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CONTROLLABILITY OF QUANTUM STOCHASTIC DIFFERENTIAL INCLUSIONS

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ABSTRACT. We investigate the controllability problem of quantum stochastic differential inclusions driven by quantum field operators. The operator-valued quantum stochastic processes involved are upper semicontinuous convex-valued multifunctions. We employed the fixed point approach to prove the result.

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1. INTRODUCTION

This paper is concerned with the controllability results for quantum stochastic differential inclusions with nonlocal conditions. In [7] the existence of mild solution for right and left quantum stochastic differential equation was established. The work is a further generalization of Hudson-Parthasarathy quantum stochastic differential equation to the case of unbounded coefficients. The multivalued generalization of the same work was established in [6] in which the existence of solution of quantum stochastic evolution inclusions was established. A further extension of the quantum stochastic calculus to problems with impulsive effects was investigated in [14] which was an extension of the work in [7] to the case of having impulsive effects. By using a directionally continuous selection strategy, we established the results on the solutions set of semicontinuous quantum stochastic differential inclusions in [15].

In the classical setting several authors have worked on the nonlocal evolution problem initiated by [4]. Some authors even investigated the nonlocal problems with impulsive effects [1], [2], [3], [5], [16], [8]. The controllability results of such work were established in [9], [12] and the references cited therein. The aim of this work is to establish a non-classical generalization of controllability results for quantum stochastic differential inclusions with non-local condition.

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The fixed point approach employed in this work is Kakutani-Ky Fan fixed point theorem [11]. It is suitable for the work as it gives a multivalued generalization of Schauder fixed point theorem in Banach spaces. The result obtained in this work is an extension of the results in [13] to controllability problem of quantum stochastic evolution inclusions. Moreover, we obtained as a corollary to the result an extension of the result in [14] to quantum stochastic evolution inclusion with non local condition.
In section 2, preliminaries on notations and definitions were stated while main result was established in section 3.

2. PRELIMINARY

Let $\mathbb{D}$ be some pre-Hilbert space whose completion is $\mathcal{R}$; γ is a fixed Hilbert space and $L^2_\gamma(\mathbb{R}_+)$ is the space of square integrable γ-valued maps on $\mathbb{R}_+$.
The inner product of the Hilbert space $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ will be denoted by $\langle .,. \rangle$ and $\| \, . \, \|$ the norm induced by $\langle .,. \rangle$. We denote by V the Banach space from the completion of induced norm $\| \, . \, \|$.
Let $\mathbb{E}$ be linear space generated by the exponential vectors in Fock space $\Gamma(L^2_\gamma(\mathbb{R}_+))$ and $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ be the set of all sequences $\xi = \{\xi_n\}_{n=1}^\infty$ of members of $\mathbb{D}\underline{\otimes}\mathbb{E}$, such that
$\Sigma_{n=1}^\infty \| x\xi_n \|^2 < \infty, \forall x \in \mathcal{B}$; where $\mathcal{B} \equiv L^+_w((\mathbb{D}\underline{\otimes}\mathbb{E})_\infty, \mathcal{R}\otimes\Gamma(L^2_\gamma(\mathbb{R}_+)))$.
Then the family of seminorms $\{\| \, . \, \|_\xi, \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty\}$, where

$$\| x \|_\xi = \left[\Sigma_{n=1}^\infty \| x\xi_n \|^2 \right]^{\frac{1}{2}}, x \in \mathcal{B},$$

generates a σ-strong topology, denoted by $\tau_{\sigma s}$. The completion of $(\mathcal{B}, \tau_{\sigma s})$ is denoted by $\widetilde{\mathcal{B}}$. The underlying elements of $\widetilde{\mathcal{B}}$ consist of linear maps from $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$ into $\mathcal{R} \otimes \Gamma(L^2_\gamma(\mathbb{R}_+))$ having domains of their adjoints containing $(\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$. By a multivalued stochastic process indexed by $I = [0, T] \subseteq \mathbb{R}_+$, we mean a multifunction on I with values in $\widetilde{\mathcal{B}}$.
If Φ is a multivalued stochastic process indexed by $I \subseteq \mathbb{R}_+$, then a selection of Φ is a stochastic process $\phi : I \to \widetilde{\mathcal{B}}$ with the property that $\varphi(t) \in \Phi(t)$ for almost all $t \in I$.
We denote by S^1_Φ the set of all selectors of $\Phi(.)$ that belong to the Lebesgue-Bochner space $L^1(I, \widetilde{\mathcal{B}})$.
Definition 1: A multivalued stochastic process $\Phi : \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ is said to be upper semicontinuous (u.s.c.) if for all nonempty closed subset $C \subseteq \widetilde{\mathcal{B}}$, the set $\Phi^-(C) = \{x \in \widetilde{\mathcal{B}} : \Phi(x) \cap C \neq \emptyset\}$ is closed in $\widetilde{\mathcal{B}}$.

