{for } t \in [0, T] \tag{2.43}$$

where the constant K_3 is independent of n. Therefore for any $T > 0$, we have that the nonlinear terms are bounded on $[0, T]$.

Now, setting $w_i = v$ in (2.9), for $v \in H_0^1$, we have

$$\begin{aligned}|\big\langle u_n''(t), v\big\rangle| \le & C_5\big(\|Au_n(t)\|_{-1,2} + \|\nabla u_n'(t)\|_2 + \|g(u_n(t))\|_2 \\ & + \int_0^t m(t-s)\|\nabla u(s)\|_2 ds\big)\|v\|_{1,2}\end{aligned} \tag{2.44}$$

Hence, from (2.42), (2.43) and (2.44), we have that

$$\|u_n''(t)\|_{-1,2} \le C_6\big(\|\nabla u_n'(t)\|_2 + \int_0^t m(t-s)\|\nabla u(s)\|_2 ds + 1\big) \tag{2.45}$$

Applying Cauchy-Schwartz inequality and Poincaré's inequality, we see that

$$\|u_n''(s)\|_{-1,2}^2 \le C_7\Big[\|\nabla u_n'(t)\|_2^2 + \int_0^t m(t-s)\|\nabla u(s)\|_2^2 ds + 1\Big] \tag{2.46}$$

integrating over t for $t \in [0, T]$, we get

$$\int_0^t \|u_n''(s)\|_{-1,2}^2 ds \le C_8 \int_0^t \Big(\|\nabla u_n'(t)\|_2^2 + \|\nabla u(s)\|_2^2 + 1\Big) ds \tag{2.47}$$

Hence using (2.28) and (2.39), we obtain

$$\int_0^t \|u_n''(s)\|_{-1,2}^2 ds \le K_4 \tag{2.48}$$

for $t \in [0,T]$ and $K_4 > 0$

The estimates above permit us to obtain a subsequence u_i of u_n and to pass the limit in the approximate problem to obtain a weak solution satisfying
(l_1) $u_i(t) \to u(t)$ weakly-star in $L^\infty([0,T]; H_0^1(\Omega) \cap H^2(\Omega))$
(l_2) $u_i'(t) \to u'(t)$ weakly-star in $L^\infty([0,T]; L^2(\Omega)) \cap L^2([0,T]; H_0^1(\Omega))$
(l_3) $u_i''(t) \to u''(t)$ weakly-star in $L^2([0,T]; H^{-1}(\Omega))$
(l_4) $Au_i(t) \to \chi(t)$ weakly-star in $L^\infty([0,T]; H^{-1}(\Omega))$
(l_5) $g(u_i(t)) \to \xi(t)$ weakly-star in $L^\infty([0,T]; L^2(\Omega))$
Using Aubin-Lion's theorem, we deduce that as $n \to \infty$
(l_6) $u_i(t) \to u(t)$ strongly in $L^\infty([0,T]; H_0^1(\Omega))$
(l_7) $u_i'(t) \to u'(t)$ strongly in $L^\infty([0,T]; L^2(\Omega))$
Letting $n \to \infty$ in (2.9), we deduce from (l_1)-(l_5) that

$$\int_0^T \left[\langle u'', v\rangle + \langle \nabla u', \nabla v\rangle + \langle \chi, v\rangle - \langle \xi, v\rangle\right] dt = 0 \tag{2.49}$$

for all $v \in L^2([0,T]; H_0^1(\Omega))$. Hence we are left to prove that $\chi = Au$ and $\xi = g(u)$

By the Sobolev embedding theorem, the continuity of g and assumption (A_2), we have that for any $t \in [0,T]$

$$\begin{aligned} &\|g(u_n(t)) - g(u(t))\|_2 \\ &\leq C_9\Big[\|u_n(t)\|_{2(p+1)}^p + \|u(t)\|_{2(p+1)}^p\Big]\|u_n(t) - u(t)\|_{2(p+1)} \\ &\leq C_9^*\Big[\|u_n(t)\|_{2(p+1)}^p + \|u(t)\|_{2(p+1)}^p\Big]\|\nabla u_n(t) - \nabla u(t)\|_2 \end{aligned} \tag{2.50}$$

hence from (2.25) (l_1) and (2.50) when $n \to \infty$, we have

$$\begin{aligned} g(u_n) \to g(u) &\qquad \text{strongly in } L^2(\Omega) \quad \text{and} \\ \xi(t) = &g(u(t)) \quad \text{in} \quad L^2(\Omega), \quad t \in [0,T], \end{aligned} \tag{2.51}$$

