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AN EFFICIENT FAMILY OF SECOND DERIVATIVE RUNGE-KUTTA COLLOCATION METHODS FOR OSCILLATORY SYSTEMS

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ABSTRACT. An efficient family of high-order second derivative Runge-Kutta collocation methods is derived for the numerical solution of oscillatory systems. The approach uses polynomial interpolation and collocation techniques to construct continuous schemes which were evaluated at both step and off-step points to obtain hybrid formulae. The hybrid formulae can be applied simultaneously as block methods for moving the integration process forward at a time, if desired. The block methods based on hybrid formulation can also be converted to second derivative Runge-Kutta collocation methods. The stability properties and order of accuracy of the methods are studied. They can also be implemented easily since they are collocation methods and provide a high order of accuracy. The methods were illustrated by the applications to some test problems of oscillatory system found in the literature and the numerical results obtained confirm the accuracy and efficiency of the methods.

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1. INTRODUCTION

Many of the most popular numerical integration methods for ordinary differential equations showed up very well in integrating oscillatory system of initial value problems of the form

$$\left\{ \begin{array}{l} y'(x) = f(x, y(x)),\ y(x_0) = y_0, \\ y : \mathbb{R} \to \mathbb{R}^m,\ f : \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}^m. \end{array} \right. \tag{1}$$

Though, methods for solving oscillatory problems can be classified into two: the first consists of methods with constant coefficients

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which can be applied to problems with periodic solutions. The second having coefficients depending on the frequency of the problem, when a good estimate of the frequency is known in advance, see[1, 2, 3]. In particular, implicit methods are suitable for the numerical integration of oscillatory system of initial value problems in ordinary differential equations, see for example [4, 5]. In this paper we shall be interested in implicit Runge-Kutta methods, especially implicit second derivative Runge-Kutta collocation(SDRKC) methods because they have high order of convergence and good stability properties [6, 7, 8, 9, 10, 11, 12, 13]. However, on the other hand, the computational cost of these methods is relatively high since they are fully implicit. The second derivative implicit Runge-Kutta collocation methods belong to the family of multi-derivative numerical integration methods and are one-step multistage methods. The stages and the final output of the SDRK collocation methods at the end of step number n are defined respectively as

$$\begin{aligned} &Y_i = y_{n-1} + h\sum_{j=0}^{s} a_{ij} f(Y_j) + h^2 \sum_{j=0}^{s} \hat{a}_{ij} g(Y_j), \\ &i = 1, 2, ..., s, \end{aligned} \tag{2}$$

$$y_n = y_{n-1} + h\sum_{i=0}^{s} b_i f(Y_i) + h^2 \sum_{i=0}^{s} \hat{b}_i g(Y_i), \tag{3}$$

where the quantities $Y_1, Y_2, Y_3, ..., Y_s$ are called internal stage values and y_n is the update at the n^{th} step, that is the numerical approximation to the exact solution *y(x)* at $x = x_n$. The integer s is the number of stages of the method. Also $a_{ij}, \hat{a}_{ij}, b_i$ and $\hat{b}_i$ are the constant coefficients which can be constructed so that y_n is a good approximation to the solution $y(x_n) = y(x_{n-1} + h)$ and h denotes the step size $x_n - x_{n-1}$ which is sometimes constant or varied during integration. In the second derivative Runge-Kutta methods, in addition to the computation of the f-values at the internal stages in the standard Runge-Kutta methods, the second derivative methods involve computing g-values, where f and g are defined as

$$y'_{n+j} = f_{n+j} \equiv f(x_n + jh, y(x_n + jh)) \tag{4}$$

and

$$y''_{n+j} = g_{n+j} \equiv f_x + f_y y' = f_x + f f_y. \tag{5}$$

According to [12] these methods can be practical if the costs of evaluating g are comparable to those in evaluating f, and can be more efficient than the standard Runge-Kutta methods if the number of function evaluations is fewer. It is convenient to rewrite the coefficients of the defining method (2) in block matrix form as,

$$Y = e \oplus y_n + h(A \oplus I_N)F(Y) + h^2(\hat{A} \oplus I_N)G(Y), \qquad (6)$$
$$y_{n+1} = y_n + h(b^T \oplus I_N)F(Y) + h^2(\hat{b}^T \oplus I_N)G(Y),$$

where the matrices $A = [a_{ij}]_{s\times s}$ and $\hat{A} = [\hat{a}_{ij}]_{s\times s}$ indicate the dependence of the stages on the derivatives found at the other stages and $b = [b_i]_{s\times 1}, \hat{b} = [\hat{b}_i]_{s\times 1}$ are vectors of quadrature weights, showing how the final result depends on the derivatives computed at the various stages, **I** is the identity matrix of size equal to the differential equation system to be solved, and N is the dimension of the system. Also $\oplus$ is the Kronecker product of two matrices and e is the $s \times 1$ vector of units [14]. For simplicity, we rewrite the method in (6) as follows

$$Y = y_n + hAF(Y) + h^2\hat{A}G(Y), \qquad (7)$$
$$y_{n+1} = y_n + hb^T F(Y) + h^2\hat{b}^T G(Y),$$

and the block vectors in R^{sN} are defined by

$$Y = \begin{bmatrix} Y_1 \\ Y_2 \\ \vdots \\ Y_s \end{bmatrix}, \; F(Y) = \begin{bmatrix} f(Y_1) \\ f(Y_2) \\ \vdots \\ f(Y_s) \end{bmatrix}, \; G(Y) = \begin{bmatrix} g(Y_1) \\ g(Y_2) \\ \vdots \\ g(Y_s) \end{bmatrix}.$$

The coefficients of the implicit second derivative Runge-Kutta collocation methods can be conveniently represented in a compact extended partitioned Butcher Tableau,

$$\begin{array}{c|c||c} c & A & \hat{A} \\ \hline & b^T & \hat{b}^T \end{array} \qquad (8)$$

where $c = [1]_{s\times 1}$ is the abscissae vectors which indicates the positions within the step of the stage values.

2. DERIVATION TECHNIQUE OF THE SDRKC METHODS

In this section we describe the derivation technique of the second derivative Runge-Kutta collocation methods for direct integration of oscillatory system of initial value problems of the form (1). We seek the approximant of the form

$$y(x) = \sum_{i=0}^{p-1} \phi_i x^i \tag{9}$$

for the numerical approximation to the exact solution $y(x)$ of equation (1). Following [15] we set the sum $p = r+s+t$, where r denotes the number of interpolation points used and $s > 0, t > 0$ are distinct collocation points. Interpolating $y(x)$ in (9) at the points $\{x_{n+j}\}$ and collocating $y'(x)$ and $y''(x)$ at the same points $\{x_{n+j}\}$ we have the following system of equations

$$\begin{aligned}
y(x_{n+j}) &= y_{n+j}, & (j = 0, 1, 2, ..., r-1), \quad (10)\\
y'(x_{n+j}) &= f_{n+j}, & (j = 0, 1, 2, ..., s-1), \quad (11)\\
y''(x_{n+j}) &= g_{n+j}, & (j = 0, 1, 2, ..., t-1). \quad (12)
\end{aligned}$$

Equations (10)-(12) can be expressed in the matrix form as:

$$V\phi = y \tag{13}$$

where the square matrix V, the vectors ϕ and y are defined as follows:

$$V = \begin{pmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & \cdots & x_n^{p-1} \\
1 & x_{n+1} & x_{n+1}^2 & x_{n+1}^3 & x_{n+1}^4 & \cdots & x_{n+1}^{p-1} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
1 & x_{n+r-1} & x_{n+r-1}^2 & x_{n+r-1}^3 & x_{n+r-1}^4 & \cdots & x_{n+r-1}^{p-1} \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & \cdots & D'x_n^{p-2} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 1 & 2x_{n+s-1} & 3x_{n+s-1}^2 & 4x_{n+s-1}^3 & \cdots & D'x_{n+s-1}^{p-2} \\
0 & 0 & 2 & 6x_n & 12x_n^2 & \cdots & D''x_n^{p-3} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 2 & 6x_{n+t-1} & 12x_{n+t-1}^2 & \cdots & D''x_{n+t-1}^{p-3}
\end{pmatrix}, \tag{14}$$

$$\phi = (\phi_0, \phi_1, \phi_2, \cdots, \phi_{p-1})^T,$$
$$y = (y_n, \cdots, y_{n+r-1}, y'_n, \cdots, y'_{n+s-1}, y''_n, \cdots, y''_{n+t-1})^T$$

where $D' = (p-1)$ and $D'' = (p-1)(p-2)$ in (14) represent the first and second derivatives respectively and correspond to the differentiation with respect to x. Similar to the Vandermonde matrix,

