Journal of the Nigerian Mathematical Society
Vol. 39, Issue 2, pp. 239-247, 2020

VARIANT OF FINITE SYMMETRIC INVERSE SEMIGROUP

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ABSTRACT. In a semigroup S fix an element $a \in S$ and, for all $x, y \in S$, define a binary operation $*_a$ on S by $x *_a y = xay$, (where the juxterposition on the left denote the original semigroup operation on S). The operation $*_a$ is clearly associative and so S forms a new semigroup under this operation, which is denoted by S^a and called *variant* of S by a. For a finite set $X_n = \{1, 2, ..., n\}$, let $\mathcal{I}_n$ be the symmetric inverse semigroup on X_n and fix an idempotent $a \in \mathcal{I}_n$. In this paper, we study the variant $\mathcal{I}_n^a$ of $\mathcal{I}_n$ by a. In particular, we characterised Green's relations and starred Green's relations $\mathcal{L}^*$, $\mathcal{R}^*$ in $\mathcal{I}_n^a$ and also showed that the variant semigroup $\mathcal{I}_n^a$ is abundant.

Keywords: Variant semigroup, Symmetric inverse semigroup, Green's relations, Starred Green's relations, Abundant semigroup.

2010 AMS Mathematics Subject Classification: 20M20

1. Introduction

Let a be a fixed element of a semigroup S. For all $x, y \in S$, we define a binary operation $*_a$ by $x *_a y = xay$, (where the juxterposition on the left denote the original semigroup operation on S). The operation $*_a$ is associative and so S forms a new semigroup under this operation, which is denoted by S^a and called *variant* of S by a. The first study of variant of an arbitrary semigroup was by Hickey [5] but the notion was originally an extension of an idea due to Lyapin [11], which can also be attributed to Magill [12] who considered the semigroup of functions from a set X into a set Y under the sandwich operation " $\cdot$ " defined by

$$f \cdot g = f \circ \mu \circ g,$$

where μ is some fixed function from Y to X (see also [1], [9], [13] and [14]). When $X = Y$, the semigroup studied by Magill [12] is

Received by the editors December 27, 2018; Revised July 15, 2020 ; Accepted: July 31, 2020

www.nigerianmathematicalsociety.org; Journal available online at https://ojs.ictp.it/jnms/

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exactly the variant of *full transformation semigroup* $\mathcal{T}_X$. The variant of finite full transformation semigroups $\mathcal{T}_n^a$ have been studied. For instance, Kudryautseva and Tsyaputa [10] classified the non-isomorphic variants of $\mathcal{T}_n$ and later in [19], Tsyaputa characterised Green's relations in such a semigroup. The structure of the variant of finite full transformation semigroup was extensively studied by Dolinka and East [8].
While a semigroup S possesses certain characteristic, its variant S^a may not have that same characteristic. In fact, it is the case that in general $S \ncong S^a$. It has been pointed out in [8] that the variants $\mathcal{T}_n$, $\mathcal{P}_n$ and $\mathcal{I}_n$ of respectively finite full transformation semigroup, finite partial transformation semigroup (consisting of all partial maps on a finite set of n elements) and finite symmetric inverse semigroup (consisting of all injective partial maps on a finite set of n elements) are non-regular even though each of these semigroups is regular. Regular semigroups are interesting classes of semigroup that have been extensively studied via the classical Green's equivalences [4], see for example Howie [7]. A generalisation of regular semigroup called abundant semigroup was introduced by Fountain [2, 3] and studied via an extended version of Green's relations known as stared Green's relations.

Two elements a and b of semigroup S are said to be $\mathcal{L}^*$-related if they are related under the standard Green's $\mathcal{L}$ relation in some oversemiroup of S. The relation $\mathcal{R}^*$ is defined dually. A semgroup S is called right abundant if every $\mathcal{L}^*$-class and every $\mathcal{R}^*$-class of S contains an idempotent. Abundant semigroup have been studied see [18, 15, 16, 17, 6]. In this paper, we investigate the variant semigroup $\mathcal{I}_n^a$ of all partial one-to-one transformation in line with [8] on a finite set $X_n = \{1, 2, ..., n\}$. In particular, we characterised Green's relations similar to the corresponding characterisation in the variant semigroup $\mathcal{T}_n^a$ obtained by [8], we also characterised starred Green's relations $\mathcal{L}^*$, $\mathcal{R}^*$ in the semigroup $\mathcal{I}_n^a$ and showed that it is an abundant semigroup.