Definition 2: A multivalued stochastic process Φ will be called (i) adapted if $\Phi(t) \subseteq \widetilde{\mathcal{A}}_t$ for each $t \in \mathbb{R}_+$; (ii) measurable if $t \mapsto d_{\eta\xi}(x, \Phi(t))$ is measurable for arbitrary $x \in \widetilde{\mathcal{B}}$, $\xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$; (iii) locally absolutely p-integrable if $t \mapsto \| \Phi(t) \|_\xi$, $t \in \mathbb{R}_+$, lies in $L^p_{loc}(\widetilde{\mathcal{B}})$ for arbitrary $\xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$

The set of all absolutely p-integrable multivalued stochastic processes will be denoted by $L^p_{loc}(\widetilde{\mathcal{B}})_{mvs}$ and for $p \in (0, \infty)$, $L^p_{loc}(I \times \widetilde{\mathcal{B}})_{mvs}$ is the set of maps $\Phi : I \times \widetilde{\mathcal{B}} \to 2^{\widetilde{\mathcal{B}}}$ such that $t \mapsto \Phi(t, X(t))$, $t \in I$ lies in $L^p_{loc}(\widetilde{\mathcal{B}})_{mvs}$ for every $X \in L^p_{loc}(\widetilde{\mathcal{B}})$.

We denote by $comp(\widetilde{\mathcal{B}})$ (resp. $comp_c(\widetilde{\mathcal{B}})$) the collection of all compact subsets (resp. compact (convex) subsets) of $\widetilde{\mathcal{B}}$. We define the Hausdorff topology on $comp(\widetilde{\mathcal{B}})$ as follows:

For $x \in \widetilde{\mathcal{B}}$, $\mathcal{M}, \mathcal{N} \in comp(\widetilde{\mathcal{B}})$ and $\xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$, define

$$\rho_\xi(\mathcal{M}, \mathcal{N}) \equiv \max(\delta_\xi(\mathcal{M}, \mathcal{N}), \delta_\xi(\mathcal{N}, \mathcal{M}))$$

where

$$\delta_\xi(\mathcal{M}, \mathcal{N}) \equiv \sup_{x \in \mathcal{M}} \mathbf{d}_\xi(x, \mathcal{N}) \text{ and}$$
$$\mathbf{d}_\xi(x, \mathcal{N}) \equiv \inf_{y \in \mathcal{N}} \| x - y \|_\xi .$$

The Hausdorff topology which shall be employed in what follows, denoted by, τ_H , is generated by the family of pseudometrics $\{\rho_\xi(.) : \xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty\}$.

Moreover, if $\mathcal{M} \in comp(\widetilde{\mathcal{B}})$, then $| \mathcal{M} |_\xi$ is defined by

$$\| \mathcal{M} \|_\xi \equiv \rho_\xi(\mathcal{M}, \{0\});$$

for arbitrary $\xi \in (\mathbb{D}\underline{\otimes}\mathbb{E})_\infty$.

Consider multivalued stochastic processes $E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{B}})_{mvs}$ and $(0, x_0)$ be a fixed point in $[0, T] \times \widetilde{\mathcal{B}}$. Then, a relation of the form

$$\begin{aligned} X(t) \in x_0 + \int_0^t (E(s, X(s))d\Lambda_\pi(s) + F(s, X(s))dA_f(s) \\ + G(s, X(s))dA^+_g(s) + H(s, X(s))ds) \; t \in [0, T] \end{aligned}$$

will be called a stochastic integral inclusion with coefficients E, F, G and H.

