

**A FORWARD-BACKWARD SPLITTING ALGORITHM
FOR QUASI-BREGMAN NONEXPANSIVE MAPPING,
EQUILIBRIUM PROBLEMS AND ACCRETIVE
OPERATORS**

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*Dedicated to professor Charles Ejike Chidume
on the occasion of his 70th birth day*

ABSTRACT. In this paper, we study a forward-backward splitting algorithm for fixed points of a quasi-Bregman nonexpansive mapping, solution of equilibrium problem and zero points of the sum of families of accretive operators and α_i -inverse-strongly accretive operators. We proved weak convergence of the sequences generated by this algorithm in reflexive Banach space. Our result extends and improves important recent results announced by many authors.

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1. INTRODUCTION

Let E be a real Banach space and C a nonempty closed convex subset of E . The normalized duality map from E to 2^{E^*} (E^* is the dual space of E) denoted by J is defined by

$$J(x) = \{f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\|\}, \forall x \in E.$$

Let $A : \text{Dom}(A) \subset E \rightarrow 2^E$ be a set-valued operator with $\text{Dom}(A) = \{z \in E : Az \neq \emptyset\}$ and the $\text{Ran}(A) = \cup\{Az : z \in D(A)\}$. A is said to be accretive if for all $\lambda > 0$ and for each $x, y \in D(A)$.

$$\|\lambda(x' - y') + (x - y)\| \geq \|x - y\|, \forall x' \in Ax, y' \in Ay.$$

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The accretive operators were introduced independently in 1967 by Browder [10] and Kato [21]. An early fundamental result in the theory of accretive operators, due to Browder, states that the initial value problem

$$Au + \frac{dv}{dt} = 0, \quad v(0) = v_0$$

is solvable if A is locally Lipschitz and accretive on E ; see Browder [11] and the references therein. It follows from Kato [21] that A is accretive if and only if, for $x, y \in \text{Dom}(A)$, there exists $j(x - y) \in J(x - y)$ such that

$\langle u - v, j(x - y) \rangle \geq 0$, where $u \in Ax$ and $v \in Ay$. A is α -inverse strongly accretive if, for $j(x - y) \in J(x - y)$

$$\langle Ax - Ay, j(x - y) \rangle \geq \alpha \|Ax - Ay\|^2 \quad (1)$$

Let I denote the identity operator on E . An accretive operator A is said to be m -accretive if the range of $(I + \lambda A)$ is the whole space E for all $\lambda > 0$. A fundamental problem is to find a zero of accretive operator A in a real Banach space E : find $x \in E$ such that $0 \in Ax$. This problem includes, as special cases, nonsmooth convex optimization and convex-concave saddle point problems. Hence this problem has many applications in scientific fields such as image processing, machine learning and signal processing. If $A = \nabla f$, the gradient of a differentiable convex function f , solving the problem is done via the following recursion: $x_0 \in E$ and $x_{n+1} = (I - \lambda_n \nabla f)x_n$, $n \geq 0$, where $\{\lambda_n\}$ is a positive number sequence. The above scheme is called steepest descent method.

If A is a monotone operator (i.e accretive on a Hilbert space), the above inclusion was investigated by Rockafellar [29] which was recognized as Rockafellar's proximal point algorithm: $x_0 \in E$, $x_{n+1} = (I + \lambda_n A)^{-1}x_n$, $\forall n \geq 0$, where $(I + \lambda_n A)^{-1}$ is called the resolvent of A . Rockafellar proved weak convergence of the sequence $\{x_n\}$ when the regularization sequence $\{\lambda_n\}$ is bounded away from zero. In many problems the operator A can be written as the sum of two accretive operators, i.e, $A = M + N$.

Many authors are constructing algorithms for solving fixed point problems for nonlinear mappings using Bregman's technique (see, e.g [4, 14, 23] and the references therein).