It is known , due to [8], that every variant S^x of a semigroup S by an elements $x \in S$ is isomorphic to a variant S^a of S by an idempotent $a \in S$. Thus, it is no loss of generality if we assume throughout this paper that the variant $\mathcal{I}_n^a$ of $\mathcal{I}_n$ is always with respect to an idempotent a of height r (where $1 \leq r \leq n-1$) with $\mathrm{dom}(a) = \mathrm{im}(a) = \{a_1, a_2, \ldots, a_r\}$. Thus, the idempotent a is a partial identity on $\{a_1, a_2, \ldots, a_r\}$ and can be written, in an array notation, as

$$a = \begin{pmatrix} a_1 & a_2 & \ldots & a_r \\ a_1 & a_2 & \ldots & a_r \end{pmatrix}.$$

For a semigroup S we denote by $E(S)$, the set of all idempotent elements in S, that is $E(S) = \{e \in S | e^2 = e\}$.

In the next lemma we characterised idempotent elements in $\mathcal{I}_n^a$.

Lemma 1.1. *Let* $\alpha \in \mathcal{I}_n^a$. *Then,* $\alpha \in E(\mathcal{I}_n^a)$ *if and only if* $\alpha \in E(\mathcal{I}_n)$ *and* $\mathrm{im}(\alpha) \subseteq \mathrm{dom(a)}$.

Proof. Suppose that $\alpha \notin E(\mathcal{I}_n^a)$ and let $x \in \mathrm{dom}(\alpha)$ be such that $x\alpha^2 \neq x$. Then, $x\alpha^2 = x(\alpha *_a \alpha) = x(\alpha a \alpha) = (x\alpha)a\alpha$.

Now, if $x\alpha \in \mathrm{dom}(a)$, then $(x\alpha)a = x\alpha$ (since a is a partial identity) and so, $x\alpha^2 = x(\alpha *_a \alpha) = (x\alpha)\alpha \neq x\alpha$ always. If $x\alpha$ is not in $\mathrm{dom}(a)$, then x is not in $\mathrm{dom} x(\alpha *_a \alpha)$ and therefore, α is again not an idempotent.

Also, suppose that $\mathrm{im}(\alpha) \nsubseteq \mathrm{dom}(a)$ and let $x \in \mathrm{im}(\alpha) \setminus \mathrm{dom}(a)$ and $y \in \mathrm{dom}(\alpha)$ be such that $y\alpha = x$. Then clearly y is not in $\mathrm{dom}(\alpha a \alpha)$ which implies that $\alpha * a\alpha \neq \alpha$.

Conversely, it is easy to see that if $\alpha \in E(\mathcal{I}_n)$ and $\mathrm{im}(\alpha) \subseteq \mathrm{dom}(a)$, then $\alpha a \alpha = \alpha$ which implies α is an idempotent in $(\mathcal{I}_n^a)$. □

2. Green's relations in $(\mathcal{I}_n^a)$

On a semigroup S (see Howie [7]), the five equivalence relations $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}$ and $\mathcal{D}$ are defined as for all $a, b \in S$, $a\mathcal{L}b$ if and only if $S^1a = S^1b$; $a\mathcal{R}b$ if and only if $aS^1 = bS^1$; $a\mathcal{J}b$ if and only if $S^1aS^1 = S^1bS^1$; $\mathcal{H} = \mathcal{L} \cap \mathcal{R}$ and $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$. These are known, in semigroup theory as Green's relations [4]. Since the appearance of these relations in 1951 they became standard tool for understanding the structure of a semigroup. In order to avoid confusion, if $\mathcal{K}$ is one of $\mathcal{L}, \mathcal{R}, \mathcal{H}, \mathcal{J}$ and $\mathcal{D}$, we will write $\mathcal{K}^a$ for Green's $\mathcal{K}$-relation on the variant S^a of S, and write K_x^a for the $\mathcal{K}^a$-class of $x \in S^a$.

As noted by Dolinka and East [8], for any semigroup S and a fixed element $a \in S$, the subsets

$$P_1(S) = \{\alpha \in S : xa\mathcal{R}x\} \tag{1}$$

$$P_2(S) = \{x \in S : ax\mathcal{L}x\} \tag{2}$$

$$P(S) = P_1(S) \cap P_2(S) \tag{3}$$

played vital role in characterising Green's relations in the variant semigroup S^a. In fact, in [8], the following characterisation of Green's relations in S^a was proved.