V in (13) is non-singular. A closed form of the solution for the system in (13) is presented which has been obtained by considering the inverse of the Vandermonde matrix, that is,

$$\phi = V^{-1} y. \tag{15}$$

From the interpolation polynomial (9) and the equations (13) -(15) we have the following multistep collocation formula of [16] which was a generalization of [17] and here we extend it to second derivative of the form

$$y(x) = \sum_{j=0}^{r-1} \phi_j(x) y_{n+j} + h \sum_{j=0}^{s-1} \psi_j(x) y'_{n+j} + h^2 \sum_{j=0}^{t-1} \gamma_j(x) y''_{n+j} \tag{16}$$

where $y_{n+j} = y(x_n + jh)$ and y'_{n+j} and y''_{n+j} are as defined in (4) and (5) respectively. Here $\phi_j(x), \psi_j(x)$ and $\gamma_j(x)$ are the continuous coefficients of the formula. They are assumed polynomials of degree *p-1* given by

$$\phi_j(x) = \sum_{i=0}^{p-1} \phi_{i+1,j} x^i,$$

$$h\psi_j(x) = h \sum_{i=0}^{p-1} \psi_{i+1,j} x^i$$

and

$$h^2 \gamma_j(x) = h^2 \sum_{i=0}^{p-1} \gamma_{i+1,j} x^i. \tag{17}$$

The numerical constant coefficients $\phi_{i+1,j}, \psi_{i+1,j}$ and $\gamma_{i+1,j}$ in (17) are to be determined. The evaluation of the matrix V and its inverse $U = V^{-1}$ are carried out with a computer algebra system, for example Maple to determine the constant coefficients $\phi_{i+1,j}, \psi_{i+1,j}$ and $\gamma_{i+1,j}$ in (17). We now state our main result in the theorem below, which is an extension of the theorem in [18].

Theorem 1: Let y(x) in (9) be differentiable on $[x_0, T]$ and has continuous second derivatives in the interval and satisfies the system of equations in(10)-(12). Then

$$V = U^{-1}$$

where V is as defined in (13) which is assumed to be non-singular.

Proof: To proof the theorem we define the basis functions x^i in (9) as

$$W_i(x) = \sum_{i=0}^{p-1} x^i.$$

Substituting (17) into (16) we have,

$$y(x) = \sum_{j=0}^{r-1}\sum_{i=0}^{p-1} \phi_{i+1,j} y_{n+j} W_i(x) + h\sum_{j=0}^{s-1}\sum_{i=0}^{p-1} \psi_{i+1,j} f_{n+j} W_i(x) + h^2 \sum_{j=0}^{t-1}\sum_{i=0}^{p-1} \gamma_{i+1,j} g_{n+j} W_i(x)$$

which is simplified to get

$$y(x) = \sum_{i=0}^{p-1}\left\{\sum_{j=0}^{r-1} \phi_{i+1,j} y_{n+j} + h\sum_{j=0}^{s-1} \psi_{i+1,j} f_{n+j} + h^2\sum_{j=0}^{t-1} \gamma_{i+1,j} g_{n+j}\right\} W_i(x). \tag{18}$$

If we let

$$Q_i = \sum_{j=0}^{r-1} \phi_{i+1,j} y_{n+j} + h\sum_{j=0}^{s-1} \psi_{i+1,j} f_{n+j} + h^2\sum_{j=0}^{t-1} \gamma_{i+1,j} g_{n+j} \tag{19}$$

then (18) becomes

$$y(x) = \sum_{i=0}^{p-1} Q_i W_i(x). \tag{20}$$

Now consider (20) in the vector form

$$\begin{gathered} y(x) = (Q_0, Q_1, Q_2, \cdots, Q_{p-1})^T \times \\ (W_0(x), W_1(x), W_2(x), \cdots, W_{p-1}(x))^T \\ y(x) = Q_i^T (W_i(x))^T. \end{gathered} \tag{21}$$

Imposing (10)-(12) on (21) we have the matrix form:

$$\sum_{i=0}^{p-1} Q_i W_i(x_{n+j}) \;=\; y_{n+j}, \qquad j \in \{0, 1, 2, ..., r-1\}, \tag{22}$$

$$\sum_{i=0}^{p-1} Q_i W_i(x_{n+j}) = f_{n+j}, \qquad (j = 0, 1, 2, ..., s-1), \tag{23}$$

$$\sum_{i=0}^{p-1} Q_i W_i(x_{n+j}) = g_{n+j}, \qquad (j = 0, 1, 2, ..., t-1), \tag{24}$$

giving

$$VQ = M. \tag{25}$$

Assuming that the matrix V is non-singular, then from (25) we have,

$$Q = V^{-1}M. \tag{26}$$

Inserting (26) into (21) and recall that $p = r + s + t$, we get the propose continuous scheme of the multistep collocation formula (16), written as,

$$\begin{aligned} y(x) &= M^T (V^{-1})^T (W_i(x))^T \\ &= (y_n, \cdots, y_{n+r-1}, f_n, \cdots, f_{n+s-1}, g_n, \cdots, g_{n+t-1})^T \times \\ & \quad U^T \left(1, x, \cdots, x^{r+s+t-1}\right)^T. \end{aligned} \tag{27}$$

Expanding (19) fully we get

$$\begin{aligned} Q_i = (&\phi_{i+1,0}, \cdots, \phi_{i+1,r-1}, h\psi_{i+1,0}, \cdots, \\ &h\psi_{i+1,s-1}, h^2\gamma_{i+1,0}, \cdots, h^2\gamma_{i+1,t-1}) \\ &M, \ j = 0, 1, 2, \cdots, p-1. \end{aligned} \tag{28}$$

Comparing the right hand side of (28) with (21) we have that

$$Q_i = U_{i+1}M. \tag{29}$$

Therefore

$$Q^T = (UM)^T \ \ (or \ Q = UM). \tag{30}$$

From (30) and(26) we get

$$U = V^{-1}. \tag{31}$$

This completes the proof. □

3. SPECIFICATION OF THE METHODS

3.1 A fourth-order SDRK collocation method

For the first second derivative Runge-Kutta collocation method of order four we define $\xi = (x - x_n)$ for the construction of the continuous scheme. We also consider the zeros of the second degree

Chebychev polynomial in the symmetric interval [-1,1], which were transformed into the standard interval $[x_n, x_{n+1}]$ by means of the following linear transformation $x \in [-1,1] \to [x_n, x_{n+1}]$. These Chebychev polynomials were chosen because of their superior convergence rate and stiffly accurate characteristic properties in relation to the approximation of functions [19]. Proceeding in the same manner as is done for the linear multistep methods, we expand (16) using the method of Taylor series expansion and collect powers in h to obtain the continuous scheme of the form in (27) as follows

$$y(x) = \phi_0(x)y_n + h[\psi_0(x)f_{n+u} + \psi_1(x)f_{n+v}] + h^2[\gamma_0(x)g_{n+u} + \gamma_1(x)g_{n+v}] \tag{32}$$

where

$$\phi_0(x) = 1,$$

$$\psi_0(x) = \left[\frac{48\sqrt{2}\xi^4 - 96\sqrt{2}h\xi^3 + 36\sqrt{2}h^2\xi^2 + (24+12\sqrt{2})h^3\xi}{48h^3}\right],$$

$$\psi_1(x) = \left[\frac{-48\sqrt{2}\xi^4 + 96\sqrt{2}h\xi^3 - 36\sqrt{2}h^2\xi^2 + (24-12\sqrt{2})h^3\xi}{48h^3}\right],$$

$$\gamma_0(x) = \left[\frac{24\xi^4 - (48+8\sqrt{2})h\xi^3 + (30+12\sqrt{2})h^2\xi^2 - (6+3\sqrt{2})h^3\xi}{48h^2}\right],$$

$$\gamma_1(x) = \left[\frac{24\xi^4 - (48-8\sqrt{2})h\xi^3 + (30-12\sqrt{2})h^2\xi^2 - (6-3\sqrt{2})h^3\xi}{48h^2}\right].$$