Splitting method has recently received much attention because many nonlinear problems arising in areas such as image processing, machine learning and signal processing are mathematically modeled as

nonlinear operator equation. Recently, several authors have extensively investigated zero points of monotone operators using splitting technique; see [16, 34].

Numerous problems in optimization, economics and physics can be reduced to finding solutions of some equilibrium problem. Various methods have been studied for solutions of some equilibrium problems, see for example [2, 3, 8, 17, 18, 19, 30, 32] and the references contained therein.

Ugwnnadi et al. [33] proved a new strong convergence theorem for a finite family of closed quasi-Bregman strictly pseudocontractive mappings and a system of equilibrium problems in a real reflexive Banach space.

Motivated and inspired by above mentioned results, we study a forward-backward splitting algorithm for finding a zero point of sum of finite family of m -accretive operators and α -inverse strongly accretive operators, solution of equilibrium problems and fixed points problems of quasi-Bregman nonexpansive mappings.

2. PRELIMINARIES

Throughout this paper, we shall assume $f : E \rightarrow (-\infty, +\infty]$ is a proper, lower semi-continuous and convex function. We denote by $\text{dom} f := \{u \in E : f(u) < +\infty\}$ the domain of f . Let $u \in \text{int}(\text{dom}(f))$; the subdifferential of f at u is the convex set defined by

$$\partial f(u) = \{u^* \in E^* : f(u) + \langle u^*, y - x \rangle \leq f(v), \forall v \in E\},$$

where the Fenchel conjugate of f is the function $f^* : E^* \rightarrow (-\infty, +\infty]$ defined by

$$f^*(u^*) = \sup\{\langle u^*, u \rangle - f(u) : u \in E\}.$$

It is known that the Young-Fenchel inequality holds:

$$\langle u^*, u \rangle \leq f(u) + f^*(u^*), \quad \forall u \in E.$$

A function f on E is coercive [20] if the sublevel sets of f are bounded; equivalently,

$$\lim_{\|u\| \rightarrow +\infty} f(u) = +\infty.$$

A function f on E is said to be strongly coercive [36] if

$$\lim_{\|u\| \rightarrow +\infty} \frac{f(u)}{\|u\|} = +\infty.$$

For any $u \in \text{int}(\text{dom}(f))$ and $v \in E$, the right-hand derivative of f at u in the direction v is defined by

$$f^\circ(u, v) := \lim_{t \rightarrow 0^+} \frac{f(u + tv) - f(u)}{t}.$$

The function f is said to be Gâteaux differentiable at u if $\lim_{t \rightarrow 0} \frac{f(u+tv) - f(u)}{t}$ exists for any v . In this case, the gradient of f at u is the function $\nabla f(u) : E \rightarrow (-\infty, +\infty]$ defined by $\langle \nabla f(u), v \rangle = f^\circ(u, v)$ for any $v \in E$. The function f is said to be Gâteaux differentiable if it is Gâteaux differentiable for any $u \in \text{int}(\text{dom}(f))$. The function f is said to be Fréchet differentiable at u if this limit is attained uniformly in v , $\|v\| = 1$. The map f is said to be uniformly Fréchet differentiable on a subset C of E if the limit is attained uniformly for $u \in C$ and $\|v\| = 1$. It is well known that if f is Gâteaux differentiable (resp. Fréchet differentiable) on $\text{int}(\text{dom}(f))$, then f is continuous and its Gâteaux derivative ∇f is norm-to-weak* continuous (resp. norm-to-norm continuous) on $\text{int}(\text{dom}(f))$ (see also [1, 9]). We will need the following results.

Lemma 1: [26] If $f : E \rightarrow \mathbb{R}$ is uniformly Fréchet differentiable and bounded on bounded subsets of E , then ∇f is uniformly continuous on bounded subsets of E from the strong topology of E to the strong topology of E^* .