Theorem 2.1. *and East* [8]*] If $x \in S$, then*

$$i.\ R_x^a = \begin{cases} R_x \cap P_1(S) & \text{if } x \in P_1(S), \\ \{x\} & \text{if } x \in S \setminus P_1(S) \end{cases}$$

$$ii.\ L_x^a = \begin{cases} L_x \cap P_2(S) & \text{if } x \in P_2(S), \\ \{x\} & \text{if } x \in S \setminus P_2(S) \end{cases}$$

$$iii.\ H_x^a = \begin{cases} H_x \cap P(S) & \text{if } x \in P(S), \\ \{x\} & \text{if } x \in S \setminus P(S) \end{cases}$$

$$iv.\ D_x^a = \begin{cases} D_x \cap P(S) & \text{if } x \in P(S), \\ R_x^a & \text{if } x \in P_1(S) \setminus P_2(S), \\ L_x^a & \text{if } x \in P_2(S) \setminus P_1(S), \\ \{x\} & \text{if } x \in S \setminus (P_1(S) \cup P_2(S)). \end{cases}$$

Therefore, it will be important to further describe the elements in the subsets $P_1(S), P_2(S)$ and $P(S)$ in the semigroup $\mathcal{I}_n$ in order to obtain characterisation of Green's relation in $\mathcal{I}_n^a$.

Lemma 2.2. *In the symmetric inverse semigroup we have:*

i. $P_1(\mathcal{I}_n) = \{\alpha \in \mathcal{I}_n : \mathrm{im}(\alpha) \subseteq \mathrm{dom}(a)\}$;
ii. $P_2(\mathcal{I}_n) = \{\alpha \in \mathcal{I}_n : \mathrm{dom}(\alpha) \subseteq \mathrm{im}(\alpha)\}$;
iii. $P(\mathcal{I}_n) = \{\alpha \in \mathcal{I}_n : (\mathrm{dom}(\alpha) \cup \mathrm{im}(\alpha)) \subseteq \mathrm{dom}(a)\}$

Proof. i.

$$\begin{aligned} P_1(\mathcal{I}_n) &= \{\alpha \in \mathcal{I}_n : \alpha a \mathcal{R} \alpha\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{dom}(\alpha) = \mathrm{dom}(\alpha a)\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{dom}(\alpha) = (\mathrm{im}(\alpha) \cap \mathrm{dom}(a))\alpha^{-1}\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{im}(\alpha) = \mathrm{im}(\alpha) \cap \mathrm{dom}(a)\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{im}(\alpha) \subseteq \mathrm{dom}(a)\}. \end{aligned}$$

ii.

$$\begin{aligned} P_2(\mathcal{I}_n) &= \{\alpha \in \mathcal{I}_n : a\alpha\mathcal{L}\alpha\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{im}(\alpha) = \mathrm{im}(a\alpha\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{im}(\alpha) = (\mathrm{im}(a) \cap \mathrm{dom}(\alpha))\alpha\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{dom}(\alpha) = \mathrm{im}(a) \cap \mathrm{dom}(\alpha)\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{dom}(\alpha) \subseteq \mathrm{im}(a)\}. \end{aligned}$$

iii.

$$\begin{aligned} P(\mathcal{I}_n) &= P_1(\mathcal{I}_n) \cap P_2(\mathcal{I}_n) \\ &= \{\alpha \in \mathcal{I}_n : \alpha a\mathcal{R}\alpha \ \text{ and } \ a\alpha\mathcal{L}\alpha\} \\ &= \{\alpha \in \mathcal{I}_n : \mathrm{im}(\alpha) \subseteq \mathrm{dom}(a) \ \text{ and } \ \mathrm{dom}(\alpha) \subseteq \mathrm{dom}(\alpha)\} \\ &= \{\alpha \in \mathcal{I}_n : (\mathrm{dom}(\alpha) \cup \mathrm{im}(\alpha)) \subseteq \mathrm{dom}(a)\} \end{aligned}$$

□

Now, to characterise Green's relations in $\mathcal{I}_n^a$, we only need to apply Theorem 2.1, to the case when $S = \mathcal{I}_n^a$.