Now for the nonlinear function σ_i, using monotonicity method, we show that $\chi = Au(t)$. From (2.49) and (2.51), we obtain

$$\begin{aligned} \int_0^t \langle \chi, v\rangle ds = \int_0^t &\Big[-\langle u'', v\rangle - \langle \nabla u', \nabla v\rangle \\ &+ \langle \int_0^s m(s-\tau)\nabla u(\tau)d\tau, \nabla v\rangle + \langle g(u), v\rangle\Big] ds \end{aligned} \tag{2.52}$$

for $t \in [0,T]$. Now, owing to the continuity of σ_i and the fact that it is a non decreasing monotone function, we have that for any

$v \in L^2([0,T]; H_0^1(\Omega))$ and $t \in [0,T]$

$$
\begin{aligned}
0 \le \chi_n(t) &= \int_0^t \big\langle Au_n(s) - Av(s), u_n(s) - v(s)\big\rangle ds \\
&= \int_0^t \big\langle Au_n(s), u_n(s)\big\rangle ds - \int_0^t \big\langle Au_n(s), v(s)\big\rangle ds \qquad (2.53) \\
&\quad - \int_0^t \big\langle Av(s), u_n(s) - v(s)\big\rangle ds
\end{aligned}
$$

setting $w_i = u_n(t)$ in (2.9) and integrating from 0 to t, we get

$$
\begin{aligned}
&\int_0^t \big\langle Au_n(s), u_n(s)\big\rangle ds \\
&= \int_0^t \big[-\big\langle u_n''(s), u_n(s)\big\rangle - \big\langle \nabla u_n'(s), \nabla u_n(s)\big\rangle\big] ds \qquad (2.54) \\
&+ \int_0^t \Big[\big\langle g(u_n(s)), u_n(s)\big\rangle + \big\langle \int_0^s m(s-\tau)\nabla u_n(\tau)d\tau, \nabla u_n(s)\big\rangle\Big] ds
\end{aligned}
$$

Hence from (2.54), passing limits and using (2.52) for $t \in [0,T]$, we have

$$
\begin{aligned}
0 &\le \limsup_{n\to\infty} \chi_n(t) \\
&\le -\int_0^t \big\langle u''(s), u(s)\big\rangle ds - \int_0^t \big\langle \nabla u'(s), \nabla u(s)\big\rangle ds \\
&+ \int_0^t \big\langle \int_0^s m(s-\tau)\nabla u(\tau)d\tau, \nabla u(s)\big\rangle ds + \int_0^t \big\langle g(u(s)), u(s)\big\rangle ds \qquad (2.55) \\
&- \int_0^t \big\langle \chi(s), v(s)\big\rangle ds - \int_0^t \big\langle Av(s), u(s) - v(s)\big\rangle ds \\
&= \int_0^t \big\langle \chi(s) - A(v(s)), u(s) - v(s)\big\rangle
\end{aligned}
$$

Now we set $v = u - \lambda w$ for any $v, w \in L^2([0,T]; H_0^1(\Omega))$, $\lambda > 0$, $u \in L^\infty([0,T]; H_0^1(\Omega)) \subset L^2([0,T]; H_0^1(\Omega))$, χ, $A(v) \in L^2([0,T]; H^{-1}(\Omega))$, and letting $\lambda \to 0$, then from (2.55) the inequality yields

$$
\int_0^t \big\langle \chi - A(u - \lambda w), w\big\rangle \ge 0 \qquad (2.56)
$$

and from the semicontinuty of the operator $A(u)$, we obtain

$$
\int_0^t \big\langle \chi - A(u), w\big\rangle \ge 0 \quad \forall w \in L^2([0,T]; H_0^1) \qquad (2.57)
$$

