Journal of the Nigerian Mathematical Society
Vol. 39, Issue 2, pp. 249-254, 2020

ON LIE $SL(n,\mathbb{R})$-FOLIATION

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ABSTRACT. In this paper, we show that any compact manifold that carries a $SL(n,\mathbb{R})$-foliation is fibered on the circle S^1. Every manifold in this paper is compact and our Lie group G is connected and simply connected.

Keywords and phrases: foliation, fibering, Lie group, manifold, compact
2010 Mathematical Subject Classification: Primary 53B05; Secondary 53B20.

1. INTRODUCTION

The foliation theory is a branch of geometry which has risen in the second half of the XX-th century on a joint of ordinary differential equations and the differential topology. Basic works on the foliation theory belong to G. Ehresmann [4], A. Haefliger [7] and G. Reeb [11]. Important contribution to foliation theory was made by R. Herman [8] and Ph. Tondeur [13].

Lie foliations have been studied by several authors (cf. [1, 2, 3, 5, 6]. To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathcal{G}$ of the Lie group on which the foliation is modeled and the structural Lie algebra $\mathcal{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the closure of any one of its leaves. In particular, it is a subalgebra of $\mathcal{G}$.

In this work we want to know when a manifold that carries a Lie foliation fiber on the circle S^1. In [12], Tischer has shown that any manifold which has a closed non-singular 1-form fiber on the circle. In [1] H. Dathe gives an example where the manifold carry a Lie $SL(2,\mathbb{R})$-foliation. In this work we consider $G = SL(n,\mathbb{R})$ and we show that the manifold which supported the Lie G-foliation fiber to the cercle S^1.

2. PRELIMINARY

Received by the editors November 07, 2018; Revised July 20, 2020 ; Accepted: July 27, 2020
www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

Let G be a simply connected Lie group and M a compact manifold of dimension n. A Lie G-foliation is the given of a set $\mathcal{F}$ of couples $(U;\, f)$, where U is an open of M and $f : U \longrightarrow G$ a submersion, having the following properties:

i) The open sets U cover M.
ii) for all $(U;\, f)$ and $(W;\, h) \in \mathcal{F}$, there exists $g \in G$ such that, for all $x \in U \cap W$, we have $f(x) = h(x)g$.

In particular, level surfaces of the submersions f, for $(U; f) \in \mathcal{F}$, recollect to form a Lie foliation on M. To avoid ambiguities, it is further assumed that $\mathcal{F}$ is maximal in the following sense: if U is an open of M and $f : U \longrightarrow G$ a submersion, if, for any $(W;\, h)$ in $\mathcal{F}$, there exists g in G with $f = h.g$ on $U \cap W$, we have $(U; f) \in \mathcal{F}$. In this case G is called the transverse group of the G-foliation.
In the general case the structure transverse of G-foliation is given by Fedida's theorem[5]:

Let $\mathcal{F}$ be a G-foliation on a compact manifold M. Let $\tilde{M}$ be the universal covering of M and $\tilde{\mathcal{F}}$ the recovery of $\mathcal{F}$ on $\tilde{M}$. Then there exists a morphism $h : \pi_1(M) \longrightarrow G$ and one locally trivial fibration $D : \tilde{M} \longrightarrow G$ whose fibers are the leaves of $\tilde{\mathcal{F}}$ and such that for any $\gamma \in \pi_1(M)$, the following diagram is commutative:

$$\begin{tikzcd} \widetilde{M} \arrow[r, "\gamma"] \arrow[d, "D"'] & \widetilde{M} \arrow[d, "D"] \\ G \arrow[r, "h(\gamma)"'] & G \end{tikzcd}$$

where the first line represents the transformation of $\gamma \in \pi_1(M)$ on $\tilde{M}$. The group $\Gamma = h\{\pi_1(M)\}$ (which is a subgroup of G) is called the holonomy group of $\mathcal{F}$ and the fibration $D : \tilde{M} \longrightarrow G$ is called the developping application of the foliation $\mathcal{F}$. This structure theorem makes it possible to build Lie foliation on a manifold.
Let $\mathcal{G}$ be the Lie algebra of G. A *Maurer-Cartan* form with values in $\mathcal{G}$ is a differential 1-form $\omega \in \Omega^1(M,\, \mathcal{G})$ with trivial formal, i.e

