CONTROLLABILITY OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. We investigate the controllability problem of quantum stochastic differential inclusions driven by quantum field operators. The operator-valued quantum stochastic processes involved are upper semicontinuous convex-valued multifunctions. We employed the fixed point approach to prove the result.

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1. INTRODUCTION

This paper is concerned with the controllability results for quantum stochastic differential inclusions with nonlocal conditions. In [7] the existence of mild solution for right and left quantum stochastic differential equation was established. The work is a further generalization of Hudson-Parthasarathy quantum stochastic differential equation to the case of unbounded coefficients. The multivalued generalization of the same work was established in [6] in which the existence of solution of quantum stochastic evolution inclusions was established. A further extension of the quantum stochastic calculus to problems with impulsive effects was investigated in [14] which was an extension of the work in [7] to the case of having impulsive effects. By using a directionally continuous selection strategy, we established the results on the solutions set of semicontinuous quantum stochastic differential inclusions in [15].

In the classical setting several authors have worked on the nonlocal evolution problem initiated by [4]. Some authors even investigated the nonlocal problems with impulsive effects [1], [2], [3], [5], [16], [8]. The controllability results of such work were established in [9], [12] and the references cited therein. The aim of this work is to establish a non-classical generalization of controllability results for quantum stochastic differential inclusions with non-local condition.

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The fixed point approach employed in this work is Kakutani-Ky Fan fixed point theorem [11]. It is suitable for the work as it gives a multivalued generalization of Schauder fixed point theorem in Banach spaces. The result obtained in this work is an extension of the results in [13] to controllability problem of quantum stochastic evolution inclusions. Moreover, we obtained as a corollary to the result an extension of the result in [14] to quantum stochastic evolution inclusion with non local condition.

In section 2, preliminaries on notations and definitions were stated while main result was established in section 3.

2. PRELIMINARY

Let $\mathbb{D}$ be some pre-Hilbert space whose completion is $\mathcal{R}$; $\gamma$ is a fixed Hilbert space and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable $\gamma$-valued maps on $\mathbb{R}_+$.

The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\| . \|$ the norm induced by $\langle .,. \rangle$. We denote by $V$ the Banach space from the completion of induced norm $\| . \|$.

Let $E$ be linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$ and $(\mathbb{D} \otimes E)_\infty$ be the set of all sequences $\xi = \{\xi_n\}_{n=1}^\infty$ of members of $\mathbb{D} \otimes E$, such that $\Sigma_{n=1}^\infty \| x_{\xi_n} \|^2 < \infty, \forall x \in \overline{B}$; where $B \equiv L^+_w((\mathbb{D} \otimes E)_\infty, \mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+)))$.

Then the family of seminorms $\{\| . \|_\xi, \xi \in (\mathbb{D} \otimes E)_\infty\}$, where

$$\| x \|_\xi = \left[ \Sigma_{n=1}^\infty \| x_{\xi_n} \|^2 \right]^{\frac{1}{2}}, x \in B,$$

generates a $\sigma$-strong topology, denoted by $\tau_{\sigma s}$ . The completion of $(B, \tau_{\sigma s})$ is denoted by $\widetilde{B}$. The underlying elements of $\widetilde{B}$ consist of linear maps from $(\mathbb{D} \otimes E)_\infty$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $(\mathbb{D} \otimes E)_\infty$. By a multivalued stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$, we mean a multifunction on $I$ with values in $\widetilde{B}$.

If $\Phi$ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of $\Phi$ is a stochastic process $\phi : I \rightarrow \widetilde{B}$ with the property that $\varphi(t) \in \Phi(t)$ for almost all $t \in I$.

We denote by $S^1_\Phi$ the set of all selectors of $\Phi(.)$ that belong to the Lebesgue-Bochner space $L^1(I, \widetilde{B})$.