The stochastic differential inclusion corresponding to the integral inclusion above is;

$$\begin{aligned} dX(t) &\in E(t,X(t))d\Lambda_\pi(t) + F(t,X(t))dA_f(t) \\ &\quad + G(t,X(t))dA_g^+(t) + H(t,X(t))dt. \\ X(0) &= x_0 \text{ almost all } t \in [0,T]. \end{aligned}$$

In this paper, we consider the following nonlocal quantum stochastic evolution system:

$$\begin{aligned} dx(t) &\in [A(t)x(t) + (Bu)(t) + H(t,x(t))]dt + E(t,x(t))d\Lambda_\pi(t) \\ &\quad + F(t,x(t))dA_f(t) + G(t,x(t))dA_g^+(t) \text{ almost all } \ t \in I. \quad (1) \\ x(0) &+ M(x) = x_0, \end{aligned}$$

where $\{A(t)\}_{t\in I}$ is a family of linear operators that generate an evolution operator

$$U : \Delta = \{(t,s) \in I \times I : 0 \leq s \leq t \leq T\} \to L(\widetilde{\mathcal{B}})$$

$E, F, G, H \in L^2_{loc}(I \times \widetilde{\mathcal{B}})_{mvs}$, $M : C(I, \widetilde{\mathcal{B}}) \to \widetilde{\mathcal{B}}$, $x_0 \in \widetilde{\mathcal{B}}$. $L^2(I,V)$ is a Banach space of admissible control function with norm $\| u \|_{L^2} = \left(\int_0^T \| u(t) \|^2 \, dt \right)^{\frac{1}{2}}$. B is a bounded linear operator from V to $\widetilde{\mathcal{B}}$.

Definition 3 :A continuous adapted stochastic process $x \in C(I, \widetilde{\mathcal{B}})$ will be said to be a mild solution of problem (2.1) if x is of the integral equation

$$\begin{aligned} x(t) = U(t,0)x(0) + \int_0^t U(t,s)\Big(&\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\ &+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu)ds\Big), \ t \in I. \end{aligned}$$

with $\Psi_P \in S^1_{P(.,x(.))}$, $P \in \{E,F,G,H\}$ and $x(0) + M(x) = x_0$.

Definition 4: The system (2.1) is said to be nonlocally controllable on I if, for every $x_0, x_1 \in \widetilde{\mathcal{B}}$, there exists a control $u \in L^2(I,V)$ such that the mild solution $x(.)$ of (2.1) satisfies $x(T) + M(x) = x_1$.

The following hypotheses shall be employed in the main result:

H(1): $\Phi : I \times \widetilde{\mathcal{B}} \to comp_c(\widetilde{\mathcal{B}})$, $\Phi \in \{E,F,G,H\}$ are multivalued stochastic processes such that:

(i) for every $x \in \widetilde{\mathcal{B}}$, $t \to \Phi(t,x)$ is measurable; (ii) for every $t \in I$, $\Phi(t,.)$ is u.s.c. on $\widetilde{\mathcal{B}}$.

(iii) $\sup\{| \Phi(t,x) |_\xi : \| x \|_\xi \leq \frac{n}{5}\} \leq \varphi_n(t)$ a.e. , with $\varphi_n(.) \in L^1(I,\mathbb{R})$ and

$$\underline{\lim} \frac{1}{n} \int_0^T \varphi_n(s)ds = 0$$

H(2): $\{A(t)\}_{t\in I}$ is a family of linear, densely defined operators that generate a strongly continuous evolution operator

$$U : \Delta = \{(t,s) \in I \times I : 0 \leq s \leq t \leq T\} \to L(\widetilde{\mathcal{B}})$$

such that $U(t,s)$ is compact for $t-s>0$.
H(3): $M : C(I,\widetilde{\mathcal{B}}) \to \widetilde{\mathcal{B}}$ is a compact operator such that

$$\lim_{\|y\|_\xi \to \infty} \frac{\| M(y) \|_\xi}{\| y \|_\xi} = 0.$$

H(4): The linear operator $W : L^2(I,V) \to \widetilde{\mathcal{B}}$ defined by

$$Wu = \int_0^T U(T,s)Bu(s)ds$$

has an invertible operator W^{-1} which takes values in $L^2(I,V) \setminus KerW$ and there exists positive constants M_1 and M_2 such that $\| B \|_\xi \leq M_1$ and $\| W^{-1} \| \leq M_2$
The following fixed point theorem shall be employed in the sequel.
Lemma 1 ([17], p. 452): Let X be a Banach space and $K \in P_{cl,c}(X)$ and suppose that the operator $G : K \to P_{cl,c}(K)$ is upper semicontinuous and the set $G(K)$ is relatively compact in X, then G has a fixed point in K..

