ON LIE $SL(n, \mathbb{R})$-FOLIATION

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ABSTRACT. In this paper, we show that any compact manifold that carries a $SL(n, \mathbb{R})$-foliation is fibered on the circle $S^1$. Every manifold in this paper is compact and our Lie group $G$ is connected and simply connected.

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1. INTRODUCTION

The foliation theory is a branch of geometry which has risen in the second half of the XX-th century on a joint of ordinary differential equations and the differential topology. Basic works on the foliation theory belong to G. Ehresmann [4], A. Haefliger [7] and G. Reeb [11]. Important contribution to foliation theory was made by R. Herman [8] and Ph. Tondeur [13].

Lie foliations have been studied by several authors (cf. [1, 2, 3, 5, 6]). To each Lie foliation are associated two Lie algebras, the Lie algebra $\mathcal{G}$ of the Lie group on which the foliation is modeled and the structural Lie algebra $\mathcal{H}$. The latter algebra is the Lie algebra of the Lie foliation $\mathcal{F}$ restricted to the closure of any one of its leaves. In particular, it is a subalgebra of $\mathcal{G}$.

In this work we want to know when a manifold that carries a Lie foliation fiber on the circle $S^1$. In [12], Tischer has shown that any manifold which has a closed non-singular 1-form fiber on the circle. In [1] H. Dathe gives an example where the manifold carry a Lie $SL(2, \mathbb{R})$-foliation. In this work we consider $G = SL(n, \mathbb{R})$ and we show that the manifold which supported the Lie $G$-foliation fiber to the cercle $S^1$.

2. PRELIMINARY
Let $G$ be a simply connected Lie group and $M$ a compact manifold of dimension $n$. A Lie $G$-foliation is the given of a set $\mathcal{F}$ of couples $(U; f)$, where $U$ is an open of $M$ and $f : U \rightarrow G$ a submersion, having the following properties:

i) The open sets $U$ cover $M$.

ii) for all $(U; f)$ and $(W; h) \in \mathcal{F}$, there exists $g \in G$ such that, for all $x \in U \cap W$, we have $f(x) = h(x)g$.

In particular, level surfaces of the submersions $f$, for $(U; f) \in \mathcal{F}$, recollect to form a Lie foliation on $M$. To avoid ambiguities, it is further assumed that $\mathcal{F}$ is maximal in the following sense: if $U$ is an open of $M$ and $f : U \rightarrow G$ a submersion, if, for any $(W; h)$ in $\mathcal{F}$, there exists $g$ in $G$ with $f = h \cdot g$ on $U \cap W$, we have $(U; f) \in \mathcal{F}$.

In this case $G$ is called the transverse group of the $G$-foliation.

In the general case the structure transverse of $G$-foliation is given by Fedida’s theorem[5]:

Let $\mathcal{F}$ be a $G$-foliation on a compact manifold $M$. Let $\tilde{M}$ be the universal covering of $M$ and $\tilde{\mathcal{F}}$ the recovery of $\mathcal{F}$ on $\tilde{M}$. Then there exists a morphism $h : \pi_1(M) \rightarrow G$ and one locally trivial fibration $D : \tilde{M} \rightarrow G$ whose fibers are the leaves of $\tilde{\mathcal{F}}$ and such that for any $\gamma \in \pi_1(M)$, the following diagram is commutative:

$$
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\gamma} & \tilde{M} \\
\downarrow D & & \downarrow D \\
G & \xrightarrow{h(\gamma)} & G
\end{array}
$$

where the first line represents the transformation of $\gamma \in \pi_1(M)$ on $\tilde{M}$. The group $\Gamma = h(\pi_1(M))$ (which is a subgroup of $G$) is called the holonomy group of $\mathcal{F}$ and the fibration $D : \tilde{M} \rightarrow G$ is called the developing application of the foliation $\mathcal{F}$. This structure theorem makes it possible to build Lie foliation on a manifold.