Theorem 2.3. *Let* $\alpha, \beta \in \mathcal{I}_n^a$. *Then,*

$$\textit{i. } \alpha\mathcal{R}^a\beta \Longleftrightarrow \begin{cases} \mathrm{dom}(\alpha) = \mathrm{dom}(\beta) & \text{if } \alpha, \beta \in P_1, \\ \\ \alpha = \beta & \text{if } \alpha, \beta \in \mathcal{I}_n \setminus P_1, \end{cases}$$

$$\textit{ii. } \alpha\mathcal{L}^a\beta \Longleftrightarrow \begin{cases} \mathrm{im}(\alpha) = \mathrm{im}(\beta) & \text{if } \alpha, \beta \in P_2, \\ \\ \alpha = \beta & \text{if } \alpha, \beta \in \mathcal{I}_n \setminus P_2, \end{cases}$$

$$\textit{iii. } \alpha\mathcal{H}^a\beta \Longleftrightarrow \begin{cases} \mathrm{dom}(\alpha) = \mathrm{dom}(\beta) & \text{if } \alpha, \beta \in P, \\ \\ \alpha = \beta & \text{if } \alpha, \beta \in \mathcal{I}_n \setminus P, \end{cases}$$

$$\textit{iv. } \alpha\mathcal{D}^a\beta \Longleftrightarrow \begin{cases} |\mathrm{im}(\alpha)| = |\mathrm{im}(\beta)| & \text{if } \alpha, \beta \in S \setminus P, \\ \\ \mathrm{dom}(\alpha) = \mathrm{dom}(\beta) & \text{if } \alpha, \beta \in P_1 \setminus P_2, \\ \\ \mathrm{im}(\alpha) = \mathrm{im}(\beta) & \text{if } \alpha, \beta \in P_2 \setminus P_1, \\ \\ \alpha = \beta & \text{if } \alpha, \beta \in \mathcal{I}_n \setminus (P_1 \cup P_2). \end{cases}$$

Proof. This is a combined effect of Theorem 2.1 and Lemma 2.2. □

3. Abundant Semigroups of $\mathcal{I}_n^a$

Recall that the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on a semiigroup S are defined, by the rule that two elements $a, b \in S$ are $\mathcal{L}^*$ and $\mathcal{R}^*$ related if and only if the two elements are respectively $\mathcal{L}^*$ and $\mathcal{R}^*$ relations in some larger semiroup containing S as a subsemigroup. These relations also have the following more transparent characterisation.

$$\mathcal{L}^*(S) = \{(a, b) : (\forall x, y \in S^1) ax = ay \Leftrightarrow bx = by\} \tag{4}$$

and

$$\mathcal{R}^*(S) = \{(a, b) : (\forall x, y \in S^1) xa = ya \Leftrightarrow xb = yb\}. \tag{5}$$

Our first result in this section characterises the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ in $\mathcal{I}_n^a$.

Theorem 3.1. *Let $\alpha, \beta \in \mathcal{I}_n^a$. Then*

(i) $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{I}_n^a)$ *if and only if* $\mathrm{im}(\alpha a) = \mathrm{im}(\beta a)$,

(ii) $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{I}_n^a)$ *if and only if* $\mathrm{dom}(a\alpha) = \mathrm{dom}(a\beta)$.

Proof. (i) Suppose that $\alpha, \beta \in \mathcal{L}^*(\mathcal{I}_n^a)$ and choose any idempotent ξ in $\mathcal{I}_n$ such that $\mathrm{dom}(\xi) = \mathrm{im}(\alpha a)$. Then,

$$\alpha *_a \xi = \alpha a \xi = \alpha a = \alpha *_a 1_{X_n}$$

if and only if

$$\beta *_a \xi = \beta a \xi = \beta a = \beta *_a 1_{X_n},$$

which implies that $\mathrm{im}\beta a \subseteq \mathrm{dom}(\xi) = \mathrm{im}\alpha a$.

Similarly, if we choose any idempotent η in $\mathcal{I}_n$ such that $\mathrm{dom}(\eta) = \mathrm{im}(\beta a)$. Then,

$$\beta *_a \eta = \beta a \eta = \beta a = \beta *_a 1_{X_n}$$

if and only if

$$\alpha *_a \eta = \alpha a \eta = \alpha a = \alpha *_a 1_{X_n},$$

which implies that $\mathrm{im}\alpha a \subseteq \mathrm{dom}(\eta) = \mathrm{im}(\beta a)$. Thus, we have $\mathrm{im}(\alpha a) = \mathrm{im}(\beta a)$.

Conversely, suppose that $\mathrm{im}(\alpha a) = \mathrm{im}(\beta a)$. Then, $(\alpha a, \beta a) \in \mathcal{L}^*(\mathcal{I}_n)$, and so, $(\alpha a)x = (\alpha a)y \iff (\beta a)x = (\beta a)y$, for all $x, y \in \mathcal{I}_n$. That is, $\alpha *_a x = \alpha *_a y \iff \beta *_a x = \beta *_a y$, for all $x, y \in \mathcal{I}_n$. Thus, $(\alpha, \beta) \in \mathcal{L}^*(\mathcal{I}_n^a)$.