3. MAIN RESULTS

Theorem 1: If hypotheses H(1)-H(4) hold, then the problem (2.1) is nonlocally controllable on I.

Proof: Let $y(.) \in C(I,\widetilde{\mathcal{B}})$, define the control

$$\begin{aligned} u_p(t) = W^{-1}\Bigg[& x_1 - M(y) - U(T,0)(x_0 - M(y)) \\ & - \int_0^T U(T,s)\Bigg(\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\ & + \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s))ds\Bigg](t), \end{aligned}$$

where $\Psi_P \in S^1_{P(.,x(.))}$, $P \in \{E,F,G,H\}$. We shall show that with this control, the multifunction $N : C(I,\widetilde{\mathcal{B}}) \to 2^{C(I,\widetilde{\mathcal{B}})}$, defined by

$$\begin{aligned} N(y) =& \{x \in C(I,\widetilde{\mathcal{B}}) : x(t) = U(t,0)(x_0 - M(y)) \\ & + \gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + Bu_p)(t), \Psi_P \in S^1_{P(.,x(.))}\} \end{aligned}$$

where $\gamma(\Psi_E+\Psi_F+\Psi_G+\Psi_H+Bu_p)\in C(I,\widetilde{\mathcal{B}})$ is defined by

$$\begin{aligned}\gamma(\Psi_E+\Psi_F+\Psi_G+\Psi_H+Bu_p)(t) &= \int_0^t U(t,s)\Big(\Psi_E(s,x(s))d\Lambda_\pi(s)\\ &+\Psi_F(s,x(s))dA_f(s)+\Psi_G(s,x(s))dA_g^+(s)\\ &+(\Psi_H(s,x(s))+Bu_p)ds\Big)\end{aligned}$$

has a fixed point. This fixed point is then a solution of the system (2.1). The proof of the theorem shall be in steps, we show that N is an upper semicontinuous compact-convex valued multifunction in $C(I,\widetilde{\mathcal{B}})$ and we conclude by Kakutani-KyFan fixed point theorem and thus the system (2.1) is nonlocally controllable. Clearly, $x_1-M(y)\in(N(y))(T)$.

Step 1 : There exists a positive integer $n_0\geq 1$ such that $N(B_{n_0})\subseteq B_{n_0}$, where $B_{n_0}=\{y\in C(I,\widetilde{\mathcal{B}}):\parallel y\parallel_C\leq n_0\}$. Suppose not, then we can find $y_n\in C(I,\widetilde{\mathcal{B}})$, $x_n\in N(y_n)$ such that $\parallel y_n\parallel_C\leq n$ and $\parallel x_n\parallel>n$. Then we have for every $n\geq 1$,

$$x_n(t)=U(t,0)(x_0-M(y_n)+\gamma(\Psi_{E,n}+\Psi_{F,n}+\Psi_{G,n}+\Psi_{H,n}+Bu_{p_n})(t)$$

for some $\Psi_{P,n}\in S^1_{P(.,y_n(.))}$. So we get

$$\begin{aligned}n &<\parallel x_n\parallel_C\\ &\leq M_3(\parallel x_0\parallel_\xi+\parallel M(y_n)\parallel_\xi)\\ &+\parallel\gamma(\Psi_{E,n}+\Psi_{F,n}+\Psi_{G,n}+\Psi_{H,n})\parallel_C+\parallel\gamma(Bu_{p_n})\parallel_C\end{aligned}\tag{2}$$