Let $\mathcal{G}$ be the Lie algebra of $G$. A Maurer-Cartan form with values in $\mathcal{G}$ is a differential 1-form $\omega \in \Omega^1(M, \mathcal{G})$ with trivial formal, i.e.

$$
d\omega + \frac{1}{2}[\omega, \omega] = 0
$$
Here \([\omega, \omega]\) is the differential 2-form on \(M\) with values in \(G\) given by \([\omega, \omega](X, Y) = [\omega(X), \omega(Y)]\). If \(\omega\) is non singular, i.e. if \(\omega_x : T_x(M) \rightarrow G\) is surjective at any point \(x \in M\), then the dimension of \(G\) is finite and \(\ker(\omega)\) is subbundle of \(T(M)\) of codimension \(\dim G\). Vanishing of the formal curvature implies that the subbundle \(\ker(\omega)\) is involutive and hence defines a foliation \(\mathcal{F}\) on \(M\), with 

\[
\ker(\omega) \subset T(\mathcal{F}) \subset \ker(\omega).
\]

A Lie foliation is a foliation defined in this way by a non-singular Maurer-Cartan form.
Lie \(G\)-foliations form a very special class of foliations and satisfy various strong properties. For example, any left invariant Riemannian metric of \(G\) gives rise to a metric on the normal bundle of \(\mathcal{F}\), invariant by the holonomy pseudogroup; that is, \(\mathcal{F}\) is a Riemannian foliation. Each leaf of \(\mathcal{F}\) has trivial holonomy and they are mutually Lipschitz diffeomorphic. Conversely by the work of P. Molino[10], the study of Riemannian foliations reduces to that of Lie foliations.

### 3. LIE-FOLIATION WITH TRANSVERSE GROUP \(SL(n, \mathbb{R})\)

In the first we want to decompose the group \(SL(n, \mathbb{R})\) such that the group \(GA\) is one of the factor. The following decomposition can motivate us.

Let \(GA\) be the Lie group of affine transformations \(x \in \mathbb{R} \mapsto ax + b \in \mathbb{R}\), where \(b \in \mathbb{R}\) and \(a \in ]0; +\infty[\). It can be embedded in the group \(SL(2, \mathbb{R})\) as follows:

\[
\begin{pmatrix} x & \mapsto ax + b \\ a \in ]0; +\infty[ \end{pmatrix} \in GA \mapsto \frac{1}{\sqrt{a}} \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} \in SL(2, \mathbb{R}).
\]

There exist [4] a manifold \(M\) equipped with a Lie \(SL(2, \mathbb{R})\)-foliation \(\mathcal{F}\) with \(GA\) as the closure of its holonomy group. Then, the basic cohomology of \(\mathcal{F}\) is the cohomology of differential forms on \(SL(2, \mathbb{R})\) invariant by \(GA\). The quotient \(SL(2, \mathbb{R})/GA\) is diffeomorphic to the circle \(S^1\).

**Example 1:** Let \(\mathcal{F}_o\) the Lie \(GA\)-foliation on a compact manifold \(M_o\). We suppose that \(\mathcal{F}_o\) can’t be obtain by inverse image of a homogenous foliation. The projection \(p : SL(2, \mathbb{R}) \rightarrow S^1\) have a section given by the decomposition \(SL(2, \mathbb{R}) \cong GA \times S^1\). Let
$D_o : \tilde{V}_o \rightarrow GA$ the the developing map of $\mathcal{F}_o$.

The map $D : \tilde{V}_o \times S^1 \rightarrow SL(2, \mathbb{R}),(\tilde{x}, y) \mapsto D_o \tilde{x} \sigma(y)$ is local trivial fibration, these fibers define a Lie $SL(2, \mathbb{R})$-foliation on $\tilde{V}_o \times S^1$ and induce a Lie $SL(2, \mathbb{R})$-foliation $\mathcal{F}$ on the manifold $V = \tilde{V}_o \times S^1$ which is not conjugate to a homogenous foliation.

**Theorem 1:**[12]
If there exists on a compact manifold $M$ a closed differential form without singularities, then $M$ is fibered on the circle.

**Proposition 1:**[1]
A compact manifold that carry a Lie $SL(2, \mathbb{R})$-foliation is fiber on the circle $S^1$.

Our aim is to generalize this proposition to Lie $SL(n, \mathbb{R})$-foliation. Before that we have

**Proposition 2:** We have the decomposition $SL(n, \mathbb{R}) \cong T^{n-1} \times G$ where $T^{n-1}$ is the maximal tore of $SL(n, \mathbb{R})$ identify by the subgroup of diagonal matrices. And $G$ is Lie group such that $\text{Lie}(G) = \bigoplus_{i \neq j} < E_{ij}>$ and $\text{dim}G = n^2 - n$. Moreover we have:

$[E_{ij}, E_{kl}] = 0$ if $i \neq l$ and $j \neq k$

$[E_{ij}, E_{jl}] = E_{il}$ if $i \neq l$

$[E_{ij}, E_{ki}] = -E_{kj}$ if $k \neq j$

$[E_{ij}, E_{ji}] = E_{ii} - E_{jj}$

**Proof:** Let $SL(n, \mathbb{R})$ the real special linear group and $T$ the maximal tore of $SL(n, \mathbb{R})$ identify with the subgroup of diagonal matrices. We denote by $X(T)$ the group of morphism $T \rightarrow \mathbb{R}^\times$.