Remark 1: If E is a reflexive Banach space, then we have the following results:

- (i) f is essentially smooth if and only if f^* is essentially strictly convex (see [7] Theorem 5.4);
- (ii) $(\partial f)^{-1} = \partial f^*$ (see [9])
- (iii) f is Legendre if and only if f^* is Legendre (see [7], Corollary 5.5)
- (iv) If f is Legendre, then ∇f is a bijection satisfying $\nabla f = (\nabla f^*)^{-1}$, $\text{ran } \nabla f = \text{dom } \nabla(f^*) = \text{int}(\text{dom}(f^*))$ and $\text{ran } \nabla f^* = \text{dom}(f) = \text{int}(\text{dom}(f))$ (see [7], Theorem 5.10).

Examples of Legendre functions were given in [7, 5]. One important and interesting Legendre function is $\frac{1}{p}\|\cdot\|^p$ ($1 < p < \infty$) when E is a smooth and strictly convex Banach space. In this case the gradient ∇f of f is coincident with the generalized duality mapping of E , i.e. $\nabla f = J_p$ ($1 < p < \infty$). In particular, a Hilbert spaces.

Let $f : E \rightarrow (-\infty, +\infty]$ be a convex and Gateaux differentiable function. The function $D_f : \text{dom}f \times \text{intdom}f \rightarrow (-\infty, +\infty]$, defined as follows:

$$D_f(u, v) := f(u) - f(v) - \langle \nabla f(v), u - v \rangle, \quad (2)$$

is called the Bregman distance with respect to f (see [15]). It is obvious from the definition of D_f that

$$D_f(z, u) = D_f(z, v) + D_f(v, u) + \langle \nabla f(v) - \nabla f(u), z - v \rangle. \quad (3)$$

A point $p \in C$ is said to be an asymptotic fixed point of a map T , if there exists a sequence $\{x_n\}$ in C which converges weakly to p such that $\lim_{n \rightarrow +\infty} \|x_n - Tx_n\| = 0$. We denote by $\hat{F}(T)$ the set of asymptotic fixed points of T . A point $p \in C$ is said to be strong asymptotic fixed point of a map T , if there exists a sequence $\{x_n\}$ in C which converges strongly to p such that $\lim_{n \rightarrow +\infty} \|x_n - Tx_n\| = 0$. We denote by $\tilde{F}(T)$ the set of strong asymptotic fixed points of T . A map $T : C \rightarrow C$ is called quasi-Bregman nonexpansive if $F(T) \neq \emptyset$ and $D_f(p, Tx) \leq D_f(p, x)$ for all $x \in C$ and $p \in F(T)$.

Recall that the Bregman projection [12] of $u \in \text{int}(\text{dom}(f))$ onto nonempty, closed and convex set $C \subset \text{dom}(f)$ is the unique vector $\Pi_C(x) \in C$ satisfying

$$D_f(\Pi_C(x), x) = \inf\{D_f(y, x) : y \in C\}.$$

Concerning the Bregman projection, the following are well known.

Lemma 2: [14] Let C be a nonempty, closed and convex subset of a reflexive Banach space E . Let $f : E \rightarrow \mathbb{R}$ be a Gateaux differentiable and totally convex function and let $u \in E$. Then

- (a) $z = \Pi_C(x)$ if and only if $\langle \nabla f(x) - \nabla f(z), y - z \rangle \leq 0$, $\forall y \in C$;
- (b) $D_f(y, \Pi_C(x)) + D_f(\Pi_C(x), x) \leq D_f(y, x)$, $\forall x \in E, y \in C$.

Lemma 3: [25] Let E be a Banach space and $f : E \rightarrow \mathbb{R}$ be a Gateaux differentiable function which is uniformly convex on

bounded subset of E . Let $\{x_n\}_{n \in \mathbb{N}}$ and $\{y_n\}_{n \in \mathbb{N}}$ be bounded sequences in E . Then

$$\lim_{n \rightarrow \infty} D_f(x_n, y_n) = 0 \quad \text{if and only if} \quad \lim_{n \rightarrow \infty} \|x_n - y_n\| = 0.$$

Lemma 4: [27] Let $f : E \rightarrow \mathbb{R}$ be Gâteaux differentiable and totally convex function. If $x_0 \in E$ and the sequence $\{D_f(x_n, x_0)\}$ is bounded, then the sequence $\{x_n\}$ is bounded too.