where $M_3>0$ is such that $\parallel U(t,s)\parallel_L\leq M_3$. Note that

$$\begin{aligned}\parallel\gamma(\Psi_{E,n}+\Psi_{F,n}+\Psi_{G,n}+\Psi_{H,n})\parallel_C &=\sup_{t\in I}\parallel\gamma(\Psi_{E,n}(t)+\Psi_{F,n}(t)\\ &+\Psi_{G,n}(t)+\Psi_{H,n}(t))\parallel_\xi\\ &\leq\sup_{t\in I}\int_0^t\parallel U(t,s)\parallel_L(\parallel\Psi_E(s,x(s))\parallel_\xi d\Lambda_\pi(s)\\ &+\parallel\Psi_F(s,x(s))\parallel_\xi dA_f(s)+\parallel\Psi_G(s,x(s))\parallel_\xi dA_g^+(s)\\ &+(\parallel\Psi_H(s,x(s))\parallel_\xi ds)\\ &\leq M_3\int_0^T\varphi_n(s)ds.\end{aligned}\tag{3}$$

$$\begin{aligned}\parallel\gamma(Bu_{p_n})\parallel_C &\leq\sup_{t\in I}\int_0^t\parallel U(t,s)\parallel_L.\parallel B\parallel_\xi.\parallel u_{p_n}(s)\parallel_\xi ds\\ &\leq M_2M_3T^{\frac{1}{2}}\parallel u_{p_n}\parallel_{L^2}\end{aligned}\tag{4}$$

and

$$\begin{aligned}\| u_{p_n} \|_{L^2} =& \| W^{-1}\Big[x_1 - M(y_n) - U(T,0)(x_0 - M(y_n))\\ &- (\gamma(\Psi_{E,n} + \Psi_{F,n} + \Psi_{G,n} + \Psi_{H,n}))(T)\Big] \|\\ &\leq M_1\Big[\| x_1 \|_\xi + M_3 \| x_0 \|_\xi + (1 + M_3) \| M(y_n) \|_\xi\\ &+ M_3 \int_0^T \varphi_n(s)ds\Big]\end{aligned} \tag{5}$$

Hence by (3.1), we have

$$\begin{aligned}n <& (M_3 + M_1M_2M_3^2b^{\frac{1}{2}}) \| x_0 \|_\xi + (M_1M_2M_3^2b^{\frac{1}{2}}) \| x_1 \|_\xi\\ &+ ((1 + M_1M_2b^{\frac{1}{2}} + M_1M_2M_3b^{\frac{1}{2}})M_3 \| M(y_n) \|_\xi\\ &+ (M_3 + M_1M_2M_3^2b^{\frac{1}{2}}) \int_0^T \varphi_n(s)ds\\ &\Rightarrow 1 < \frac{1}{n}\Big[C_1 + C_2 \| M(y_n) \|_\xi + C_3 \int_0^T \varphi_n(s)ds\Big],\end{aligned} \tag{6}$$

where $C_1 = M_3 + M_1M_2M_3^2b^{\frac{1}{2}} \| x_0 \|_\xi + (M_1M_2M_3^2b^{\frac{1}{2}}) \| x_1 \|_\xi$, $C_2 = (1+M_1M_2b^{\frac{1}{2}}+M_1M_2M_3b^{\frac{1}{2}})M_3$ and $C_3 = (M_3+M_1M_2M_3^2b^{\frac{1}{2}})$. By passing to the limit as $n \to \infty$ in inequality (3.5), we get $1 \leq 0$, a contradiction. Thus we conclude that there is $n_0 \geq 1$ such that $N(B_{n_0}) \subseteq B_{n_0}$.

Step 2: We shall show that $N(B_{n_0})$ is equicontinuous. Let $x \in N(B_{n_0})$ and let $t_2, t_1 \in I$, $t_2 > t_1 > 0$. We have $y \in N(B_{n_0})$,