Evaluating the continuous scheme y(x) in (32) at the points $x = x_{n+1}, x_{n+u}$ and x_{n+v} (where u and v are the zeros of the second degree Chebychev polynomial) we obtain symmetric block hybrid formula,

$$y_{n+1} = y_n + \frac{h}{48}[24f_{n+u} + 24f_{n+v}] + \frac{h^2}{48}[\sqrt{2}g_{n+u} - \sqrt{2}g_{n+v}]$$

$$y_{n+u} = y_n + \frac{h}{384}[(96-30\sqrt{2})f_{n+u} + (96-66\sqrt{2})f_{n+v}] + \frac{h^2}{384}[(11-4\sqrt{2})g_{n+u} + (5-4\sqrt{2})g_{n+v}]$$

$$y_{n+v} = y_n + \frac{h}{384}[(96+66\sqrt{2})f_{n+u} + (96+30\sqrt{2})f_{n+v}] + \frac{h^2}{384}[(5+4\sqrt{2})g_{n+u} - (11+4\sqrt{2})g_{n+v}].$$

Converting the block hybrid formula to second derivative Runge-Kutta collocation method and using (7) we write the method as,

$$y_n = y_{n-1} + h\left(\frac{1}{2}\right)F_1 + h\left(\frac{1}{2}\right)F_2 + h^2\left(\frac{\sqrt{2}}{48}\right)G_1 - h^2\left(\frac{\sqrt{2}}{48}\right)G_2. \tag{33}$$

The internal stage values at the n^{th} step are computed as,

$$Y_1 = y_{n-1} + h\left(\frac{1}{4} - \frac{5\sqrt{2}}{64}\right)F_1 + h\left(\frac{1}{4} - \frac{11\sqrt{2}}{64}\right)F_2 - h^2\left(\frac{11}{384} - \frac{\sqrt{2}}{96}\right)G_1 + h^2\left(\frac{5}{384} - \frac{\sqrt{2}}{96}\right)G_2$$

$$Y_2 = y_{n-1} + h\left(\frac{1}{4} + \frac{11\sqrt{2}}{64}\right)F_1 + h\left(\frac{1}{4} + \frac{5\sqrt{2}}{64}\right)F_2 + h^2\left(\frac{5}{384} + \frac{\sqrt{2}}{96}\right)G_1 - h^2\left(\frac{11}{384} + \frac{\sqrt{2}}{96}\right)G_2$$

$$Y_3 = y_{n-1} + h\left(\frac{1}{2}\right)F_1 + h\left(\frac{1}{2}\right)F_2 + h^2\left(\frac{\sqrt{2}}{48}\right)G_1 - h^2\left(\frac{\sqrt{2}}{48}\right)G_2$$

with the stage derivatives as follows
$F_1 = f\left(x_{n-1} + h\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right), Y_1\right)$,
$F_2 = f\left(x_{n-1} + h\left(\frac{1}{2} + \frac{\sqrt{2}}{4}\right), Y_2\right)$,
$F_3 = f(x_{n-1} + h(1), Y_3)$.

We write the coefficients of the SDRK collocation method (33) in an extended Butcher Tableau (8) as follows:

$$
\begin{array}{c|c||c}
c & A & \hat{A} \\
\hline
 & b^T & \hat{b}^T
\end{array}
=
\begin{array}{c|cc||cc}
0 & 0 & 0 & 0 & 0 \\
\frac{2-\sqrt{2}}{4} & \frac{16-5\sqrt{2}}{64} & \frac{16-11\sqrt{2}}{64} & \frac{-11+4\sqrt{2}}{384} & \frac{5-4\sqrt{2}}{384} \\
\frac{2+\sqrt{2}}{4} & \frac{16+11\sqrt{2}}{64} & \frac{16+5\sqrt{2}}{64} & \frac{5+4\sqrt{2}}{384} & \frac{-11-4\sqrt{2}}{384} \\
1 & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{48} & \frac{-\sqrt{2}}{48} \\
\hline
 & \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{48} & \frac{-\sqrt{2}}{48}
\end{array}
$$

3.2 A sixth-order SDRK collocation method

For the second method of the second derivative Runge-Kutta collocation method, we consider the zeros of the Chebychev polynomial of degree 3 in the symmetric interval [-1,1], transformed into the standard interval $[x_n, x_{n+1}]$ using the same linear transformation of the form in method (33), that is, $x \in [-1, 1] \to [x_n, x_{n+1}]$. Again expanding (16) we have the continuous scheme of the form in (27) as follows,

$$
y(x) = \phi_0(x)y_n + h[\psi_0(x)f_{n+u} + \psi_1(x)f_{n+w} + \psi_2(x)f_{n+v}] +
$$

$$
h^2[\gamma_0(x)g_{n+u} + \gamma_1(x)g_{n+w} + \gamma_2(x)g_{n+v}] \tag{34}
$$

where

$$
\phi_0(x) = 1,
$$

$$
\psi_0(x) = \left[\frac{\begin{array}{l} 640\sqrt{3}\xi^6 - (384 + 1920\sqrt{3})h\xi^5 + (960 + 2100\sqrt{3})h^2\xi^4 - \\ (720 + 1000\sqrt{3})h^3\xi^3 + (120 + 150\sqrt{3})h^4\xi^2 + (60 + 30\sqrt{3})h^5\xi \end{array}}{135h^5} \right],
$$

$$
\psi_1(x) = \left[\frac{256\xi^5 - 640h\xi^4 + 480h^2\xi^3 - 80h^3\xi^2 + 5h^4\xi}{45h^4} \right],
$$

$$
\psi_2(x) = \left[\frac{\begin{array}{l} -640\sqrt{3}\xi^6 - (384 - 1920\sqrt{3})h\xi^5 + (960 - 2100\sqrt{3})h^2\xi^4 - \\ (720 - 1000\sqrt{3})h^3\xi^3 + (120 - 150\sqrt{3})h^4\xi^2 + (60 - 30\sqrt{3})h^5\xi \end{array}}{135h^5} \right],
$$

$$
\gamma_0(x) = \left[\frac{\begin{array}{l} 640\xi^6 - (1920 + 192\sqrt{3})h\xi^5 + (2220 + 480\sqrt{3})h^2\xi^4 - \\ (1240 + 420\sqrt{3})h^3\xi^3 + (330 + 150\sqrt{3})h^4\xi^2 - (30 + 15\sqrt{3})h^5\xi \end{array}}{540h^4} \right],
$$

$$\gamma_1(x) = \left[\frac{256\xi^6 - 768h\xi^5 + 816h^2\xi^4 - 352h^3\xi^3 + 51h^4\xi^2 - 3h^5\xi}{54h^4}\right],$$

$$\gamma_2(x) = \left[\frac{\begin{array}{l} 640\xi^6 - (1920 - 192\sqrt{3})h\xi^5 + (2220 - 480\sqrt{3})h^2\xi^4 - \\ (1240 - 420\sqrt{3})h^3\xi^3 + (330 - 150\sqrt{3})h^4\xi^2 - (30 - 15\sqrt{3})h^5\xi \end{array}}{540h^4}\right].$$