Definition 1: A multivalued stochastic process $\Phi : \widetilde{B} \rightarrow 2^{\widetilde{B}}$ is said to be upper semicontinuous (u.s.c.) if for all nonempty closed subset $C \subseteq \widetilde{B}$, the set $\Phi^-(C) = \{ x \in \widetilde{B} : \Phi(x) \cap C \neq \emptyset \}$ is closed in $\widetilde{B}$. 
Definition 2: A multivalued stochastic process $\Phi$ will be called
(i) adapted if $\Phi(t) \subseteq \tilde{A}$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto d_{\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \tilde{B}, \xi \in (\mathcal{D} \otimes \mathbb{E})_{\infty}$; (iii) locally absolutely $p$-integrable if $t \mapsto \| \Phi(t) \|_{\xi}, t \in \mathbb{R}_+$, lies in $L^p_{loc}(\tilde{B})$ for arbitrary $\xi \in (\mathcal{D} \otimes \mathbb{E})_{\infty}$.

The set of all absolutely $p$-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\tilde{B})_mvs$ and for $p \in (0, \infty)$, $L^p_{loc}(I \times \tilde{B})_mvs$ is the set of maps $\Phi : I \times \tilde{B} \to 2^{\tilde{B}}$ such that $t \mapsto \Phi(t, X(t)), t \in I$ lies in $L^p_{loc}(\tilde{B})_mvs$ for every $X \in L^p_{loc}(\tilde{B})$.

We denote by $comp(\tilde{B})$ (resp. $comp_c(\tilde{B})$) the collection of all compact subsets (resp. compact (convex) subsets) of $\tilde{B}$. We define the Hausdorff topology on $comp(\tilde{B})$ as follows:

For $x \in \tilde{B}, M, N \in comp(\tilde{B})$ and $\xi \in (\mathcal{D} \otimes \mathbb{E})_{\infty}$, define

$$\rho_{\xi}(M, N) \equiv \max(\delta_{\xi}(M, N), \delta_{\xi}(N, M))$$

where

$$\delta_{\xi}(M, N) \equiv \sup_{x \in M} d_{\xi}(x, N)$$

and

$$d_{\xi}(x, N) \equiv \inf_{y \in N} \| x - y \|_{\xi}.$$  

The Hausdorff topology which shall be employed in what follows, denoted by, $\tau_H$, is generated by the family of pseudometrics $\{\rho_{\xi}(\cdot) : \xi \in (\mathcal{D} \otimes \mathbb{E})_{\infty}\}$.

Moreover, if $M \in comp(\tilde{B})$, then $| M |_{\xi}$ is defined by

$$| M |_{\xi} \equiv \rho_{\xi}(M, \{0\});$$

for arbitrary $\xi \in (\mathcal{D} \otimes \mathbb{E})_{\infty}$.

Consider multivalued stochastic processes $E, F, G, H \in L^2_{loc}(I \times \tilde{B})_mvs$ and $(0, x_0)$ be a fixed point in $[0, T] \times \tilde{B}$. Then, a relation of the form

$$X(t) \in x_0 + \int_0^t \left( E(s, X(s))d\Lambda^+(s) + F(s, X(s))dA_f(s) \right. \left. + G(s, X(s))dA^+(s) + H(s, X(s))ds \right) t \in [0, T]$$

will be called a stochastic integral inclusion with coefficients $E, F, G$ and $H$.

The stochastic differential inclusion corresponding to the integral
integral equation will be said to be a mild solution of problem (2.1) if

\[
\int
\]

Definition 4

(iii) \( \sup_{t,x} \| x(t) \| \leq \varphi_n(t) \) a.e., with \( \varphi_n(\cdot) \in L^1(\Theta, \mathbb{R}) \) and

\[
\lim_{n \to \infty} \frac{1}{n} \int_0^T \varphi_n(s) \, ds = 0
\]
H(2): \( \{A(t)\}_{t \in I} \) is a family of linear, densely defined operators that generate a strongly continuous evolution operator

\[
U : \Delta = \{(t, s) \in I \times I : 0 \leq s \leq t \leq T \} \rightarrow L(\tilde{B})
\]
such that \( U(t, s) \) is compact for \( t - s > 0 \).