The following definition is slightly different from that in Butnariu and Iusem [13].

Definition 2:[22] Let E be a Banach space. A function $f : E \rightarrow \mathbb{R}$ is said to be a Bregman function if the following conditions are satisfied:

- (i) f is continuous, strictly convex and Gâteaux differentiable;
- (ii) the set $\{y \in E : D_f(x, y) \leq r\}$ is bounded for all $x \in E$ and $r > 0$.

The following lemma follows from Butnariu and Iusem [13] and Zălinescu [36].

Lemma 5: Let E be a reflexive Banach space and $f : E \rightarrow \mathbb{R}$ be a strongly coercive Bregman function. Then

- (i) $\nabla f : E \rightarrow E^*$ is one-to-one, onto and norm-to-weak* continuous;
- (ii) $\langle x - y, \nabla f(x) - \nabla f(y) \rangle = 0$ if and only if $x = y$;
- (iii) $\{x \in E : D_f(x, y) \leq r\}$ is bounded for all $y \in E$ and $r > 0$;
- (iv) $\text{dom } f^* = E^*$, f^* is Gâteaux differentiable and $\nabla f^* = (\nabla f)^{-1}$.

The following two results are well known; see [36]

Theorem 1: Let E be a reflexive Banach space and let $f : E \rightarrow \mathbb{R}$ be a convex function which is bounded on bounded subsets of E . Then the following assertions are equivalent:

- (1) f is strongly coercive and uniformly convex on bounded subsets of E ;
- (2) $\text{dom } f^* = E^*$, f^* is bounded on bounded subsets and uniformly smooth on bounded subsets of E^* ;
- (3) $\text{dom } f^* = E^*$, f^* is Frechet differentiable and ∇f^* is uniformly norm-to-norm continuous on bounded subsets of E^* .

Theorem 2: Let E be a reflexive Banach space and let $f : E \rightarrow \mathbb{R}$ be a continuous convex function which is strongly coercive. Then the following assertions are equivalent:

- (1) f is bounded on bounded subsets and uniformly smooth on bounded subsets of E ;

- (2) f^* is Frechet differentiable and f^* is uniformly norm-to-norm continuous on bounded subsets of E^* ;
- (3) $\text{dom} f^* = E^*$, f^* is strongly coercive and uniformly convex on bounded subsets of E^* .

Lemma 6: [22] (see also [25]) Let E be a reflexive Banach space, let $f : E \rightarrow \mathbb{R}$ be a strongly coercive Bregman function and let V_f be the function defined by

$$V_f(x, x^*) = f(x) - \langle x, x^* \rangle + f^*(x^*), \quad x \in E, \quad x^* \in E^*.$$

Then the following assertions hold:

- (1) $D_f(x, \nabla f^*(x^*)) = V_f(x, x^*)$ for all $x \in E$ and $x^* \in E^*$.
- (2) $V_f(x, x^*) + \langle \nabla f^*(x^*) - x, y^* \rangle \leq V_f(x, x^* + y^*)$ for all $x \in E$ and $x^*, y^* \in E^*$.

Lemma 7: [25] Let E be a Banach space and $f : E \rightarrow \mathbb{R}$ be a convex function which is uniformly convex on bounded subsets of E . Let $r > 0$ be a constant and ρ_r be the gauge of uniform convexity of f . Then

- (i) For any $x, y \in B_r$ and $\alpha \in (0, 1)$,

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y) - \alpha(1 - \alpha)\rho_r(\|x - y\|).$$