Evaluating the continuous scheme y(x) in (34) at the point $x = x_{n+1}$ and at some off-grid points x_{n+u}, x_{n+w} and x_{n+v} (where u, w and v are the zeros of the third degree Chebychev polynomial) we have the following symmetric block hybrid formula,

$$y_{n+1} = y_n + \tfrac{h}{180}[48f_{n+u} + 84f_{n+w} + 48f_{n+v}] + \tfrac{h^2}{180}[\sqrt{3}g_{n+u} - \sqrt{3}g_{n+v}]$$

$$y_{n+u} = y_n + \tfrac{h}{34560}[(4608 - 1396\sqrt{3})f_{n+u} + (8064 - 4608\sqrt{3})f_{n+w} + (4608 - 2636\sqrt{3})f_{n+v}] + \tfrac{h^2}{34560}[-(271 - 96\sqrt{3})g_{n+u} - 40g_{n+w} + (161 - 96\sqrt{3})g_{n+v}]$$

$$y_{n+w} = y_n + \tfrac{h}{4320}[(576 + 280\sqrt{3})f_{n+u} + 1008f_{n+w} + (576 - 280\sqrt{3})f_{n+v}] + \tfrac{h^2}{4320}[(10 + 12\sqrt{3})g_{n+u} - 140g_{n+w} + (10 - 12\sqrt{3})g_{n+v}]$$

$$y_{n+v} = y_n + \tfrac{h}{34560}[(4608 + 2636\sqrt{3})f_{n+u} + (8064 + 4608\sqrt{3})f_{n+w} + (4608 + 1396\sqrt{3})f_{n+v}] + \tfrac{h^2}{34560}[(161 + 96\sqrt{3})g_{n+u} - 40g_{n+w} - (271 + 96\sqrt{3})g_{n+v}].$$

Converting the block hybrid formula to second derivative Runge-Kutta collocation method and writing the method in the form of (7) we have,

$$y_n = y_{n-1} + h\left(\frac{4}{15}\right)F_1 + h\left(\frac{7}{15}\right)F_2 + h\left(\frac{4}{15}\right)F_3 +$$

$$h^2\left(\frac{\sqrt{3}}{180}\right)G_1 - h^2\left(\frac{\sqrt{3}}{180}\right)G_3 \tag{35}$$

where the internal stage values at the n^{th} step are computed as,

$$Y_1 = y_{n-1} + h\left(\frac{2}{15} - \frac{349\sqrt{3}}{8640}\right)F_1 + h\left(\frac{7}{30} - \frac{4\sqrt{3}}{30}\right)F_2 + \left(\frac{2}{15} - \frac{659\sqrt{3}}{8640}\right)F_3$$
$$-h^2\left(\frac{271}{34560} - \frac{\sqrt{3}}{360}\right)G_1 - h^2\left(\frac{1}{864}\right)G_2 + h^2\left(\frac{161}{34560} - \frac{\sqrt{3}}{360}\right)G_3$$

$$Y_2 = y_{n-1} + h\left(\frac{2}{15}+\frac{7\sqrt{3}}{108}\right)F_1 + h\left(\frac{7}{30}\right)F_2 + \left(\frac{2}{15}-\frac{7\sqrt{3}}{108}\right)F_3$$
$$+h^2\left(\frac{1}{432}+\frac{\sqrt{3}}{360}\right)G_1 - h^2\left(\frac{7}{216}\right)G_2 + h^2\left(\frac{1}{432}-\frac{\sqrt{3}}{360}\right)G_3$$

$$Y_3 = y_{n-1} + h\left(\frac{2}{15}+\frac{659\sqrt{3}}{8640}\right)F_1 + h\left(\frac{7}{30}+\frac{4\sqrt{3}}{30}\right)F_2 + \left(\frac{2}{15}+\frac{349\sqrt{3}}{8640}\right)F_3$$
$$+h^2\left(\frac{161}{34560}+\frac{\sqrt{3}}{360}\right)G_1 - h^2\left(\frac{1}{864}\right)G_2 - h^2\left(\frac{271}{34560}+\frac{\sqrt{3}}{360}\right)G_3$$

$$Y_4 = y_{n-1} + h\left(\frac{4}{15}\right)F_1 + h\left(\frac{7}{15}\right)F_2 + h\left(\frac{4}{15}\right)F_3 +$$
$$h^2\left(\frac{\sqrt{3}}{180}\right)G_1 - h^2\left(\frac{\sqrt{3}}{180}\right)G_3$$

and the stage derivatives are as follows

$F_1 = f\left(x_{n-1} + h\left(\frac{1}{2}-\frac{\sqrt{3}}{4}\right), Y_1\right),$

$F_2 = f\left(x_{n-1} + h\left(\frac{1}{2}\right), Y_2\right),$

$F_3 = f\left(x_{n-1} + h\left(\frac{1}{2}+\frac{\sqrt{3}}{4}\right), Y_3\right),$

$F_4 = f(x_{n-1} + h(1), Y_4).$

Using the extended Butcher Tableau (8) we present the coefficients of the SDRK collocation method (35) as follows,

$$\begin{array}{c|c||c} c & A & \hat{A} \\ \hline & b^T & \hat{b}^T \end{array} =$$

$$\begin{array}{c|ccc||ccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2-\sqrt{3}}{4} & \frac{1152-349\sqrt{3}}{8640} & \frac{7-4\sqrt{3}}{30} & \frac{1152-659\sqrt{3}}{8640} & \frac{-271+96\sqrt{3}}{34560} & \frac{-1}{864} & \frac{161-96\sqrt{3}}{34560} \\
\frac{1}{2} & \frac{144+70\sqrt{3}}{1080} & \frac{7}{30} & \frac{144-70\sqrt{3}}{1080} & \frac{10+12\sqrt{3}}{4320} & \frac{-7}{216} & \frac{10-12\sqrt{3}}{4320} \\
\frac{2+\sqrt{3}}{4} & \frac{1152+659\sqrt{3}}{8640} & \frac{7+4\sqrt{3}}{30} & \frac{1152+349\sqrt{3}}{8640} & \frac{161+96\sqrt{3}}{34560} & \frac{-1}{864} & \frac{-271-96\sqrt{3}}{34560} \\
1 & \frac{4}{15} & \frac{7}{15} & \frac{4}{15} & \frac{\sqrt{3}}{180} & 0 & \frac{-\sqrt{3}}{180} \\
\hline
 & \frac{4}{15} & \frac{7}{15} & \frac{4}{15} & \frac{\sqrt{3}}{180} & 0 & \frac{-\sqrt{3}}{180}
\end{array}$$

3.3 An eighth-order SDRK collocation method

In this section we introduce collocation at the two end points of the standard interval $[x_n, x_{n+1}]$ in addition to the interior collocation points of the zeros of the second degree Chebychev polynomial of method (33). Expanding (16) we obtain the proposed continuous scheme of the form in (27) as follows,

$$\begin{aligned} y(x) &= \phi_0(x)y_n + h[\psi_0(x)f_n + \psi_1(x)f_{n+u} + \psi_2(x)f_{n+v} + \psi_3(x)f_{n+1}] \\ &\quad + h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+u} + \gamma_2(x)g_{n+v} + \gamma_3(x)g_{n+1}] \end{aligned} \tag{36}$$

Evaluating the proposed continuous scheme y(x) in (36) at the points $x = x_{n+1}, x_{n+u}$ and x_{n+v} (where u and v are the zeros of the second degree Chebychev polynomial) we obtain the block hybrid formula which were converted to second derivative Runge-Kutta collocation method and using the extended Butcher Tableau (8) we present the coefficients of the SDRK collocation method (36) as follows,

$$\begin{array}{c|c} c & A \\ \hline & b^T \end{array} = \begin{array}{c|cccc} 0 & 0 & 0 & 0 & 0 \\ \frac{2-\sqrt{2}}{4} & \frac{13739-8448\sqrt{2}}{26880} & \frac{-3072+2929\sqrt{2}}{13440} & \frac{-3072+2159\sqrt{2}}{13440} & \frac{11989-8448\sqrt{2}}{26880} \\ \frac{2+\sqrt{2}}{4} & \frac{13739+8448\sqrt{2}}{26880} & \frac{-3072-2159\sqrt{2}}{13440} & \frac{-3072-2929\sqrt{2}}{13440} & \frac{11989+8448\sqrt{2}}{26880} \\ 1 & \frac{67}{70} & \frac{-16}{35} & \frac{-16}{35} & \frac{67}{70} \\ \hline & \frac{67}{70} & \frac{-16}{35} & \frac{-16}{35} & \frac{67}{70} \end{array}$$

$$\begin{array}{||c} \hat{A} \\ \hline \hat{b}^T \end{array} = \begin{array}{||cccc} 0 & 0 & 0 & 0 \\ \frac{1251-832\sqrt{2}}{53760} & \frac{-375+256\sqrt{2}}{5376} & \frac{361-256\sqrt{2}}{5376} & \frac{-1181+832\sqrt{2}}{5376} \\ \frac{1251+832\sqrt{2}}{53760} & \frac{361+256\sqrt{2}}{5376} & \frac{-375-256\sqrt{2}}{5376} & \frac{-1181-832\sqrt{2}}{5376} \\ \frac{19}{420} & \frac{2\sqrt{2}}{21} & \frac{-2\sqrt{2}}{21} & \frac{-19}{420} \\ \hline \frac{19}{420} & \frac{2\sqrt{2}}{21} & \frac{-2\sqrt{2}}{21} & \frac{-19}{420} \end{array}$$