$P \in S^1(P(.,y(.)))$, for any $\epsilon > 0$, such that $t_1 - \epsilon > 0$,

$$
\begin{aligned}
&\|x(t_2) - x(t_1)\|_\xi = \| \gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + Bu_p)(t_2) \\
&- \gamma(\Psi_E + \Psi_F + \Psi_G + \Psi_H + Bu_p)(t_1) \|_\xi \\
&= \| \int_0^{t_2} U(t_2,s)\Big[\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\
&+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds\Big] \\
&- \int_0^{t_1} U(t_1,s)\Big[(\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\
&+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds \|_\xi \\
&\le \int_{t_2}^{t_1} \| U(t_2,s) \|_L . \| \Big[\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\
&+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds\Big] \|_\xi \\
&+ \int_0^{t_1} \| U(t_2,s) - U(t_1,s) \|_L \| \Big[\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\
&+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s)) \|_\xi \, ds \\
&\le M_3 \int_{t_1}^{t_2} [\varphi_{n_0}(s) + M_2 \| u_f(s) \|_\xi] ds \\
&+ \int_t^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L \| \Big[\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\
&+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s)) \|_\xi \, ds \\
&+ \int_0^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L \| \Big[\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\
&+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s)) \|_\xi \, ds \\
&\le M_3 \int_{t_1}^{t_2} [\varphi_{n_0}(s) + M_2 \| u_f(s) \|_\xi] ds + 2M_3 \int_{t_1-\epsilon}^{t_1} [\varphi_{n_0}(s) + M_2 \| u_f(s) \|_\xi] ds \\
&+ \int_0^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L \, [\varphi_{n_0}(s) + M_2 \| u_p(s) \|_\xi] ds \\
&\le M_3 \int_{t_1}^{t_2} \varphi_{n_0}(s) ds + 2M_3 \int_{t_1-\epsilon}^{t_1} \varphi_{n_0}(s) ds + \int_0^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L \, \varphi_{n_0}(s) ds \\
&+ \Big(\int_0^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L^2 \, ds \Big)^{\frac{1}{2}} M_2 \| u_p \|_{L^2} \\
&+ M_2 M_3 (t_2 - t_1)^{\frac{1}{2}} \| u_p \|_{L^2} + 2M_2 M_3 \epsilon^{\frac{1}{2}} \| u_p \|_{L^2}
\end{aligned}
$$

and

$$
\begin{aligned}
\| u_p \|_{L^2} \le M_1 \Big[& M_3 \| x_0 \|_\xi + \| x_1 \|_\xi + (M_3 + 1) \| M(B_{n_0}) \|_\xi \\
& + M_2 \int_0^T \varphi_{n_0}(s) ds \Big] = M_4
\end{aligned}
$$

where $\| \quad M \|_\xi = \sup[\| M(y) \|_\xi \colon y \in B_{n_0}]$ is bounded (since M is a compact operator). Moreover, $t \to U(t,s)$ is continuous in the operator norm topology, uniformly $s \in T$ such that $t - s$ is bounded away from zero. Given $\epsilon_1 > 0$, by absolute continuity of the Lebesgue integral we can choose $\epsilon > 0$, such that

$$2M_3 \int_{t_1-\epsilon}^{t_1} \varphi_m(s)ds + 2M_2M_3M_4\epsilon^{\frac{1}{2}} < \frac{\epsilon_1}{2}.$$

By the continuity property of $U(.,s)$ and absolute continuity of the Lebesgue integral, we can find $\delta > 0$ such that if $t_2 - t_1 < \delta$, we have

$$M_3 \int_{t_1}^{t_2} \varphi_{n_0}(s)ds + \int_0^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L \varphi_{n_0}(s)ds$$
$$+\Big(\int_0^{t_1-\epsilon} \| U(t_2,s) - U(t_1,s) \|_L^2 \, ds\Big)^{\frac{1}{2}} M_2M_4 + M_2M_3M_4(t_2-t_1)^{\frac{1}{2}} < \frac{\epsilon_1}{2}$$

So $N(B_{n_0})$ is equicontinuous. Also,

$$K = \{\gamma(\Psi_E+\Psi_F+\Psi_G+\Psi_H+Bu_p) : y \in B_{n_0}, \Psi_P \in S^1_{P(.,x(.))}\} \subseteq C(I,\widetilde{\mathcal{B}})$$

is equicontinuous.