- (ii) For any $x, y \in B_r$,

$$\rho_r(\|x - y\|) \leq D_f(x, y)$$

- (iii) If, in addition, f is bounded on bounded subsets and uniformly convex on bounded subsets of E then, for any $x \in E, y^*, z^* \in B_r^*$ and $\alpha \in (0, 1)$,

$$V_f(x, \alpha y^* + (1 - \alpha)z^*) \leq \alpha V_f(x, y^*) + (1 - \alpha)V_f(x, z^*) - \alpha(1 - \alpha)\rho_r^*(\|y^* - z^*\|).$$

In order to solve equilibrium problems, we shall assume that the bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfies the following conditions [8] :

- (A1) $G(x, x) = 0, \forall x \in C$;
- (A2) G is monotone, i.e., $G(x, y) + G(y, x) \leq 0 \quad \forall x, y \in C$;
- (A3) $\limsup_{t \rightarrow 0} G(tz + (1 - t)x, y) \leq G(x, y), \quad \forall x, y, z \in C$;
- (A4) the function $y \mapsto G(x, y)$ is convex and lower semi-continuous.

For $r > 0$ the resolvent of a bifunction G [35] is the operator $\text{Res}_G^f : E \rightarrow C$ defined by

$$\text{Res}_G^f(x) = \left\{ z \in C : G(z, y) + \frac{1}{r} \langle \nabla f(z) - \nabla f(x), y - z \rangle \geq 0 \quad \forall y \in C \right\}. \quad (4)$$

From Lemma 1, in [28] if $f : (-\infty, +\infty] \rightarrow \mathbb{R}$ is strongly coercive and Gâteaux differentiable function, and G satisfies condition

(A1) – (A4), then $\text{dom}(\text{Res}_G^f) = E$.

The following lemma gives some characterization of the resolvent Res_G^f .

Lemma 8: [28] Let E be a real reflexive Banach space and C be a nonempty closed convex subset of E . Let $f : E \rightarrow (-\infty, +\infty]$ be a function. If the bifunction $G : C \times C \rightarrow \mathbb{R}$ satisfies the condition (A1)-(A4), then the following hold:

- (i) Res_G^f is singled-valued;
- (ii) Res_G^f is a Bregman firmly nonexpansive operator
- (iii) $F(\text{Res}_G^f) = \text{EP}(G)$;
- (iv) $\text{EP}(G)$ is closes and convex subset of C ; for all $x \in E$ and for all $q \in F(\text{Res}_G^f)$

$$D_f(q, \text{Res}_G^f(x)) + D_f(\text{Res}_G^f(x), x) \leq D_f(q, x). \quad (5)$$

1. MAIN RESULT

It is easy to see that for each $i = 1, 2, \dots, k$

$$(N_i + M_i)^{-1}(0) = \text{Fix}((I + \lambda M_i)^{-1}(I - \lambda N_i)).$$

Lemma 9: Let $M_i : D(M) \subset E \rightarrow 2^E$, $i = 1, 2, \dots, k$ and $N_i : C \rightarrow E$, $i = 1, 2, \dots, k$ be finite family m -accretive operators and α_i -inverse strongly accretive respectively. Let $\pi = ((I + \lambda M_k)^{-1}(I - \lambda N_k)) \circ ((I + \lambda M_{k-1})^{-1}(I - \lambda N_{k-1})) \circ \dots \circ ((I + \lambda M_1)^{-1}(I - \lambda N_1))$. Then π is nonexpansive.

Proof:

To show that $\pi = ((I + \lambda M_k)^{-1}(I - \lambda N_k)) \circ ((I + \lambda M_{k-1})^{-1}(I - \lambda N_{k-1})) \circ \dots \circ ((I + \lambda M_1)^{-1}(I - \lambda N_1))$ is nonexpansive, consider the mapping $(I + \lambda M_i)^{-1}(I - \lambda N_i)$, for any $i \in \{1, 2, \dots, k\}$ we have.