$$d\omega + \frac{1}{2}[\omega,\, \omega] = 0$$

Here $[\omega, \omega]$ is the differential 2-form on M with values in $\mathcal{G}$ given by $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$. If ω is non singular, i.e. if $\omega_x : T_x(M) \longrightarrow \mathcal{G}$ is surjective at any point $x \in M$, then the dimension of $\mathcal{G}$ is finite and $\ker(\omega)$ is subbundle of $T(M)$ of codimension $\dim \mathcal{G}$. Vanishing the of the formal curvature implies that the subbundle $\ker(\omega)$ is involutive and hence defines a foliation $\mathcal{F}$ on M, with $codim \mathcal{F} = \dim \mathbf{g}$ and

$$T(\mathcal{F}) = \ker(\omega).$$

A Lie foliation is a foliation defined in this way by a non-singular Maurer-Cartan form.
Lie G-foliations form a very special class of foliations and satisfy various strong properties. For example, any left invariant Riemannian metric of G gives rise to a metric on the normal bundle of $\mathcal{F}$, invariant by the holonomy pseudogroup; that is,$\mathcal{F}$ is a Riemannian foliation. Each leaf of $\mathcal{F}$ has trivial holonomy and they are mutually Lipshitz diffeomorphic. Conversely by the work of P. Molino[10], the study of Riemannian foliations reduces to that of Lie foliations.

3. LIE-FOLIATION WITH TRANSVERSE GROUP $SL(n, \mathbb{R})$

In the first we want to decompose the group $SL(n, \mathbb{R})$ such that the group GA is one of the factor. The following decomposition can motivate us.

Let GA be the Lie group of affine transformations $x \in \mathbb{R} \longmapsto ax + b \in \mathbb{R}$, where $b \in \mathbb{R}$ and $a \in]0; +\infty[$. It can be embedded in the group $SL(2, \mathbb{R})$ as follows:

$$\begin{pmatrix} x \longmapsto ax + b \\ a > O \end{pmatrix} \in GA \longmapsto \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{R})$$

There exist [4] a manifold M equipped with a Lie $SL(2, \mathbb{R})$-foliation $\mathcal{F}$ with GA as the closure of its holonomy group. Then, the basic cohomology of $\mathcal{F}$ is the cohomology of differential forms on $SL(2, \mathbb{R})$ invariant by GA. The quotient $SL(2; \mathbb{R})/GA$ is diffeomorphic to the circle S^1.

Example 1: Let $\mathcal{F}_o$ the Lie GA-foliation on a compact manifold M_o. We suppose that $\mathcal{F}_o$ can't be obtain by inverse image of a homogenous foliation. The projection $p : SL(2, \mathbb{R} \longrightarrow S^1$ have a section given by the decomposition $SL(2, \mathbb{R}) \cong GA \times S^1$. Let

$D_o : \widetilde{V}_o \longrightarrow GA$ the the developing map of $\mathcal{F}_o$.

The map $D : \widetilde{V}_o \times S^1 \longrightarrow SL(2,\mathbb{R}),(\widetilde{x}, y) \longmapsto D_o\widetilde{x}.\sigma(y)$ is local trivial fibration, these fibers define a Lie $SL(2,\mathbb{R})$-foliation on $\widetilde{V}_o \times S^1$ and induce a Lie $SL(2,\mathbb{R})$-foliation $\mathcal{F}$ on the manifold $V = \widetilde{V}_o \times S^1$ which is not conjugate to a homogenous foliation.

Theorem 1:[12]
If there exists on a compact manifold M a closed differential form without singularities, then M is fibered on the circle.

Proposition 1:[1]
A compact manifold that carry a Lie $SL(2,\mathbb{R})$-foliation is fiber on the circle S^1.