Hence we conclude that $\chi = A(u)$

Uniqueness. Let u_1 and u_2 be two solutions of (2.9) and let $z = u_1 - u_2$ then, this satisfies the equation

$$\begin{aligned}&\big\langle z''(t), w_i\big\rangle + \sum_{i=1}^{n}\big\langle \sigma_i(u_{1x_i}) - \sigma_i(u_{2x_i}), w_{ix_i}\big\rangle + \big\langle \nabla z'(t), \nabla w_i\big\rangle \\ &\quad = \big\langle g(u_1) - g(u_2), w_i\big\rangle + \big\langle \int_0^t m(t-s)\nabla z(s)ds, \nabla w_i\big\rangle\end{aligned} \tag{2.58}$$

for $w \in H_0^1(\Omega)$

$$\begin{aligned} z(x,0) &= 0 \qquad x \in \Omega \\ z(x,t) &= 0 \qquad x \in \delta\Omega, \quad t \geq 0\end{aligned}$$

setting $w_i = z'(t)$ in (2.58), we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\Big(\|z'(t)\|_2^2\Big) + \|\nabla z'(t)\|_2^2 \\ &\quad = \big\langle g(u_1) - g(u_2), z'(t)\big\rangle - \sum_{i=1}^{n}\big\langle [\sigma_i(u_{1x_i}) - \sigma_i(u_{2x_i})], {z'}_{x_i}(t)\big\rangle \\ &\quad + \int_0^t m(t-s)\langle \nabla z(s), \nabla z'(t)\rangle ds\end{aligned} \tag{2.59}$$

From Hölder and Young's inequality, estimating the first term on the right hand side, we have

$$\begin{aligned}&\int_0^t m(t-s)\int_\Omega \nabla z(s)\nabla z'(t)dxds \\ &\leq c(\epsilon_3)\int_0^t m(t-s)\|\nabla z(s)\|_2^2 ds + \epsilon_3\int_0^t m(s)ds\|\nabla z'(t)\|_2^2\end{aligned} \tag{2.60}$$

and for the second term on the right hand side, using (2.25), assumption (A_1) and Holder's inequality we have

$$\begin{aligned}&\big\langle g(u_1(t)) - g(u_2(t)), z'(t)\big\rangle \\ &\leq C_{10}\left(\|u_1(t)\|_{2p+2}^p + \|u_2(t)\|_{2p+2}^p\right)\|z(t)\|_{2p+2}\|z'(t)\|_2 \\ &\leq C_{10}^*\Big(\|\nabla u_1(t)\|_2^p + \|\nabla u_2(t)\|_2^p\Big)\|\nabla z(t)\|_2\|z'(t)\|_2 \\ &\leq C_{11}c(\epsilon_3)\|z'(t)\|_2^2 + C_{11}\epsilon_3\|\nabla z(t)\|_2^2\end{aligned} \tag{2.61}$$

Also, from the continuity property of σ_i, we obtain,

$$\begin{aligned}&\sum_{i=1}^{n}\int_\Omega[\sigma_i(u_{1x_i}) - \sigma_i(u_{2x_i})]\ {z'}_{x_i}(t)dx \\ &\quad \leq C_{12}\|\nabla z(t)\|_2\|\nabla z'(t)\|_2 \\ &\quad \leq C_{12}^*c(\epsilon_3)\|\nabla z(t)\|_2^2 + C_{12}^*\epsilon_3\|\nabla z'(t)\|_2^2\end{aligned} \tag{2.62}$$