H(3): \( M : C(I, \tilde{B}) \rightarrow \tilde{B} \) is a compact operator such that

\[
\lim_{\|y\|_\xi \rightarrow \infty} \frac{\|M(y)\|_\xi}{\|y\|_\xi} = 0.
\]

H(4): The linear operator \( W : L^2(I, V) \rightarrow \tilde{B} \) defined by

\[
W u = \int_0^T U(T, s)B u(s)ds
\]
has an invertible operator \( W^{-1} \) which takes values in \( L^2(I, V) \setminus \text{Ker} W \) and there exists positive constants \( M_1 \) and \( M_2 \) such that \( \|B\|_\xi \leq M_1 \) and \( \|W^{-1}\| \leq M_2 \).

The following fixed point theorem shall be employed in the sequel.

**Lemma 1** ([17], p. 452): Let \( X \) be a Banach space and \( K \in P_{cl,c}(X) \) and suppose that the operator \( G : K \rightarrow P_{cl,c}(K) \) is upper semicontinuous and the set \( G(K) \) is relatively compact in \( X \), then \( G \) has a fixed point in \( K \).

### 3. MAIN RESULTS

**Theorem 1**: If hypotheses H(1)-H(4) hold, then the problem (2.1) is nonlocally controllable on \( I \).

**Proof**: Let \( y(.,) \in C(I, \tilde{B}) \), define the control

\[
u_p(t) = W^{-1}\left[ x_1 - M(y) - U(T, 0)(x_0 - M(y)) \right. \\
- \int_0^T U(T, s)\left( \Psi_E(s, x(s))dA_x(s) + \Psi_F(s, x(s))dA_f(s) \\
+ \Psi_G(s, x(s))dA_y(s) + (\Psi_H(s, x(s))ds \right](t),
\]
where \( \Psi_P \in S_{P(.,x(.,))}^{1}, P \in \{E, F, G, H\} \). We shall show that with this control, the multifunction \( N : C(I, \tilde{B}) \rightarrow 2^{C(I, \tilde{B})} \), defined by

\[
N(y) = \{ x \in C(I, \tilde{B}) : x(t) = U(t, 0)(x_0 - M(y) \\
+ \gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + B u_p)(t), \Psi_p \in S_{P(.,x(.,))}^{1} \}
\]
where $\gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + Bu_p) \in C(I, \tilde{B})$ is defined by

$$\gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + Bu_p)(t) = \int_0^t U(t, s) \left( \Psi_E(s, x(s)) d\Lambda_\pi(s) + \Psi_F(s, x(s)) dA_f(s) + \Psi_G(s, x(s)) dA_g(s) + (\Psi_H(s, x(s)) + Bu_p) ds \right)$$

has a fixed point. This fixed point is then a solution of the system (2.1). The proof of the theorem shall be in steps, we show that $N$ is an upper semicontinuous compact-convex valued multifunction in $C(I, \tilde{B})$ and we conclude by Kakutani-KyFan fixed point theorem and thus the system (2.1) is nonlocally controllable. Clearly, $x_1 - M(y) \in (N(y))(T)$.