$$\begin{aligned} & \| (I + \lambda M_i)^{-1}(I - \lambda N_i)x - (I + \lambda M_i)^{-1}(I - \lambda N_i)y \|^q \\ & \leq \| (I - \lambda N_i)x - (I - \lambda N_i)y \|^q \\ & = \| (x - y) - \lambda(N_i x - N_i y) \|^q \\ & = \| x - y \|^q - q\lambda \langle N_i x - N_i y, j(x - y) \rangle + d_q \lambda^q \| N_i x - N_i y \|^q \\ & \leq \| x - y \|^q - q\lambda \alpha \| N_i x - N_i y \|^q + d_q \lambda^q \| N_i x - N_i y \|^q \\ & = \| x - y \|^q - \lambda(q\alpha - d_q \lambda^{q-1}) \| N_i x - N_i y \|^q \\ & \leq \| x - y \|^q. \end{aligned}$$

Hence, the mapping $(I + \lambda M_i)^{-1}(I - \lambda N_i)$ is nonexpansive for any $i \in \{1, 2, \dots, k\}$. This implies that $\pi = ((I + \lambda M_k)^{-1}(I - \lambda N_k)) \circ ((I + \lambda M_{k-1})^{-1}(I - \lambda N_{k-1})) \circ \dots \circ ((I + \lambda M_1)^{-1}(I - \lambda N_1))$ is also nonexpansive as a composition of finite nonexpansive mappings.

We assume that the nonexpansive mapping π above is also Bregman nonexpansive with respect to f in the sequel.

Theorem 3: Let E be q -uniformly smooth space. C a nonempty closed convex subset of E . Let $f : E \rightarrow \mathbb{R}$ be a strongly coercive, Legendre function, which is bounded, uniformly Fréchet differentiable and totally convex on bounded subset of E . Let G be a bifunction from $C \times C$ to \mathbb{R} satisfying (A1)-(A4). Let M_i and N_i be as in Lemma 9 above. Let $T : C \rightarrow C$ be a quasi-Bregman nonexpansive mapping such that $\Omega := F(T) \cap EP(G) \cap (\cap_{i=1}^k (N_i + M_i)^{-1}(0)) \neq \emptyset$. Assume $(I - T)$ is demiclosed at the origin and let $\{x_n\}$ be a sequence generated by

$$\begin{cases} x_0 \in C, \\ z_n = \nabla f^*(\beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n)), \\ G(z_n, y) + \frac{1}{r_n} \langle y - z_n, \nabla f(z_n) - \nabla f(u_n) \rangle \geq 0 \quad \forall y \in C, \\ w_n = \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(\pi u_n)), \\ x_{n+1} = \Pi_C w_n, \quad n \geq 0. \end{cases} \quad (6)$$

Assume $Fix(\pi) = \cap_{i=1}^k (N_i + M_i)^{-1}(0)$. Let $\alpha = \min_{1 \leq i \leq k} \{\alpha_i\}$, $\lambda > 0$, with $\lambda \in (0, (\frac{q\alpha}{d_q})^{\frac{1}{q-1}})$ and $\{\beta_n\}, \{\alpha_n\} \subseteq [a, b]$ with $0 < a < b < 1$, let $r_n \in (0, 1)$. Then $\{x_n\}$ converges weakly to some point in Ω .

Proof:

Let $p \in \Omega$. Then we have

$$\begin{aligned}
D_f(p, z_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n))) \\
&= V_f(p, \beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n)) \\
&= f(p) - \langle p, \beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n) \rangle \\
&\quad + f^*(\beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n)) \\
&= \beta_n f(p) + (1 - \beta_n) f(p) - \beta_n \langle p, \nabla f(Tx_n) \rangle \\
&\quad - (1 - \beta_n) \langle p, \nabla f(x_n) \rangle \\
&\quad + \beta_n f^*(\nabla f(Tx_n)) + (1 - \beta_n) f^*(\nabla f(x_n)) \\
&\leq \beta_n (f(p) - \langle p, \nabla f(Tx_n) \rangle + f^*(\nabla f(Tx_n))) \\
&\quad + (1 - \beta_n) (f(p) - \langle p, \nabla f(x_n) \rangle + f^*(\nabla f(x_n))) \\
&= \beta_n V_f(p, \nabla f(Tx_n)) + (1 - \beta_n) V_f(p, \nabla f(x_n)) \\
&= \beta_n D_f(p, Tx_n) + (1 - \beta_n) D_f(p, x_n) \\
&\leq \beta_n D_f(p, x_n) + (1 - \beta_n) D_f(p, x_n) \\
&\leq D_f(p, x_n).
\end{aligned}$$