3.4 A tenth-order SDRK collocation method

Using the three zeros of the Chebychev polynomial in method (35) as the interior collocation points we introduce collocation at the two end points of the standard interval $[x_n, x_{n+1}]$. Expanding (16) we have the proposed continuous scheme of the form in (27) as follows,

$$
\begin{aligned}
y(x) &= \phi_0(x)y_n \\
&+h[\psi_0(x)f_n + \psi_1(x)f_{n+u} + \psi_2(x)f_{n+w} + \psi_3(x)f_{n+v} + \psi_4(x)f_{n+1}] \\
&+h^2[\gamma_0(x)g_n + \gamma_1(x)g_{n+u} + \gamma_2(x)g_{n+w} + \gamma_3(x)g_{n+v} + \gamma_4(x)g_{n+1}]
\end{aligned} \tag{37}
$$

Evaluating the proposed continuous scheme y(x) in (37) at the points $x = x_{n+1}, x_{n+u}, x_{n+w}$, and x_{n+v} we obtain the block hybrid formula which were converted to second derivative Runge-Kutta collocation method. Displaying the coefficients of the method in the form of the extended Butcher Tableau (8) we have

$$
\begin{array}{c|c} c & A \\ \hline & b^T \end{array} =
$$

$$
\begin{array}{c|ccccc}
0 & 0 & 0 & 0 & 0 & 0 \\
\frac{2-\sqrt{3}}{4} & \frac{128150-71424\sqrt{3}}{143360} & \frac{-262144+158889\sqrt{3}}{362880} & \frac{608-351\sqrt{3}}{3240} & \frac{-262144+151287\sqrt{3}}{362880} & \frac{61877-35712\sqrt{3}}{71680} \\
\frac{1}{2} & \frac{1453}{1120} & \frac{-2048-462\sqrt{3}}{2835} & \frac{76}{405} & \frac{2048+462\sqrt{3}}{2835} & \frac{103}{224} \\
\frac{2+\sqrt{3}}{4} & \frac{128150+71424\sqrt{3}}{143360} & \frac{-262144-151287\sqrt{3}}{362880} & \frac{608+351\sqrt{3}}{3240} & \frac{-262144-158889\sqrt{3}}{362880} & \frac{61877+35712\sqrt{3}}{71680} \\
1 & \frac{123}{70} & -\frac{4096}{2835} & \frac{152}{405} & -\frac{4096}{2835} & \frac{123}{70} \\
\hline
 & \frac{123}{70} & -\frac{4096}{2835} & \frac{152}{405} & -\frac{4096}{2835} & \frac{123}{70}
\end{array}
$$

$$
\begin{array}{||c} \hat{A} \\ \hline \hat{b}^T \end{array} =
\begin{array}{||ccccc}
0 & 0 & 0 & 0 & 0 \\
\frac{54650-31104\sqrt{3}}{2580480} & \frac{-71637+40960\sqrt{3}}{1451520} & \frac{-1}{138240} & \frac{70923-40960\sqrt{3}}{1451520} & \frac{-26947+15552\sqrt{3}}{1290240} \\
\frac{613}{20160} & \frac{147+60\sqrt{3}}{5670} & \frac{-23}{1080} & \frac{147-160\sqrt{3}}{5670} & \frac{-47}{4032} \\
\frac{54650+31104\sqrt{3}}{2580480} & \frac{-70923+40960\sqrt{3}}{1451520} & \frac{-1}{138240} & \frac{-71637-40960\sqrt{3}}{1451520} & \frac{-26947-15552\sqrt{3}}{1290240} \\
\frac{53}{1260} & \frac{32\sqrt{3}}{567} & 0 & \frac{-32\sqrt{3}}{567} & \frac{-53}{1260} \\
\hline
\frac{53}{1260} & \frac{32\sqrt{3}}{567} & 0 & \frac{-32\sqrt{3}}{567} & \frac{-53}{1260}
\end{array}
$$

4. ANALYSIS OF THE SDRK COLLOCATION METHODS

4.1 Order, Consistency, Zero-stability and Convergence of the SDRKC Methods

With the multistep collocation formula (16) we associate the linear difference operator ℓ defined by

$$
\ell[y(x);h] = \sum_{j=0}^{r} \phi_j(x)y(x+jh) + h\sum_{j=0}^{s} \psi_j(x)y'(x+jh) + h^2\sum_{j=0}^{t} \gamma_j(x)y''(x+jh) \tag{38}
$$

where *y(x)* is an arbitrary function, continuously differentiable on

$[x_0, T]$. Following [20], we can write the terms in (38) as a Taylor series expansion about the point x to obtain the expression,

$$\ell[y(x); h] = C_0 y(x) + C_1 h y'(x) + C_2 h^2 y''(x) + \cdots + C_p h^p y^{(p)}(x) + \cdots \tag{39}$$

where the constant coefficients $C_p, p = 0, 1, 2, \cdots$ are given as follows,

$$C_0 = \sum_{j=0}^{r} \phi_j$$

$$C_1 = \sum_{j=0}^{r} j\phi_j$$

$$C_2 = \frac{1}{2!}\left[\sum_{j=1}^{r} j\phi_j - 2\sum_{j=0}^{s} \psi_j\right]$$

$$C_3 = \frac{1}{3!}\left[\sum_{j=1}^{r} j^2\phi_j - 3\sum_{j=1}^{s} j\psi_j - \sum_{j=0}^{t} \gamma_j\right]$$

$$\vdots$$

$$C_p = \frac{1}{p!}\left[\sum_{j=1}^{r} j^p\phi_j - \frac{1}{(p-1)!}\sum_{j=1}^{s} j^{p-1}\psi_j - \frac{1}{(p-2)!}\sum_{j=0}^{t} j^{p-2}\gamma_j\right],$$

$$p = 3, 4, \cdots$$

According to [20], the multistep collocation formula (16) has order p if

$$\ell[y(x); h] = \bigcirc\left(h^{(p+1)}\right), \; C_0 = C_1 = \cdots = C_p = 0, \; C_{p+1} \neq 0. \tag{40}$$

Therefore, C_{p+1} is the error constant and $C_{p+1}h^{p+1}y^{(p+1)}(x)$ is the principal local truncation error. Hence, from our calculation the order and error constants for the constructed methods are presented in Table 1. It is clear from the Table that the second derivative Runge-Kutta collocation methods are of high order with smaller error constants and hence more accurate than the conventional Gauss and Lobatto-Runge-Kutta methods of the same order of convergence.

Table 1: Order and error constants of the SDRKC methods

Method	Order	Error constant
Method (33)	p = 4,	$C_5 = 1.4583 \times 10^{-2}$
Method (35)	p = 6,	$C_7 = 1.1858 \times 10^{-4}$
Method (36)	p = 8,	$C_9 = 3.3585 \times 10^{-6}$
Method (37)	p = 10,	$C_{11} = 4.9761 \times 10^{-8}$

Definition 4.1:(Consistency) The second derivative of high-order accuracy Runge-Kutta collocation methods (33), (35), (36) and (37) are said to be consistent if the order of the individual method is greater than or equal to one, that is, if $p \geq 1$.
(i)$\rho(1) = 0$ and
(ii)$\rho'(1) = \sigma(1)$, where $\rho(z)$ and $\sigma(z)$ are respectively the 1st and 2nd characteristic polynomials.

From Table 1 and definition 4.1 we can attest that the second derivative Runge-Kutta collocation methods are consistent.