Our aim is to generalize this proposition to Lie $SL(n,\mathbb{R})$-foliation. Before that we have

Proposition 2: We have the decomposition $SL(n,\mathbb{R}) \cong T^{n-1} \times G$ where T^{n-1} is the maximal tore of $SL(n,\mathbb{R})$ identify by the subgroup of diagonal matrices. And G is Lie group such that $Lie(G) = \bigoplus_{i\neq j} < E_{ij} >$ and $dimG = n^2 - n$. Moreover we have:

$$[E_{ij}, E_{kl}] = 0 \ if \ i \neq l \ and \ j \neq k$$

$$[E_{ij}, E_{jl}] = E_{il} \ if \ i \neq l$$

$$[E_{ij}, E_{ki}] = -E_{kj} \ if \ k \neq j$$

$$[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$$

Proof: Let $SL(n,\mathbb{R})$ the real special linear group and T the maximal tore of $SL(n,\mathbb{R})$ identify with the subgroup of diagonal matrices. We denote by $X(T)$ the group of morphism $T \longrightarrow \mathbb{R}^{\times}$.
Then T act on the Lie algebra $\mathcal{G}$ of $SL(n,\mathbb{R})$ by conjugaison and we have

$$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{i\neq j} \lambda E_{ij},$$

$\lambda \in \mathbb{R}$, where $\mathcal{H}$ is the subset of diagonal matrices of vanishing trace.
Using this splitting we then can decompose $SL(n,\mathbb{R}) \cong T \times G$ such that $LieT = \mathcal{H}$ and $Lie(G) = \bigoplus_{i\neq j} < E_{ij} >$, E_{ij} being the $n \times n$ matrix where the element in the ith line and the jth culumb is equal

to 1 and the other elements are vanish.
Let $Y = (a_{ij}), i, j = 1, ..., n$ in $\mathcal{H}$, thus we have

$$\sum_{i}^{n} a_{ii} = 0 \Rightarrow a_{11} = -\sum_{i\neq 1} a_{ii}$$

then

$$Y = \sum_{i\neq 1} a_{ii} Y_i$$

where $Y_i = (b_{kl})$ is the matrix with $b_{11} = -1$, $b_{kk} = 1$, $k \neq 1$ and $b_{kl} = 0$ for $k \neq l$.

We can easily note that the $(Y_i), i = 2, ..., n$ are also linearily independant, so $\mathcal{H} =< Y_i, i = 2, ..., n >$ and then $dim\mathcal{H} = n-1$. This imply than $dimT = n-1$, therefore $dimG = (n^2-1)-(n-1) = n^2 - n$.

And by a simple calculation using matrix product, we have the value of the Lie bracket $[E_{ij}, E_{kl}]$, this finish then the proof.

Remark 1: The group G in the previous proposition can be identify with the group $SO(n) \times SO(n)$ and the maximal tore T is isomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}-1}/SO(n)$ and then we have

$$SL(n,\mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}-1}$$

Proposition 3: Let $\mathcal{F}$ be a Lie G-foliation on a compact manifold M, with $G = G_1 \times G_2$. There exists a Lie G_i-foliation $\mathcal{F}_i$ on M induced by the foliation $\mathcal{F}$.

Proof: Let G_1 and G_2 be two Lie groups and $\mathcal{F}$ be a Lie G-foliation on a compact manifold M, where $G = G_1 \times G_2$.

If D is the developing map of $\mathcal{F}$ on the universal cover $\widetilde{M}$ of M, then the simple foliation defined by $p_i \circ D$ (where p_i, $i = 1, 2$ is the projection of G on G_i, $i = 1, 2$), pass in quotient and induces a foliation $\mathcal{F}_i$ on M.

Theorem 2: A compact manifold that carry a Lie $SL(n,\mathbb{R})$-foliation fiber on the circle S^1.

Proof: Let M be a compact manifold with a Lie $SL(n,\mathbb{R})$-foliation. We have also

$$SL(n,\mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}-1}$$

$$SL(n,\mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}-3} \times \mathbb{R}^2$$

Now we take $G = SL(n,\mathbb{R})$, $G_1 = SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}-3}$ and $G_2 = \mathbb{R}^2$ so using the proposition, the Lie $SL(n,\mathbb{R})$-foliation induces a Lie $\mathbb{R}^2$-foliation on M. Since $\mathbb{R}^2$ is abelian the structures equations of the Lie $\mathbb{R}^2$-foliation are closed 1-forms on M, then using the Tischler theorem, M is a fibration over the circle.

4. CONCLUSION

Thus, in this work we have showing that every compact manifold which carrying a Lie foliation with transverse group $SL(n, \mathbb{R})$, fiber on the circle. We have using the decomposition of this group and the Tischler' theorem for solving the problem studied. Next we want to generalize this methode with all semi-simple Lie group.

ACKNOWLEDGEMENTS

The author would like to thank the anonymous referee whose comments improved the original version of this manuscript.

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