Since from (6) $u_n = Res_G^f(z_n)$ and using (5), we get

$$\begin{aligned}
D_f(p, u_n) &\leq D_f(p, z_n) - D_f(u_n, z_n) \\
&\leq D_f(p, z_n) \\
&= D_f(p, x_n).
\end{aligned}$$

But

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq D_f(p, w_n) = D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(\pi u_n))) \\
&\leq \alpha_n D_f(p, x_n) + (1 - \alpha_n) D_f(p, \pi u_n) \\
&\leq \alpha_n D_f(p, x_n) + (1 - \alpha_n) D_f(p, u_n) \\
&\leq D_f(p, x_n) + (1 - \alpha_n) D_f(p, x_n) \\
&= D_f(p, x_n),
\end{aligned}$$

which implies that $\lim_{n \rightarrow \infty} D_f(p, x_n)$ exists and consequently $\{D_f(p, x_n)\}$ is bounded. Furthermore, by Lemma 4 $\{x_n\}$ is bounded, hence $\{w_n\}$, $\{u_n\}$ and $\{z_n\}$ are bounded.

Let $\rho_r^* : E \rightarrow \mathbb{R}$ be the guange function of uniform convexity of the conjugate function f^* , $r > 0$ (chosen appropriately). By Lemmas 6 and 7, we obtain

$$\begin{aligned}
D_f(p, z_n) &= D_f(p, \nabla f^*(\beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n))) \\
&= V_f(p, \beta_n \nabla f(Tx_n) + (1 - \beta_n) \nabla f(x_n))
\end{aligned}$$

$$\begin{aligned}
&\leq \beta_n V_f(p, \nabla f(Tx_n)) + (1 - \beta_n) V_f(p, \nabla f(x_n)) \\
&\quad - \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|) \\
&= \beta_n D_f(p, Tx_n) + (1 - \beta_n) D_f(p, x_n) \\
&\quad - \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|) \\
&\leq \beta_n D_f(p, x_n) + (1 - \beta_n) D_f(p, x_n) \\
&\quad - \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|) \\
&= D_f(p, x_n) - \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|) \quad (7) \\
&\leq D_f(p, x_n).
\end{aligned}$$

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq D_f(p, w_n) = D_f(p, \nabla f^*(\alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(\pi u_n))) \\
&= V_f(p, \alpha_n \nabla f(x_n) + (1 - \alpha_n) \nabla f(\pi u_n))
\end{aligned}$$

$$\begin{aligned}
&\leq \alpha_n V_f(p, \nabla f(x_n)) + (1 - \alpha_n) V_f(p, \nabla f(\pi u_n)) \\
&\quad - \alpha_n(1 - \alpha_n) \rho_r^*(\|\nabla f(x_n) - \nabla f(\pi u_n)\|) \\
&= \alpha_n D_f(p, x_n) + (1 - \alpha_n) D_f(p, \pi u_n) \\
&\quad - \alpha_n(1 - \alpha_n) \rho_r^*(\|\nabla f(x_n) - \nabla f(\pi u_n)\|) \\
&\leq \alpha_n D_f(p, x_n) + (1 - \alpha_n) D_f(p, u_n) \quad (8) \\
&\quad - \alpha_n(1 - \alpha_n) \rho_r^*(\|\nabla f(x_n) - \nabla f(\pi u_n)\|) \\
&= D_f(p, x_n) - \alpha_n(1 - \alpha_n) \rho_r^*(\|\nabla f(x_n) - \nabla f(\pi u_n)\|). \quad (9)
\end{aligned}$$