Definition 4.2: (Zero-stability) The second derivative Runge-Kutta collocation methods (33), (35), (36) and (37) are said to be zero-stable if the roots condition of the methods are satisfied, that is, if

$$\rho(\lambda) = det\left[\sum_{i=0}^{k} A^i \lambda^{k-1}\right] = 0$$

satisfies $|\lambda_j| \leq 1,\ j = 1, 2, \cdots, k$ and for those roots with $|\lambda_j| = 1$, the multiplicity does not exceed 2, (see [20]).

Definition 4.3: (Convergence) The necessary and sufficient conditions for the second derivative Runge-Kutta collocation methods (33), (35), (36) and (37) to be convergent are that they most be consistent and zero-stable (see [20] theorem 2.1 page 33 and [21]).

From definitions 4.1 and 4.2 the second-derivative Runge-Kutta collocation methods are convergent.

4.2 Regions of absolute stability of the SDRKC methods

For any new derived method for the solution of ordinary differential equations, linear stability is very important aspect to consider.

Therefore we consider for the new second derivative Runge-Kutta collocation methods the test equation of the form

$$\frac{dy}{dx} = \lambda y, \qquad \lambda \in \mathbb{C} \textit{ and } \Re\lambda < 0,$$

with a fixed positive step size h. Since the new methods contain second derivative *g(x,y)*, it is natural to suppose that $g(x,y) = \lambda^2 y$. Therefore reformulating the block hybrid formula in (33), (35), (36) and (37) as general linear methods (see [1]) which is represented by a partitioned $(s+r) \times (s+r)$ matrices containing A, U, B and V. Here, for convenience we replaced U with C and V with D. The elements of the matrices *A, C, B* and *D* are substituted into the stability matrix,

$$y^{[n-1]} = M(z)y^{[n]},\ n = 1, 2, 3, \cdots, \mathbb{N} - 1,\ z = \lambda h,$$

where

$$M(z) = D + zB(I - zA)^{-1}C$$

and the stability polynomial of each method can easily be obtain as follows,

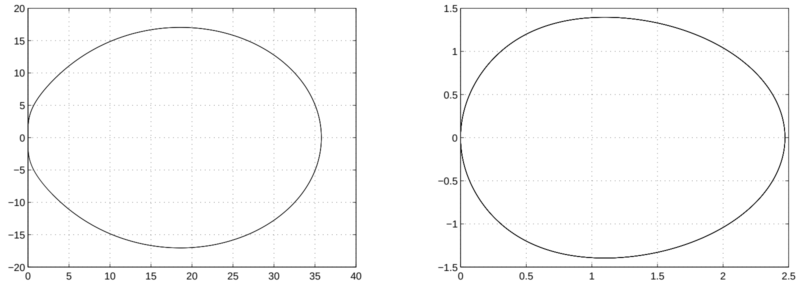
$$\rho(\eta, z) = det(r(A - Cz - D1z^2) - B).$$

The region of absolute stability (RAS)of the method is defined as

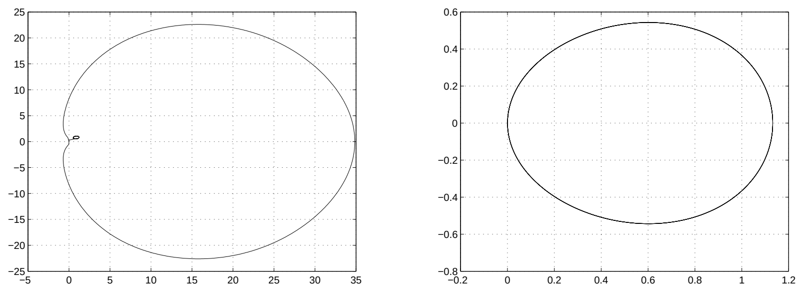
$$\Re = x \in \mathbb{C} : \rho(\eta, z) = 1 \implies |\eta| \leq 1.$$

Computing the stability function give the stability polynomial of the method, which is plotted to produce the required graph of the absolute stability of each method as shown in Figure 1.

Remark 4.1: *In the stable second derivative Runge-Kutta collocation methods we added the matrix D1 obtained from the coefficients of h^2 to the matrices A, C, B and D which enabled us to plot the regions of absolute stability of the new methods.*
The regions of absolute stability of methods (33),(35) and (37) are A-stable, since the regions consist of the complex plane outside the enclosed Figures, while method (36) is $A(\alpha)-$stable.



RAS of Method (33) and (35) in 2D respectively



RAS of Method (36) and (37) in 2D respectively

Figure 1: Regions of Absolute Stability of the SDRKC methods

5. NUMERICAL EXPERIMENTS

In this section, practical performance of the new methods are examined on some test examples. We present the results obtained from the test examples which include linear and nonlinear stiff and highly oscillatory system of initial value problems found in the literature. The results are compared with the exact solutions (Ext). The results or absolute errors $|y(x) - y_n(x)|$ are presented side by side in the Table of values. In each presentation, *nfe* denotes the number of function evaluations in the Figures. We used MATLAB codes for the computational purposes.

Example 1: In the first example we consider a system whose first component is slowly varying in the specified interval while the second component decays rapidly in the transient phase,

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -10^{-5} & 100 \\ -100 & -10^{-5} \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix}, \quad \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

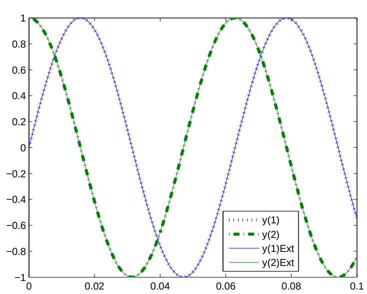
The exact solution is,

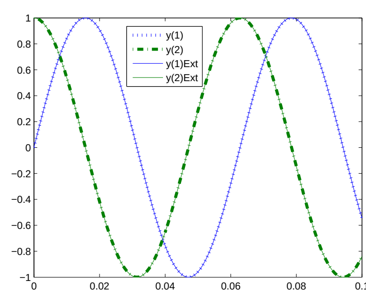
$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} e^{-10^{-5x}} sin(100x) \\ e^{-10^{-5x}} cos(100x) \end{bmatrix}.$$

We solve this problem using the newly derived methods and the results obtained are presented in Table 2 while the solution curves are displayed in Figure 2.

Table 2: Absolute errors in the numerical integration of example 1

x	y_i	Method (33) $\lvert y(x) - y_n(x)\rvert$	Method(35) $\lvert y(x) - y_n(x)\rvert$
5	y_1	0	$1.387778780781446 \times 10^{-17}$
	y_2	0	$1.110223024625157 \times 10^{-16}$
50	y_1	$1.110223024625157 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
	y_2	$1.110223024625157 \times 10^{-16}$	$5.551115123125783 \times 10^{-16}$
250	y_1	$4.440892098500626 \times 10^{-16}$	$3.330669073875470 \times 10^{-16}$
	y_2	$1.665334536937735 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
500	y_1	$6.661338147750939 \times 10^{-16}$	$6.661338147750939 \times 10^{-16}$
	y_2	$1.221245327087672 \times 10^{-15}$	$2.220446049250313 \times 10^{-16}$





Solution curves of example 1 using methods (33) and (35)respectively, with nfe =500

Figure 2: Graphical plots of example 1 using SDRKC methods

Example 2: We consider a linear system,

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \end{bmatrix} = \begin{bmatrix} -2y_1(x) \ + \ y_2(x) \\ 998y_1(x) - 999y_2(x) \end{bmatrix} +$$

$$\begin{bmatrix} 2sin(x) \\ 999(cos(x) - sin(x)) \end{bmatrix}, \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

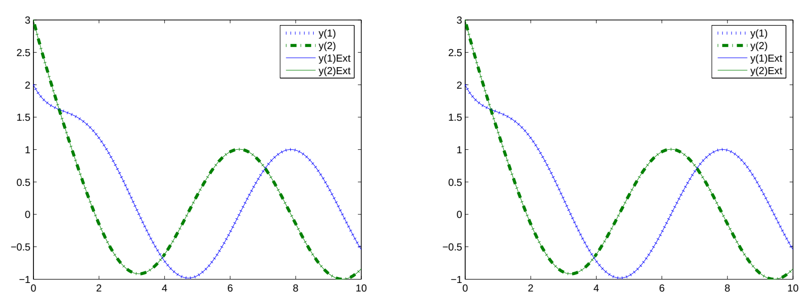
The exact solution is,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \end{bmatrix} = \begin{bmatrix} 2exp(-x) + sin(x) \\ 2exp(-x) + cos(x) \end{bmatrix}.$$

We solve this problem using the derived methods and the results obtained are presented in Table 3 while the solution curves are displayed in Figure 3. For fair comparison with method of the same order in the literature see [22] Table 2, page 131.