Adding the term $\langle \nabla z(t), \nabla z'(t) \rangle$ to both sides of (2.59), and using the estimates (2.6), (2.60), (2.61) and (2.62), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\Big(\|z'(t)\|_2^2 + \|z(t)\|_{1,2}^2\Big) + (1 - \gamma\epsilon_3 + C_{12}^* c(\epsilon_3))\|z'(t)\|_{1,2}^2 \\
&\leq C_{11}c(\epsilon_3)\|z'(t)\|_2^2 + c(\epsilon)\int_0^t m(t-s)\|z(s)\|_{1,2}^2 ds \\
&+ (C_{11}\epsilon_3 + C_{12}^* c(\epsilon_3))\|z(t)\|_{1,2}^2
\end{aligned} \tag{2.63}$$

then integrating both sides and using Gronwall's lemma , we get

$$\|z'(t)\|_2^2 + \|z(t)\|_{1,2}^2 + \int_0^t \|z'(s)\|_{1,2}^2 dt \leq K_6 \tag{2.64}$$

Hence we have that

$$\|z'(t)\|_2^2 = \|z(t)\|_{1,2}^2 = 0 \tag{2.65}$$

for all $t \in [0, T]$, which gives the desired result.

3. Blow up result

In this section we consider the blow up property of the solution to (1.1), we use the concavity method of Levin [10, 11] in obtaining blow up results for negative initial energy. For the proof of this result, we will need the following lemma

Lemma 3.1. *Let $u(x,t)$ be a solution of the problem (1.1). Then the energy equation of the problem (1.1) is defined by*

$$\begin{aligned}
E(t) =& \frac{1}{2}\|u_t(t)\|_2^2 + \sum_{i=1}^n \int_\Omega \int_0^{u_{x_i}} \sigma_i(y)dydx - \frac{1}{2}\int_0^t m(s)ds\|\nabla u(t)\|_2^2 \\
&+ \frac{1}{2}\big(m \circ \nabla u\big)(t) - \int_\Omega \int_0^u g(y)dydx
\end{aligned} \tag{3.1}$$

In addition, $E(t)$ is non increasing and satisfies

$$\begin{aligned}
E'(t) =& -\|\nabla u_t(t)\|_2^2 + \frac{1}{2}\int_0^t m'(t-s)\|\nabla u(s) - \nabla u(t)\|_2^2 ds \\
&- \frac{1}{2}m(t)\|\nabla u(t)\|_2^2 \leq 0
\end{aligned} \tag{3.2}$$

where $\big(m \circ v\big)(t) = \int_0^t m(t-s)\|v(t,\cdot) - v(s,\cdot)\|_2^2 ds$

Proof: Multiplying (1.1) by $u_t(t)$ and integrating over Ω, we obtain

$$\begin{aligned}\frac{d}{dt}\Big[\frac{1}{2}\|u_t(t)\|_2^2+\sum_{i=1}^n\int_\Omega\int_0^{u_{x_i}}\sigma_i(y)dydx-\int_\Omega\int_0^u g(y)dydx\Big]\\ -\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u_t(t)dxds=-\|\nabla u_t(t)\|_2^2\end{aligned}\tag{3.3}$$

Now for the term $\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u_t(t)dxds$, we have the estimate

$$\begin{aligned}&\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u_t(t)dxds\\ =&-\frac{1}{2}\int_0^t m(t-s)\big(\frac{d}{dt}\int_\Omega|\nabla u(s)-\nabla u(t)|^2dx\big)ds\\ &+\frac{1}{2}\int_0^t m(s)\big(\frac{d}{dt}\int_\Omega|\nabla u(t)|^2dx\big)ds\\ =&-\frac{1}{2}\frac{d}{dt}\big(\int_0^t m(t-s)\int_\Omega|\nabla u(s)-\nabla u(t)|^2dxds\big)\\ &+\frac{1}{2}\int_0^t m'(t-s)\int_\Omega|\nabla u(s)-\nabla u(t)|^2dxds\\ &+\frac{1}{2}\frac{d}{dt}\big(\int_0^t m(s)\int_\Omega|\nabla u(t)|^2dxds\big)-\frac{1}{2}m(t)\int_\Omega|\nabla u(t)|^2dx\end{aligned}\tag{3.4}$$