(ii) Suppose that $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{I}_n^a)$ and choose any idempotent ξ in $\mathcal{I}_n$ such that $\mathrm{im}(\xi) = \mathrm{dom}(a\alpha)$. Then,

$$\xi *_a \alpha = \xi a \alpha = a\alpha = 1_{X_n} *_a \alpha$$

if and only if

$$\xi *_a \beta = \xi a \beta = a\beta = 1_{X_n} *_a \beta$$

which implies that $\mathrm{dom}(a\beta) \subseteq \mathrm{im}(\xi) = \mathrm{dom}(a\alpha)$.

Similarly, if we choose any idempotent η in $\mathcal{I}_n$ such that $\mathrm{im}(\eta) = \mathrm{dom}(a\beta)$. Then,

$$\eta *_a \beta = \eta \beta a = a\beta = 1_{X_n} *_a \beta$$

if and only if

$$\eta *_a \alpha = \eta a \alpha = a\alpha = 1_{X_n} *_a \alpha,$$

which implies that $\mathrm{dom}(a\alpha) \subseteq \mathrm{im}(\xi) = \mathrm{dom}(a\beta)$. Thus, we have $\mathrm{dom}(a\beta) = \mathrm{dom}(a\beta)$.

Conversely, Suppose that $\mathrm{dom}(a\alpha) = \mathrm{dom}(a\beta)$, then by definition, it implies that $(a\alpha, a\beta) \in \mathcal{R}^*(\mathcal{I}_n)$. And so, by definition of $\mathcal{R}^*$, we have $x(a\alpha) = y(a\alpha) \iff x(a\beta) = y(a\beta)$, for all $x, y \in \mathcal{I}_n$. That is, $x *_a \alpha = y *_a \alpha \iff x *_a \beta = y *_a \beta$, for all $x, y \in \mathcal{I}_n$. Thus, $(\alpha, \beta) \in \mathcal{R}^*(\mathcal{I}_n^a)$. □

Lemma 3.2. *Let $\alpha \in \mathcal{I}_n^a$. Then, for some idempotents $e, f \in E_a(\mathcal{I}_n^a)$, $(\alpha, e) \in \mathcal{L}^*(\mathcal{I}_n)$ and $(\alpha, f) \in \mathcal{R}^*(\mathcal{I}_n)$.*

Proof. Let $a = \begin{pmatrix} a_1 & \cdots & a_r \\ a_1 & \cdots & a_r \end{pmatrix}$. Since $\mathrm{im}(a) = \{a_1, a_2, \ldots, a_r\}$ and $\mathrm{im}(\alpha a) \subseteq \mathrm{im}(a)$, then suppose without loss of generality, that $\mathrm{im}(\alpha a) = \{a_1, \ldots, a_k\}$, where $1 \leq k \leq r$. Then, the map $e = \begin{pmatrix} a_1 & \cdots & a_{k-1} & a_k \\ a_1 & \cdots & a_{k-1} & a_k \end{pmatrix}$ is such that $ea = e$. Then, since e is an idempotent, that is, $e^2 = e *_a e = eae = ee = e$ and $\mathrm{im}(ea) = \mathrm{im}(e) = \{a_1, \ldots, a_k\} = \mathrm{im}(\alpha a)$, we have that $e \in E_a(\mathcal{I}_n^a)$ and $(\alpha, e) \in \mathcal{L}^*(\mathcal{I}_n^a)$.

Similarly, if we suppose, without loss of generality, that $\mathrm{dom}(\alpha a) = \{a_1, \ldots, a_k\}$, where $1 \leq k \leq r$. Then, the map $f = \begin{pmatrix} a_1 & \cdots & a_{k-1} & a_k \\ a_1 & \cdots & a_{k-1} & a_k \end{pmatrix}$ is such that $fa = af = f$. Thus, $f \in E_a(\mathcal{I}_n^a)$ and $(\alpha, f) \in \mathcal{R}^*(\mathcal{I}_n^a)$. □

The above lemma shows that every $\mathcal{L}^*$ and every $\mathcal{R}^*$ in variant of $\mathcal{I}_n^a$ contains an idempotent.

Thus, we have the following result;

Theorem 3.3. *The variant semigroup $\mathcal{I}_n^a$ is abundant.*

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