Then $T$ act on the Lie algebra $\mathcal{G}$ of $SL(n, \mathbb{R})$ by conjugaison and we have

$\mathcal{G} = \mathcal{H} \oplus \bigoplus_{i \neq j} \lambda E_{ij}$,

$\lambda \in \mathbb{R}$, where $\mathcal{H}$ is the subset of diagonal matrices of vanishing trace.

Using this splitting we then can decompose $SL(n, \mathbb{R}) \cong T \times G$ such that $\text{Lie}T = \mathcal{H}$ and $\text{Lie}(G) = \bigoplus_{i \neq j} < E_{ij}>$, $E_{ij}$ being the $n \times n$ matrix where the element in the $i$th line and the $j$th culumb is equal
to 1 and the other elements are vanish.
Let $Y = (a_{ij}), i, j = 1, \ldots, n$ in $\mathcal{H}$, thus we have
$$\sum_{i=1}^{n} a_{ii} = 0 \Rightarrow a_{11} = -\sum_{i \neq 1} a_{ii}$$
then
$$Y = \sum_{i \neq 1} a_{ii}Y_i$$
where $Y_i = (b_{kl})$ is the matrix with $b_{11} = -1, b_{kk} = 1, k \neq 1$ and 
$b_{kl} = 0$ for $k \neq l$.

We can easily note that the $(Y_i), i = 2, \ldots, n$ are also linearily independant, so $\mathcal{H} = \langle Y_i, i = 2, \ldots, n \rangle$ and then $\dim \mathcal{H} = n - 1$.
This imply than $\dim T = n - 1$, therefore $\dim G = (n^2 - 1) - (n - 1) = n^2 - n$.
And by a simple calculation using matrix product, we have the value of the Lie bracket $[E_{ij}, E_{kl}]$, this finish then the proof.

**Remark 1:** The group $G$ in the previous proposition can be identify with the group $SO(n) \times SO(n)$ and the maximal tore $T$ is isomorphic to $\mathbb{R}^{\frac{n(n+1)}{2}} / SO(n)$ and then we have
$$SL(n, \mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$$

**Proposition 3:** Let $\mathcal{F}$ be a Lie $G$-foliation on a compact manifold $M$, with $G = G_1 \times G_2$. There exists a Lie $G_i$-foliation $\mathcal{F}_i$ on $M$ induced by the foliation $\mathcal{F}$.

**Proof:** Let $G_1$ and $G_2$ be two Lie groups and $\mathcal{F}$ be a Lie $G$-foliation on a compact manifold $M$, where $G = G_1 \times G_2$.

If $D$ is the developing map of $\mathcal{F}$ on the universal cover $\tilde{M}$ of $M$, then the simple foliation defined by $p_i \circ D$ (where $p_i, i = 1, 2$ is the projection of $G$ on $G_i$, $i = 1, 2$), pass in quotient and induces a foliation $\mathcal{F}_i$ on $M$.

**Theorem 2:** A compact manifold that carry a Lie $SL(n, \mathbb{R})$-foliation fiber on the circle $S^1$.

**Proof:** Let $M$ be a compact manifold with a Lie $SL(n, \mathbb{R})$-foliation. We have also
$$SL(n, \mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$$
$$SL(n, \mathbb{R}) \cong SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}} \times \mathbb{R}^{2}$$
Now we take $G = SL(n, \mathbb{R})$, $G_1 = SO(n) \times \mathbb{R}^{\frac{n(n+1)}{2}}$ and $G_2 = \mathbb{R}^{2}$ so using the proposition, the Lie $SL(n, \mathbb{R})$-foliation induces a Lie $\mathbb{R}^{2}$-foliation on $M$. Since $\mathbb{R}^{2}$ is abelian the structures equations
of the Lie $\mathbb{R}^2$-foliation are closed 1-forms on $M$, then using the Tischler theorem, $M$ is a fibration over the circle.

4. CONCLUSION

Thus, in this work we have showing that every compact manifold which carrying a Lie foliation with transverse group $SL(n, \mathbb{R})$, fiber on the circle. We have using the decomposition of this group and the Tischler’ theorem for solving the problem studied. Next we want to generalize this method with all semi-simple Lie group.

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REFERENCES


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