Hence from (3.3) and (3.4), we have

$$\begin{aligned}&\frac{d}{dt}\Big[\frac{1}{2}\|u_t\|_2^2+\sum_{i=1}^n\int_\Omega\int_0^{u_{x_i}}\sigma_i(y)dydx-\frac{1}{2}\int_0^t m(s)ds\|\nabla u(t)\|_2^2\\ &\quad+\frac{1}{2}\big(m\circ\nabla u\big)(t)-\int_\Omega\int_0^u g(y)dydx\Big]\\ &=-\|\nabla u_t(t)\|_2^2+\frac{1}{2}\int_0^t m'(t-s)\|\nabla u(s)-\nabla u(t)\|_2^2ds\\ &\quad-\frac{1}{2}m(t)\|\nabla u(t)\|_2^2\end{aligned}\tag{3.5}$$

which gives the desired result for any regular solution. This result remains valid for weak solutions by a simple density argument. Moreover this satisfies

$$E(t)+\int_0^t\|u_s(s)\|_{1,2}^2ds\le E(0)\tag{3.6}$$

where

$$E(0) = \frac{1}{2}\|u_1\|_2^2 + \sum_{i=1}^{n}\int_\Omega\int_0^{u_{x_i}(0)}\sigma_i(y)dydx - \int_\Omega\int_0^{u_0} g(y)dydx$$

Definition 3.1. *A solution $u(x,t)$ of (1.1) is said to blow-up in finite time, if there exists a finite time T^* such that*

$$\lim_{t\to T^*}\Big(\int_\Omega |\nabla u|^2 dx\Big)^{-1} = 0 \tag{3.7}$$

Theorem 3.1. *Let $u(x,t)$ be a solution of the problem (1.1), assume that the conditions of Lemma 2.3 hold, in addition, suppose that $\sigma_i(s)$ satisfies*

$$2\sum_{i=1}^{n}\Big[(\alpha+2)\int_\Omega\int_0^{u_{x_i}}\sigma_i(y)dydx - \int_\Omega \sigma_i(u_{x_i})u_{x_i}dx\Big] \geq M\|u\|_{1,2}^2 \tag{3.8}$$

where $M > 0$ is a constant and $g(s)$ satisfies

$$\int_\Omega ug(u)dx \geq (\alpha+2)\int_\Omega\int_0^u g(y)dydx \tag{3.9}$$

and that

$$\int_0^\infty m(s)ds \leq \frac{M}{(\alpha+1+\epsilon)} < 1 \tag{3.10}$$

then the solution $u(x,t)$ of (1.1) blow up in finite time.

Proof: Let u be a solution of (1.1) and define the functional

$$a(t) = \|u\|_2^2 + \int_0^t \|\nabla u\|_2^2 dt + (T-t)\|\nabla u_0\|_2^2 + \beta(t+\tau)^2 \tag{3.11}$$

where $t \in [0,T]$ and $\beta > 0$, then Differentiating (3.11), we obtain

$$\begin{aligned} a'(t) &= 2\int_\Omega uu_t dx + \|\nabla u\|_2^2 - \|\nabla u_0\|_2^2 + 2\beta(t+\tau) \\ &= 2\int_\Omega uu_t dx + 2\int_0^t\int_\Omega \nabla u\nabla u_t dxdt + 2\beta(t+\tau) \end{aligned} \tag{3.12}$$

Differentiating again and using (1.1), we have,

$$a''(t) = 2\|u_t\|_2^2 + 2\int_\Omega uu_{tt}dx + 2\int_\Omega \nabla u\cdot\nabla u_t dx + 2\beta$$

which yields

$$\begin{aligned} a''(t) =& 2\|u_t\|_2^2 - 2\sum_{i=1}^{n}\int_\Omega \sigma_i(u_{x_i})u_{x_i}dx + 2\int_\Omega ug(u)dx + 2\beta \\ &+ 2\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds \end{aligned} \tag{3.13}$$