Step 1: There exists a positive integer $n_0 \geq 1$ such that $N(B_{n_0}) \subseteq B_{n_0}$, where $B_{n_0} = \{ y \in C(I, \tilde{B}) : \| y \|_C \leq n_0 \}$. Suppose not, then we can find $y_n \in C(I, \tilde{B})$, $x_n \in N(y_n)$ such that $\| y_n \|_C \leq n$ and $\| x_n \| > n$. Then we have for every $n \geq 1$,

$$x_n(t) = U(t, 0)(x_0 - M(y_n)) + \gamma(\Psi_{E,n} + \Psi_{F,n} + \Psi_{G,n} + \Psi_{H,n} + Bu_{p,n})(t)$$

for some $\Psi_{P,n} \in S^1_{P(\gamma(y_n))}$. So we get

$$n < \| x_n \|_C$$

$$\leq M_3(\| x_0 \|_C + \| M(y_n) \|_\xi) + \| \gamma(\Psi_{E,n} + \Psi_{F,n} + \Psi_{G,n} + \Psi_{H,n}) \|_C + \| \gamma(Bu_{p,n}) \|_C$$

$$\leq M_3 \sup_{t \in I} \int_0^t \| U(t, s) \|_L \left( \| \Psi_E(s, x(s)) \|_\xi d\Lambda_\pi(s) + \| \Psi_F(s, x(s)) \|_\xi dA_f(s) + \| \Psi_G(s, x(s)) \|_\xi dA_g(s) + (\| \Psi_H(s, x(s)) \|_\xi + Bu_{p,n}) ds \right)$$

$$\leq M_3 \int_0^T \varphi_n(s) ds.$$  \hspace{1cm} (3)

$$\| \gamma(Bu_{p,n}) \|_C \leq \sup_{t \in I} \int_0^t \| U(t, s) \|_L \cdot \| B \|_\xi \cdot \| u_{p,n}(s) \|_\xi ds$$

$$\leq M_2 M_3 T^{\frac{1}{2}} \| u_{p,n} \|_{L^2}.$$  \hspace{1cm} (4)
and

\[ \| u_{p_n} \|_{L^2} = \| W^{-1}\left[ x_1 - M(y_n) - U(T, 0)(x_0 - M(y_n)) \\
- (\gamma(\Psi_{E,n} + \Psi_{F,n} + \Psi_{G,n} + \Psi_{H,n}))(T) \right] \| \leq M_1 \left[ \| x_1 \|_\xi + M_3 \| x_0 \|_\xi + (1 + M_3) \| M(y_n) \|_\xi \\
+ M_3 \int_0^T \varphi_n(s) ds \right] \]

Hence by (3.1), we have

\[ n < (M_3 + M_1 M_2 M_3^2 b^\frac{1}{2}) \| x_0 \|_\xi + (M_1 M_2 M_3^2 b^\frac{1}{2}) \| x_1 \|_\xi \\
+ ((1 + M_1 M_2 b^\frac{1}{2} + M_1 M_2 M_3 b^\frac{1}{2}) M_3 \| M(y_n) \|_\xi \\
+ (M_3 + M_1 M_2 M_3^2 b^\frac{1}{2}) \int_0^T \varphi_n(s) ds \]

\[ \Rightarrow 1 < \frac{1}{n} \left[ C_1 + C_2 \| M(y_n) \|_\xi + C_3 \int_0^T \varphi_n(s) ds \right], \]

where \( C_1 = M_3 + M_1 M_2 M_3^2 b^\frac{1}{2} \| x_0 \|_\xi + (M_1 M_2 M_3^2 b^\frac{1}{2}) \| x_1 \|_\xi, \)
\( C_2 = (1 + M_1 M_2 b^\frac{1}{2} + M_1 M_2 M_3 b^\frac{1}{2}) M_3 \) and \( C_3 = (M_3 + M_1 M_2 M_3^2 b^\frac{1}{2}). \)

By passing to the limit as \( n \to \infty \) in inequality (3.5), we get \( 1 \leq 0 \), a contradiction. Thus we conclude that there is \( n_0 \geq 1 \) such that \( N(B_{n_0}) \subseteq B_{n_0}. \)