Table 3: Absolute errors in the numerical integration of example 2

x	y_i	Method (33)$\lvert y(x) - y_n(x)\rvert$	Method(35) $\lvert y(x) - y_n(x)\rvert$
5	y_1	$1.274596866234656 \times 10^{-3}$	$1.19662193702241 \times 10^{-3}$
	y_2	$1.350208214789817 \times 10^{-6}$	$1.2675300864462 \times 10^{-6}$
50	y_1	$3.706881694870087 \times 10^{-5}$	$3.27152236946661 \times 10^{-5}$
	y_2	$1.109910142960136 \times 10^{-7}$	$1.01934331558917 \times 10^{-7}$
250	y_1	$4.172319945853576 \times 10^{-12}$	$3.68202972340515 \times 10^{-12}$
	y_2	$2.916279550339200 \times 10^{-11}$	$2.70423036448451 \times 10^{-11}$
500	y_1	$8.599721463637592 \times 10^{-21}$	$7.58922919034555 \times 10^{-21}$
	y_2	$1.340548156227326 \times 10^{-15}$	$1.22760970021559 \times 10^{-15}$



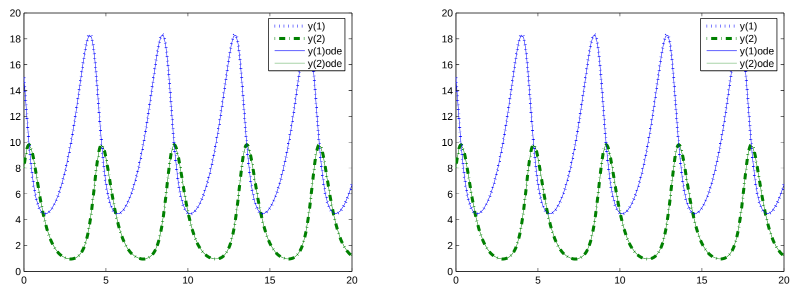
Solution curves of example 2 using methods (33) and (35)respectively, with nfe =500

Figure 3: Graphical plots of example 2 using SDRKC methods

Example 3: Lotka-Volterra system
In this example we consider a real life problem of mathematical model for predicting the population dynamic of biological system. The populations of the pair of species is described by the system,

$$y_1'(t) = 0.95y_1(t) - 0.25y_1(t)y_2(t), \qquad y_1(0) = 15,$$
$$y_2'(t) = 0.25y_1(t)y_2(t) - 2.45y_2(t), \qquad y_2(0) = 8.$$

We applied the newly derived SDRK collocation methods to the system of the Lotka-Volterra equation, subject to the given initial conditions and display the solution curves obtained in Figure 4. The solution curves generated by the new methods are in good agreement with the solutions obtained from the ode solver of MatLab.



Solution curves of example 3 using methods(33) and (36)respectively, and the ode of MatLab

Figure 4: Graphical plots of example 3 using SDRKC methods and the ODE code of MatLab

Example 4: We consider the system,

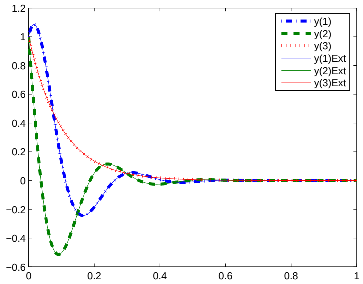
$$y_1'(x) = -10y_1(x) + \beta y_2(x), \; y_1(0) = 1,$$
$$y_2'(x) = -\beta y_1(x) - 10y_2(x), \; y_2(0) = 1,$$
$$y_3'(x) = \gamma y_3(x), \; y_3(0) = 1,$$

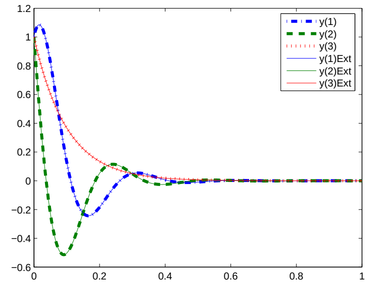
where $\beta = 21$ and $\gamma = 10$.

The exact solution of this example is given by

$$y_1(x) = e^{-\gamma x}(cos(\beta x) + sin(\beta x)),$$
$$y_2(x) = e^{-\gamma x}(cos(\beta x) - sin(\beta x)),$$
$$y_3(x) = e^{-\gamma x}.$$

We solve the problem using the new stable SDRK collocation methods in the interval [0,1] and the results obtained are presented side by side in Table 4, while the solution curves are displayed in Figure 5.





Solution curves of example 4 using methods(33) and (35)respectively, with nfe =500

Figure 5: Graphical plots of example 4 using SDRKC methods

Table 4: Absolute errors in the numerical integration of example 4

x	y_i	Method (33) $\lvert y(x) - y_n(x)\rvert$	Method(35) $\lvert y(x) - y_n(x)\rvert$
5	y_1	$8.881784197001252 \times 10^{-16}$	$2.220446049250313 \times 10^{-16}$
	y_2	$1.443289932012704 \times 10^{-15}$	$1.110223024625157 \times 10^{-16}$
	y_3	0	0
50	y_1	$7.799316747991725 \times 10^{-15}$	0
	y_2	$2.553512956637860 \times 10^{-15}$	$5.551115123125783 \times 10^{-16}$
	y_3	$2.775557561562891 \times 10^{-16}$	$1.110223024625157 \times 10^{-16}$
250	y_1	$1.665334536937735 \times 10^{-16}$	$2.081668171172169 \times 10^{-17}$
	y_2	$7.346553920761778 \times 10^{-16}$	$6.505213034913027 \times 10^{-18}$
	y_3	$6.938893903907228 \times 10^{-18}$	$7.806255641895632 \times 10^{-18}$
500	y_1	$9.666339994066076 \times 10^{-18}$	$3.320369153236857 \times 10^{-19}$
	y_2	$2.751163012681968 \times 10^{-18}$	$8.131516293641283 \times 10^{-20}$
	y_3	$2.710505431213761 \times 10^{-20}$	$6.776263578034403 \times 10^{-21}$

Example 5: Linear problem

The fifth example is a stiff system of three linear ordinary differential equations with corresponding initial conditions [23],

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \end{bmatrix} = \begin{bmatrix} 42.2 & 50.1 & -42.1 \\ -66.1 & -58 & 58.1 \\ 26.1 & 42.1 & -34 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix},$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

The exact solution is,

$$\begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \end{bmatrix} = \begin{bmatrix} exp(0.1x)sin(8x) + exp(-50x) \\ exp(0.1x)cos(8x) - exp(-50x) \\ exp(0.1x)(cos(8x) + sin(8x)) + exp(-50x) \end{bmatrix}.$$

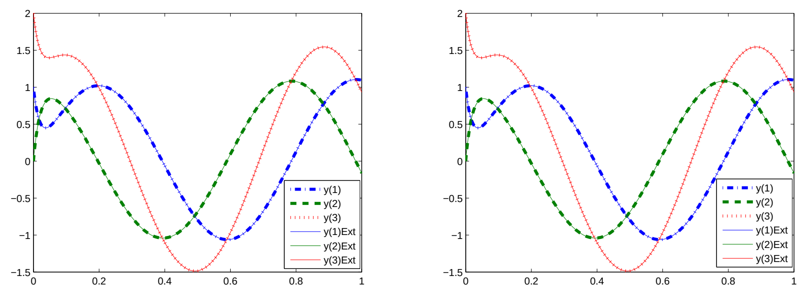
The results of using the newly constructed methods and the exact solutions on the interval [0,1] are presented in Table 5 and the solution curves are shown in Figure 6.