From (3.12), we have that

$$\begin{aligned} &a''(t)a(t) - \frac{\alpha+4}{4}a'(t)^2 \\ &= a(t)a''(t) - (\alpha+4)\Big[\int_\Omega uu_tdx + \int_0^t \nabla u\nabla u_tds + \beta(t+\tau)\Big]^2 \\ &= a(t)\Big[a''(t) - (\alpha+4)\Big(r(t) - \big\{a(t) - (T-t)\|\nabla u_0\|_2^2\big\} \\ &\qquad\qquad \times \big\{\|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2dt + \beta\big\}\Big)\Big] \end{aligned} \tag{3.14}$$

for $t \le T$, where

$$\begin{aligned} r(t) =& \Big[\|u\|_2^2 + \int_0^t \|\nabla u\|_2^2dt + \beta(t+\tau)^2\Big]\Big[\|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2dt + \beta\Big] \\ &-\Big[\int_\Omega uu_tdx + \int_0^t\int_\Omega \nabla u\cdot\nabla u_tdxdt + \beta(t+\tau)\Big] \end{aligned} \tag{3.15}$$

Using Schwartz's inequality, we have the following estimates

$$\begin{aligned} \Big[\int_\Omega uu_tdx\Big]^2 &\le \|u\|_2^2\|u_t\|_2^2, \\ \Big[\int_0^t \nabla u\nabla u_tds\Big]^2 &\le \int_0^t \|\nabla u\|_2^2ds\int_0^t \|\nabla u_t\|_2^2ds \end{aligned} \tag{3.16}$$

and

$$\begin{aligned} &\int_\Omega uu_tdx\int_0^t\int_\Omega \nabla u\nabla u_tdxdt \\ &\le \|u\|_2\Big[\int_0^t \|\nabla u_t\|_2^2ds\Big]^{\frac12}\|u_t\|_2\Big[\int_0^t \|\nabla u\|_2^2ds\Big]^{\frac12} \\ &\le \frac12\|u\|_2^2\int_0^t \|\nabla u_t\|_2^2ds + \frac12\|u_t\|_2^2\int_0^t \|\nabla u\|_2^2ds \end{aligned} \tag{3.17}$$

hence from the estimate (3.16) and (3.27), we see that $r(t) > 0$ for $t \in [0,T]$. Thus

$$a''(t)a(t) - \frac{\alpha+4}{4}a'(t)^2 \ge a(t)\eta(t) \tag{3.18}$$

where

$$\begin{aligned}\eta(t) = \Big[&2\|u_t\|_2^2 - 2\sum_{i=1}^{n}\int_\Omega \sigma_i(u_{x_i})u_{x_i}dx + 2\int_\Omega ug(u)dx \\ &+ 2\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds + 2\beta \\ &- (\alpha+4)\big(\|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2 dt + \beta\big)\Big]\end{aligned} \tag{3.19}$$

From (3.6) and assumption (3.9), we have that

$$\begin{aligned}&2\int_\Omega ug(u)dx \\ \geq& 2(\alpha+2)\int_\Omega\int_0^u g(y)dydx \\ \geq& (\alpha+2)\|u_t\|_2^2 + 2(\alpha+2)\sum_{i=1}^{n}\int_\Omega\int_0^{u_{x_i}} \sigma_i(y)dydx \\ &- (\alpha+2)\int_0^t m(s)ds\|\nabla u(t)\|_2^2 + 2(\alpha+2)\int_0^t \|\nabla u_t\|_2^2 ds \\ &+ (\alpha+2)\int_0^t m(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds - 2(\alpha+2)E(0)\end{aligned} \tag{3.20}$$

hence substituting (3.20) in (3.19), we obtain

$$\begin{aligned}\eta(t) \geq& 2\sum_{i=1}^{n}\Big[(\alpha+2)\int_\Omega\int_0^{u_{x_i}} \sigma_i(y)dydx - \int_\Omega \sigma_i(u_{x_i})u_{x_i}dx\Big] \\ &+ (\alpha+4)\|u_t\|_2^2 - (\alpha+2)\int_0^t m(s)ds\|\nabla u(t)\|_2^2 \\ &+ 2\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds - 2(\alpha+2)E(0) \\ &+ (\alpha+2)\int_0^t m(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds + 2\beta \\ &+ 2(\alpha+2)\int_0^t \|\nabla u_t\|_2^2 ds - (\alpha+4)\big(\|u_t\|_2^2 + \int_0^t \|\nabla u_t\|_2^2 dt + \beta\big)\end{aligned}$$