Step 2: We shall show that \( N(B_{n_0}) \) is equicontinuous. Let \( x \in N(B_{n_0}) \) and let \( t_2, t_1 \in I, \ t_2 > t_1 > 0. \) We have \( y \in N(B_{n_0}), \)
\( P \in S^1(P(., y(\cdot))) \), for any \( \epsilon > 0 \), such that \( t_1 - \epsilon > 0 \),

\[
\|x(t_2) - x(t_1)\|_{\xi} \leq \|x(t_2) - x(t_1)\|_{\xi} + \gamma(\Psi_E + \Psi_F + \Psi_H + B_{u_p})(t_2) - \gamma(\Psi_E + \Psi_F + \Psi_H + B_{u_p})(t_1) \|
\]

\[
\leq \| U(t_2, s) [\Psi_E(s, x(s))d\Lambda_x(s) + \Psi_F(s, x(s))dA_f(s) + \Psi_H(s, x(s)) + B_{u_p}(s)]ds \|
\]

\[
- \int_0^1 U(t_1, s) [\Psi_E(s, x(s))d\Lambda_x(s) + \Psi_F(s, x(s))dA_f(s) + \Psi_H(s, x(s)) + B_{u_p}(s)]ds \| \xi
\]

\[
\leq \int_0^{t_2} \| U(t_2, s) \| \| \Psi_E(s, x(s))d\Lambda_x(s) + \Psi_F(s, x(s))dA_f(s) + \Psi_H(s, x(s)) + B_{u_p}(s) \| ds \| \xi
\]

\[
\leq M_3 \int_{t_1}^{t_2} [\varphi_{n_0}(s) + M_2 \| u_f(s) \| \xi]ds
\]

\[
+ \int_0^{t_1 - \epsilon} \| U(t_2, s) - U(t_1, s) \| \| \Psi_E(s, x(s))d\Lambda_x(s) + \Psi_F(s, x(s))dA_f(s) + \Psi_H(s, x(s)) + B_{u_p}(s) \| ds \| \xi
\]

\[
\leq M_3 \int_{t_1}^{t_2} \varphi_{n_0}(s)ds + 2M_3 \int_{t_1 - \epsilon}^{t_1} [\varphi_{n_0}(s) + M_2 \| u_f(s) \| \xi]ds
\]

\[
+ \int_0^{t_1 - \epsilon} \| U(t_2, s) - U(t_1, s) \| \varphi_{n_0}(s) + M_2 \| u_f(s) \| \xi]ds
\]

\[
\leq M_3 \int_{t_1}^{t_2} \varphi_{n_0}(s)ds + 2M_3 \int_{t_1 - \epsilon}^{t_1} [\varphi_{n_0}(s) + M_2 \| u_f(s) \| \xi]ds + \int_0^{t_1 - \epsilon} \| U(t_2, s) - U(t_1, s) \| \varphi_{n_0}(s)ds
\]

\[
+ \left( \int_0^{t_1 - \epsilon} \| U(t_2, s) - U(t_1, s) \| \xi^2 ds \right)^{\frac{1}{2}} M_2 \| u_p \| \xi
\]

\[
+ M_2M_3(t_2 - t_1)^{\frac{1}{2}} \| u_p \| \xi + 2M_2M_3^{\frac{1}{2}} \| u_p \| \xi
\]

and

\[
\| u_p \| \xi \leq M_1 \left[ M_3 \| x_0 \| \xi + \| x_1 \| \xi + (M_3 + 1) \| M(B_{n_0}) \| \xi \right]
\]

\[
+ M_2 \int_0^T \varphi_{n_0}(s)ds = M_4
\]

where \( \| M \|_{\xi} = \sup \{ \| M(y) \|_{\xi} : y \in B_{n_0} \} \) is bounded (since \( M \) is a compact operator). Moreover, \( t \to U(t, s) \) is continuous in the operator norm topology, uniformly \( s \in T \) such that \( t - s \) is bounded away from zero. Given \( \epsilon_1 > 0 \), by absolute continuity of
the Lebesgue integral we can choose $\epsilon > 0$, such that

$$2M_3 \int_{t_1-\epsilon}^{t_1} \varphi_m(s)ds + 2M_2M_3M_4\epsilon^\frac{1}{2} < \frac{\epsilon_1}{2}.$$ 