Step 3: We shall show that $N(B_{n_0})(t) = \{x(t) : x \in N(B_{n_0})\}$ is a relatively compact subset of $\widetilde{\mathcal{B}}$ for every $t \in I$.
$N(B_{n_0})$ is bounded since B_{n_0} is bounded and $N(B_{n_0}) \subset B_{n_0}$. Let $t \in (0,T]$ be fixed and ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_{n_0}$ and $h \in N(y)$ there exist functions $\Psi_P \in S^1_{P(.,x(.))}$, $P \in \{E,F,G,H\}$, such that

$$\begin{aligned} h(t) &= U(t,0)(y_0 - M(y)) + \int_0^{t-\epsilon} U(t,s)\Big[(\Psi_E(s,x(s))d\Lambda_\pi(s) \\ &+ \Psi_F(s,x(s))dA_f(s) + \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds\Big] \\ &+ \int_{t-\epsilon}^{t} U(t,s)\Big[(\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s) \\ &+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds\Big] \end{aligned}$$

Define

$$\begin{aligned} h_\epsilon(t) &= U(t,0)(y_0 - M(y)) + \int_0^{t-\epsilon} U(t,s)\Big[\Psi_E(s,x(s))d\Lambda_\pi(s) \\ &+ \Psi_F(s,x(s))dA_f(s) + \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds\Big] \\ &= U(t,0)(y_0 - M(y)) + U(\epsilon,0)\int_0^{t-\epsilon} U(t-\epsilon,s)\Big[\Psi_E(s,x(s))d\Lambda_\pi(s) \\ &+ \Psi_F(s,x(s))dA_f(s) + \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)) + Bu_p(s))ds\Big] \end{aligned}$$

Since $U(t,s)$, is compact, the set $H_\epsilon(t) = \{h_\epsilon(t) : h_\epsilon \in N(y)\}$ is precompact in $\widetilde{\mathcal{B}}$ for every ϵ, $0 < \epsilon < t$. Moreover, for every

$h \in N(y)$,

$$\begin{aligned}\mid h(t) - h_\epsilon(t) \mid &\leq M \int_{t-\epsilon}^{t} (\varphi(x(s)) + c) ds \\ &\leq M \int_{t-\epsilon}^{t} (\varphi(\alpha(s)) + c) ds\end{aligned}$$

Therefore, there are precompact sets arbitrarily close to the set $\{h(t) : h \in N(y)\}$. Hence the set $\{h(t) : h \in N(y)\}$ is precompact in $\widetilde{\mathcal{B}}$.

Step 4: We shall show that N has closed graph.
Let $y_n \to y_*$, $h_n \in N(y_n)$, $y_n \in K$ and $h_n \to h_*$. We shall prove that $h_* \in N(y_*)$, $h_n \in N(y_n)$ means that there exists $\Psi_{n,P} \in S^1_{P(.,y_n(.))}$, such that for each $t \in I$,

$$\begin{aligned}h_n(t) = U(t,0)\Big[y_0 - M(y_n)\Big] &+ \int_0^t U(t,s)\Big[(\Psi_{*,E}(s,x(s))d\Lambda_\pi(s) \\ &+ \Psi_{*,F}(s,x(s))dA_f(s) + \Psi_{*,G}(s,x(s))dA_g^+(s) \\ &+ (\Psi_{*,H}(s,x(s)) + Bu_{y_*}(s))ds\Big]\end{aligned}$$

We have that

$$\begin{aligned}\| h_n - U(t,0)[y_0 - M(y_n)] - \int_0^t U(t,s)[Bu_n(s)]ds \\ - \Big(h_* - U(t,0)[y_0 - M(y_*)] \\ - \int_0^t U(t,s)Bu_{y_*}(s)ds\Big) \|_C \to 0 \text{ as } n \to \infty.\end{aligned}$$

Consider the linear operator

$$\Gamma : L^1(I, \widetilde{\mathcal{B}}) \to C(I, \widetilde{\mathcal{B}}),$$

$$v \mapsto \Gamma(v)(t) = \int_0^t U(t,s)v(s)ds$$

From [11], it follows that $\Gamma \circ S_P$ is a closed graph operator. Moreover, we have that

$$h_n(t) - U(t,0)\Big[y_0 - M(y_*)\Big] - \int_0^t U(t,s)Bu_{y_*}(s)ds \in \Gamma(S_{P,y_n}).$$

Since $y_n \to y_*$, it follows that

$$h_*(t) - U(t,0)[y_0 - M(y_*)] - \int_0^t U(t,s)Bu_{y_*}(s)ds$$
$$= \int_0^t U(t,s)v_*(s)ds$$

for some $v_* \in S_{P,y_*}$. As a consequence of Lemma 1, we deduce that N has a fixed point and therefore the system (2.1) is nonlocally controllable on I.