using assumption (3.8), we have

$$\begin{aligned}\eta(t) \geq & \Big[M-(\alpha+2)\int_0^t m(s)ds\Big]\|u(t)\|_{1,2}^2-2(\alpha+2)E(0)\\ & +2\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds-(\alpha+2)\beta \\ & +(\alpha+2)\int_0^t m(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds\end{aligned} \tag{3.21}$$

to estimate the term $\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds$, we use Young's inequality and Schwartz inequality to get

$$\begin{aligned}&\int_0^t m(t-s)\int_\Omega \nabla u(s)\nabla u(t)dxds\\ &=\int_0^t m(t-s)\int_\Omega \nabla u(t)\big(\nabla u(s)-\nabla u(t)\big)dxds+\int_0^t m(s)ds\|\nabla u(t)\|_2^2\\ &\geq -\Big[c(\epsilon)\int_0^t m(t-s)\|\nabla u(t)-\nabla u(s)\|_2^2 ds+\epsilon\int_0^t m(s)ds\|\nabla u(t)\|_2^2\Big]\\ &\quad+\int_0^t m(s)ds\|\nabla u(t)\|_2^2\end{aligned} \tag{3.22}$$

using the estimate (3.22) in (3.21), we have

$$\begin{aligned}\eta(t) \geq & \big(M-(\alpha+1+\epsilon)\int_0^t m(s)ds\big)\|u(t)\|_{1,2}^2\\ & -(\alpha+2)\beta-2(\alpha+2)E(0)\\ \geq & -(\alpha+2)\beta-2(\alpha+2)E(0)\geq 0\end{aligned} \tag{3.23}$$

for $E(0)\leq \frac{-\beta}{2}$ and $m(t)$ satisfying (3.10). Hence we obtain

$$\big(a(t)^{\frac{-\alpha}{2}}\big)''=\tfrac{-\alpha}{4}a(t)^{-\frac{\alpha+8}{4}}\big(a(t)a''(t)-\tfrac{\alpha+4}{4}\left(a'(t)\right)^2\big)\leq 0 \tag{3.24}$$

setting $y(t)=a(t)^{-\frac{\alpha}{4}}$, then we have

$$y''(t)\leq -\tfrac{\alpha}{4}y(t)^{1+\frac{8}{\alpha}} \tag{3.25}$$

Thus $y(t)$ tends to zero in finite time , say T^* where we assume $T^*<T$, since T^* is independent of the initial choice of T. Hence

$$\lim_{t\to T^*} a(t)=\infty \tag{3.26}$$

which implies that

$$\lim_{t\to T^*}\|\nabla u(t)\|_2^2=\infty$$

Example

In this article, we consider quasilinear wave equations where the nonlinearities take the form $\sigma_i(s) = (1+s)^{q+1}$, and $g(s) = |s|^p s$. Hence equation (1.1) becomes

$$u_{tt} - \Delta u_t - \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\big(1+u_{x_i}\big)^{q+1} + \int_0^t m(t-s)\Delta u(s)ds = |u|^p u \quad (3.27)$$

It is easy to see that the assumptions of Lemma 2.3 and Theorem 3.1 are satisfied, if we choose the initial data $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$, $u_1 \in L^2(\Omega)$. Therefore, for $u \in C^0\big([0,T]; H_0^1 \cap H^2(\Omega)\big) \cap C^1\big([0,T]; L^2(\Omega)\big) \cap C^2\big([0,T], H^{-1}(\Omega)\big)$ with $u_t \in L^2\big([0,T]; H_0^1(\Omega)$, we have from Lemma 2.3 that for $T > 0$, the corresponding problem (3.27) has a weak solution which according to Theorem 3.1 blow up in finite time.

Acknowledgment

The author would like to thank the referee for the careful reading of this paper and for the valuable comments and suggestions to improve the presentation of this paper.

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