By the continuity property of $U(.,s)$ and absolute continuity of the Lebesgue integral, we can find $\delta > 0$ such that if $t_2 - t_1 < \delta$, we have

$$M_3 \int_{t_1}^{t_2} \varphi_{n_0}(s)ds + \int_{0}^{t_1-\epsilon} \| U(t_2, s) - U(t_1, s) \| L \varphi_{n_0}(s)ds + \left( \int_{0}^{t_1-\epsilon} \| U(t_2, s) - U(t_1, s) \|^2_L ds \right)^\frac{1}{2} M_2M_4 + M_2M_3M_4(t_2 - t_1)^\frac{1}{2} < \frac{\epsilon_1}{2}$$

So $N(B_{n_0})$ is equicontinuous. Also,

$$K = \{ \gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + Bu_p) : y \in B_{n_0}, \Psi_P \in S_{P, (.,x,.)}^1 \} \subseteq C(I, \bar{B})$$

is equicontinuous.

Step 3: We shall show that $N(B_{n_0})(t) = \{ x(t) : x \in N(B_{n_0}) \}$ is a relatively compact subset of $\bar{B}$ for every $t \in I$.

$N(B_{n_0})$ is bounded since $B_{n_0}$ is bounded and $N(B_{n_0}) \subset B_{n_0}$. Let $t \in (0, T]$ be fixed and $\epsilon$ be a real number satisfying $0 < \epsilon < t$.

For $y \in B_{n_0}$ and $h \in N(y)$ there exist functions $\Psi_P \in S_{P, (.,x,.)}^1$, $P \in \{E, F, G, H\}$, such that

$$h(t) = U(t, 0)(y_0 - M(y)) + \int_{0}^{t_1-\epsilon} U(t, s) \left[ (\Psi_E(s, x(s))d\Lambda_x(s) 
+ \Psi_F(s, x(s))dA_f(s) + \Psi_G(s, x(s))dA_{\gamma}^g(s) + (\Psi_H(s, x(s)) + Bu_p(s))ds \right]$$

$$+ \int_{t_1-\epsilon}^{t} U(t, s) \left[ (\Psi_E(s, x(s))d\Lambda_x(s) 
+ \Psi_F(s, x(s))dA_f(s) + \Psi_G(s, x(s))dA_{\gamma}^g(s) + (\Psi_H(s, x(s)) + Bu_p(s))ds \right]$$

Define

$$h_\epsilon(t) = U(t, 0)(y_0 - M(y)) + \int_{0}^{t_1-\epsilon} U(t, s) \left[ (\Psi_E(s, x(s))d\Lambda_x(s) 
+ \Psi_F(s, x(s))dA_f(s) + \Psi_G(s, x(s))dA_{\gamma}^g(s) + (\Psi_H(s, x(s)) + Bu_p(s))ds \right]$$

$$= U(t, 0)(y_0 - M(y)) + U(t, 0) \int_{t_1-\epsilon}^{t} U(t - \epsilon, s) \left[ (\Psi_E(s, x(s))d\Lambda_x(s) 
+ \Psi_F(s, x(s))dA_f(s) + \Psi_G(s, x(s))dA_{\gamma}^g(s) + (\Psi_H(s, x(s)) + Bu_p(s))ds \right]$$

Since $U(t, s)$ is compact, the set $H_\epsilon(t) = \{ h_\epsilon(t) : h_\epsilon \in N(y) \}$ is precompact in $\bar{B}$ for every $\epsilon, 0 < \epsilon < t$. Moreover, for every...
\( h \in N(y), \)
\[
\left| h(t) - h_{\epsilon}(t) \right| \leq M \int_{t-\epsilon}^{t} (\varphi(x(s)) + c)ds
\]
\[
\leq M \int_{t-\epsilon}^{t} (\varphi(x(s)) + c)ds
\]
Therefore, there are precompact sets arbitrarily close to the set \( \{ h(t) : h \in N(y) \} \). Hence the set \( \{ h(t) : h \in N(y) \} \) is precompact in \( \tilde{B} \).