Table 5: Absolute errors in the numerical integration of example 5

x	y_i	Method(33)$\lvert y(x) - y_n(x)\rvert$	Method(37)$\lvert y(x) - y_n(x)\rvert$
	y_1	0	$1.110223024625157 \times 10^{-16}$
5	y_2	$8.150232377879263 \times 10^{-9}$	0
	y_3	$8.148960173315345 \times 10^{-9}$	$2.220446049250313 \times 10^{-16}$
	y_1	$1.097286705942224 \times 10^{-9}$	$1.665334536937735 \times 10^{-15}$
50	y_2	$1.098710566971306 \times 10^{-9}$	$1.554312234475219 \times 10^{-15}$
	y_3	$1.107702152225443 \times 10^{-9}$	$2.220446049250313 \times 10^{-16}$
	y_1	$5.9301001854811826 \times 10^{-11}$	$4.218847493575595 \times 10^{-15}$
250	y_2	$5.865408159166918 \times 10^{-11}$	$5.551115123125783 \times 10^{-16}$
	y_3	$6.468159341466162 \times 10^{-13}$	$4.884981308350689 \times 10^{-15}$
	y_1	$1.185984643825577 \times 10^{-11}$	$3.99680288650564 \times 10^{-15}$
500	y_2	$1.753213685340427 \times 10^{-10}$	$4.024558464266193 \times 10^{-15}$
	y_3	$1.871810484388448 \times 10^{-10}$	$4.440892098500626 \times 10^{-16}$

Example 6:

We consider another linear problem which is particularly referred to by some eminent authors, (see [2,24]) as a troublesome problem for some existing methods. This is because some of the eigenvalues lying close to the imaginary axis, a case where some stiff integrators were known to be inefficient. The reference solutions at the end point of integration interval [0,1] are shown in Table 6 while the solution curves are plotted and displayed in Figure 7.



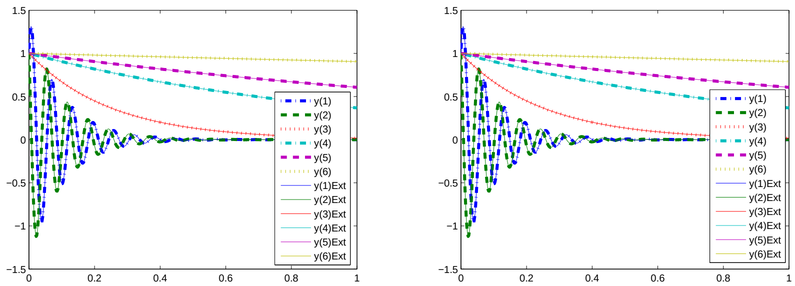
Solution curves of example 5 using methods(33) and (37) respectively, with nfe =500

Figure 6: Graphical plots of example 5 using SDRKC methods

Thus, only the first four components $\{y_1, y_2, y_3, y_4\}$ are shown in Table 6.

$$\begin{bmatrix} y_1'(x) \\ y_2'(x) \\ y_3'(x) \\ y_4'(x) \\ y_5'(x) \\ y_6'(x) \end{bmatrix} = \begin{bmatrix} -10 & 100 & 0 & 0 & 0 & 0 \\ -100 & -10 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.5 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 \end{bmatrix} \begin{bmatrix} y_1(x) \\ y_2(x) \\ y_3(x) \\ y_4(x) \\ y_5(x) \\ y_6(x) \end{bmatrix},$$

$$\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \\ y_4(0) \\ y_5(0) \\ y_6(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$



Solution curves of example 6 using method (33) and method(35), with nfe =500

Figure 7: Graphical plots of example 6 using SDRKC methods

Table 6: Absolute errors in the numerical integration of example 6

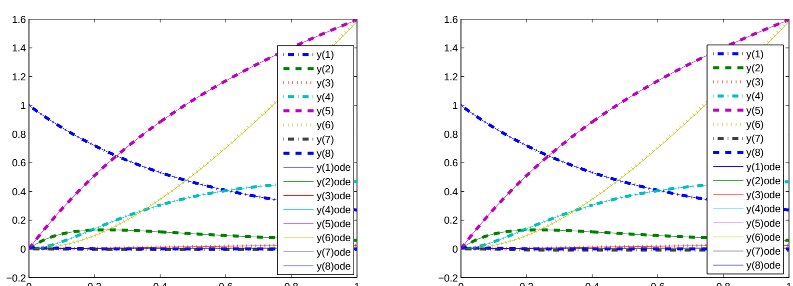
x	y_i	Method(33)$\|y(x) - y_n(x)\|$	Method(35)$\|y(x) - y_n(x)\|$
5	y_1	$2.22044604925031 \times 10^{-16}$	$4.44089209850063 \times 10^{-16}$
	y_2	$1.74166236988071 \times 10^{-15}$	$2.56739074444567 \times 10^{-16}$
	y_3	$3.33066907387547 \times 10^{-16}$	0
	y_4	$2.22044604925031 \times 10^{-16}$	0
50	y_1	$1.666533453693773 \times 10^{-15}$	$3.33066907387547 \times 10^{-16}$
	y_2	$8.24340595784179 \times 10^{-15}$	$1.94289029309402 \times 10^{-16}$
	y_3	$2.55351295663786 \times 10^{-15}$	$4.44089209850063 \times 10^{-16}$
	y_4	$3.77475828372553 \times 10^{-15}$	0
250	y_1	$5.86336534880161 \times 10^{-16}$	$2.42861286636753 \times 10^{-17}$
	y_2	$4.18068357710411 \times 10^{-16}$	$1.73472347597681 \times 10^{-18}$
	y_3	$3.63598040564739 \times 10^{-15}$	$2.77555756156289 \times 10^{-17}$
	y_4	$5.10702591327572 \times 10^{-15}$	$8.88178419700125 \times 10^{-16}$
500	y_1	$8.73121562029733 \times 10^{-18}$	$5.99699326656045 \times 10^{-19}$
	y_2	$4.43167638003450 \times 10^{-18}$	$4.06575814682064 \times 10^{-20}$
	y_3	$8.15320033709099 \times 10^{-16}$	$2.42861286636753 \times 10^{-17}$
	y_4	$1.66533453693773 \times 10^{-16}$	$5.55111512312578 \times 10^{-16}$

Example 7: Hires Problem

The high irradiance responses problem consists of system of 8 nonlinear ordinary differential equations which originates from plant physiology [25].

$$\begin{cases} y_1'(t) = -1.71y_1(t) + 0.43y_2(t) \\ +8.32y_3(t) + 7 \times 10^{-4},\ y_1(0) = 1, \\ y_2'(t) = 1.71y_1(t) - 8.75y_2(t),\ y_2(0) = 0, \\ y_3'(t) = -10.03y_3(t) + 0.43y_4(t) + 0.035y_5(t),\ y_3(0) = 0, \\ y_4'(t) = 8.32y_2(t) + 1.71y_3(t) - 1.12y_4(t),\ y_4(0) = 0, \\ y_5'(t) = -1.745y_5(t) + 0.43y_6(t) + 0.43y_7(t),\ y_5(0) = 0, \\ y_6'(t) = -280y_6(t)y_8(t) + 0.69y_4(t) + 1.71y_5(t) \\ -0.43y_6(t) + 0.69y_7(t),\ y_6(0) = 0, \\ y_7'(t) = 280y_6(t)y_8(t) - 1.81y_7(t),\ y_7(0) = 0, \\ y_8'(t) = -y_7'(t),\ y_8(0) = 0.0057. \end{cases}$$

The reference solution at the end point of the integration interval of [0, 1] are plotted and displayed in Figure 8. We observe that the graphs of the approximated solutions and the graphs of the solutions from the ode solver coincide with each other.



Solution curves of example 7 using methods(36) and (37)respectively and the ode of MatLab

Figure 8: Graphical plots of example 7 using SDRKC methods and the ode code of MatLab

6. CONCLUDING REMARKS

From the examples so far solved by the new methods we can conclude that the new methods are effective in treating stiff and highly oscillatory system of initial value problems in ordinary differential equations. This fact is clearly seen from the accuracy of results presented in the Table of values, which are supported by the solution curves displayed in Figures.

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