If $B \equiv 0$ in the system (2.1), we have quantum stochastic evolution inclusions with non local condition. Therefore, the next corollary gives a result on the existence of mild solution of quantum stochastic evolution inclusions with non local condition. However, in this case, the mild solution will be of the form :

$$x(t) = U(t,0)x(0) + \int_0^t U(t,s)\Big(\Psi_E(s,x(s))d\Lambda_\pi(s) + \Psi_F(s,x(s))dA_f(s)$$
$$+ \Psi_G(s,x(s))dA_g^+(s) + (\Psi_H(s,x(s)))ds\Big),\ t \in I.$$

with $\Psi_P \in S^1_{P(.,x(.))}$, $P \in \{E,F,G,H\}$ and $x(0) + M(x) = x_0$.

Corollary If hypotheses $(H_1)-(H_3)$ are satisfied then there exist a mild solution to the non local problem

$$dx(t) \in [A(t)x(t) + H(t,x(t))]dt + E(t,x(t))d\Lambda_\pi(t)$$
$$+ F(t,x(t))dA_f(t) + G(t,x(t))dA_g^+(t),\ \text{almost all}\ \ t \in I.$$
$$x(0) + M(x) = x_0.$$

REFERENCES

[1] A. Anguraj and M. Mallika Arjunan, *Existence and uniqueness of mild and classical solutions of impulsive evolution equations*, Electronic Journal of Differential Equations, **111** (2005), 1-8.

[2] M. Malika Arjunan, V. Kavitha and S. Selvi, *Existence results for impulsive differential equations with nonlocal conditions via measures of noncompactness*, J. Nonlinear Sci. Appl. **5** , 2012, 195-205.

[3] M. Benchohra and S. K. Ntouyas, *Existence of mild solutions of semilinear evolution inclusions with nonlocal conditions*, Georgian Math. J., **7**, 2000, 221-230.

[4] L. Byszewski, *Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162**, 1991, 494-505.

[5] L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of solutions of nonlocal Cauchy problem in a Banach space*, Appl. Anal. **40**, 1990, 11-19.

[6] G.O.S. Ekhaguere, *Quantum stochastic evolutions*, International Journal of Theoretical Physics, **35**, 1996, 9, 1909-1946.

[7] F. Fagnola and S.J. Wills, *Solving quantum stochastic differential equations with unbounded coefficients*, J. Funct. Anal. **198**, 2003, 279-310.

[8] Z. Fan, *Impulsive problems for semilinear differential equations with nonlocal conditions*, Nonlinear Anal. **72** , 2010 , 1104-1109.

[9] E.P. Gatsori, *Controllability results for nondensely defined evolution differential inclusions with nonlocal conditions*, J. Math. Anal. Appl. **297**, 2004, 194-211.

[10] R.L. Hudson and K.R. Parthasarathy, *Quantum Ito's formula and stochastic evolutions*, Comm. Math. Phys., **93**, 1984, 3, 301-323.

[11] A. Lasota, Z. Opial, *An application of the KakutaniKy Fan theorem in the theory of ordinary differential equations*, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. **13**, 1965, 781786.

[12] G. Li and X. Xue,*Controllability of evolution inclusions with nonlocal conditions*, Appl. Math. Computation, **141**, 2003, 375-384.

[13] M. O. Ogundiran, *On the mild solutions of Quantum stochastic evolution inclusions*, Communications in Appl. Anal., **19**, 2015, 2, 307-318.

[14] M. O. Ogundiran and V. F. Payne, *On the existence of solutions of Impulsive Quantum stochastic differential equations*, Journal of Differential Equations and Control processes, 2013, **2**, 62-73.

[15] M. O. Ogundiran and V. F. Payne, *On the solution sets of semicontinuous quantum stochastic differential inclusions*, Applied Mathematics E-Notes, **14**, 2014, 135-143.

[16] X. Xue, *Existence of semilinear differential equations with nonlocal initial conditions*, Acta Math. Sinica, **23**, 6, 2007, 983-988.

[17] E, Zeidler, *Nonlinear Functional Analysis and Applications, Fixed Point Theorems*, Springer-Verlag, New York, 1986.

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