Step 4: We shall show that \( N \) has closed graph.
Let \( y_n \to y_* \), \( h_n \in N(y_n) \), \( y_n \in K \) and \( h_n \to h_* \). We shall prove that \( h_* \in N(y_*) \), \( h_n \in N(y_n) \) means that there exists \( \Psi_{n,P} \in S^1_{P(y_n,\cdot)} \) such that for each \( t \in I \),
\[
h_n(t) = U(t,0)\left[ y_0 - M(y_n) \right] + \int_{0}^{t} U(t,s) \left[ (\Psi_{*,E}(s,x(s))d\Lambda_x(s) + (\Psi_{*,F}(s,x(s))dA_f(s) + (\Psi_{*,G}(s,x(s))dA_g(s) + (\Psi_{*,H}(s,x(s)) + Bu_{y_*(s)})ds
\]
We have that
\[
\| h_n - U(t,0)[y_0 - M(y_n)] - \int_{0}^{t} U(t,s)[Bu_{n}(s)]ds
\]
\[
- \left( h_* - U(t,0)[y_0 - M(y_*)] - \int_{0}^{t} U(t,s)Bu_{y_*}(s)ds \right) \|_{C} \to 0 \text{ as } n \to \infty.
\]
Consider the linear operator
\[
\Gamma : L^1(I, \tilde{B}) \to C(I, \tilde{B}),
\]
\[
v \mapsto \Gamma(v)(t) = \int_{0}^{t} U(t,s)v(s)ds
\]
From [11], it follows that \( \Gamma \circ S_P \) is a closed graph operator. Moreover, we have that
\[
h_n(t) - U(t,0)\left[ y_0 - M(y_n) \right] - \int_{0}^{t} U(t,s)Bu_{y_*(s)}ds \in \Gamma(S_{P,y_n}).
\]
Since \( y_n \to y^* \), it follows that
\[
h^*_u(t) - U(t,0)[y_0 - M(y^*)] - \int_0^t U(t,s)Bu_y(s)ds = \int_0^t U(t,s)v(s)ds
\]
for some \( v_s \in \mathcal{S}_{P,y^*} \). As a consequence of Lemma 1, we deduce that \( N \) has a fixed point and therefore the system (2.1) is nonlocally controllable on \( I \).

If \( B \equiv 0 \) in the system (2.1), we have quantum stochastic evolution inclusions with non local condition. Therefore, the next corollary gives a result on the existence of mild solution of quantum stochastic evolution inclusions with non local condition. However, in this case, the mild solution will be of the form:
\[
x(t) = U(t,0)x(0) + \int_0^t U(t,s)\left( \Psi_E(s,x(s))d\Lambda^x_\pi(s) + \Psi_F(s,x(s))dA_f(s) + \Psi_G(s,x(s))dA^+_x(t) + (\Psi_H(s,x(s)))ds \right), \quad t \in I.
\]
with \( \Psi_P \in \mathcal{S}^1_{P,(x(\cdot)),P} \), \( P \in \{E,F,G,H\} \) and \( x(0) + M(x) = x_0 \).

**Corollary** If hypotheses \((H_1)-(H_3)\) are satisfied then there exist a mild solution to the non local problem
\[
dx(t) \in [A(t)x(t) + H(t,x(t))]dt + E(t,x(t))d\Lambda^x_\pi(t)
+ F(t,x(t))dA_f(t) + G(t,x(t))dA^+_x(t), \quad \text{almost all } t \in I.
\]
\[
x(0) + M(x) = x_0.
\]

**REFERENCES**


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