From (8) we have

$$\alpha_n(1 - \alpha_n) \rho_r^*(\|\nabla f(x_n) - \nabla f(\pi u_n)\|) \leq D_f(p, x_n) - D_f(p, x_{n+1})$$

and so using the property of ρ_r^* we have

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n) - \nabla f(\pi u_n)\| = 0, \quad (10)$$

since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^* , we have

$$\lim_{n \rightarrow \infty} \|x_n - \pi u_n\| = 0. \quad (11)$$

From (6) and (7) we have

$$\begin{aligned}
D_f(p, x_{n+1}) &\leq \alpha_n D_f(p, x_n) + (1 - \alpha_n) [D_f(p, x_n) \\
&\quad - \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|)] \\
&= D_f(p, x_n) - (1 - \alpha_n) \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|),
\end{aligned}$$

which implies

$$(1 - \alpha_n) \beta_n(1 - \beta_n) \rho_r^*(\|\nabla f(Tx_n) - \nabla f(x_n)\|) \leq D_f(p, x_n) - D_f(p, x_{n+1})$$

and so

$$\lim_{n \rightarrow \infty} \|\nabla f(Tx_n) - \nabla f(x_n)\| = 0. \quad (12)$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^* , we have

$$\lim_{n \rightarrow \infty} \|Tx_n - x_n\| = 0. \quad (13)$$

From (6) we have

$$\|\nabla f(x_n) - \nabla f(z_n)\| = \beta_n \|\nabla f(x_n) - \nabla f(Tx_n)\|,$$

which implies from (11) that

$$\lim_{n \rightarrow \infty} \|\nabla f(x_n) - \nabla f(z_n)\| = 0. \quad (14)$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^* , we have

$$\lim_{n \rightarrow \infty} \|x_n - z_n\| = 0. \quad (15)$$

From (6) and Lemma 2 (b) we have

$$\begin{aligned} \lim_{n \rightarrow \infty} D_f(z_n, u_n) &= \lim_{n \rightarrow \infty} D_f(z_n, Res_G^f(z_n)) \\ &\leq \lim_{n \rightarrow \infty} (D_f(z_n, z_n) - D_f(Res_G^f(z_n), z_n)) \\ &\leq 0, \end{aligned}$$

by Lemma 3 we have

$$\lim_{n \rightarrow \infty} \|z_n - u_n\| = 0. \quad (16)$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^* , we have

$$\lim_{n \rightarrow \infty} \|\nabla f(z_n) - \nabla f(u_n)\| = 0. \quad (17)$$

But

$$\|u_n - x_n\| \leq \|u_n - z_n\| + \|z_n - x_n\|,$$

which from (14) and (15) implies

$$\lim_{n \rightarrow \infty} \|u_n - x_n\| = 0. \quad (18)$$

Also,

$$\begin{aligned} \|x_n - \pi x_n\| &\leq \|x_n - \pi u_n\| + \|\pi u_n - \pi x_n\| \\ &\leq \|x_n - \pi u_n\| + \|u_n - x_n\|, \end{aligned}$$

which from (10) and (17) implies

$$\lim_{n \rightarrow \infty} \|x_n - \pi x_n\| = 0. \quad (19)$$

Again, from (6) we have

$$\|\nabla f(\pi u_n) - \nabla f(w_n)\| = \beta_n \|\nabla f(\pi u_n) - \nabla f(x_n)\|,$$

which also implies from (9)

$$\lim_{n \rightarrow \infty} \|\nabla f(\pi u_n) - \nabla f(w_n)\| = 0.$$

Since ∇f^* is uniformly norm-to-norm continuous on bounded subset of E^* , we have

$$\lim_{n \rightarrow \infty} \|\pi u_n - w_n\| = 0.$$

Also

$$\|u_n - \pi u_n\| \leq \|u_n - x_n\| + \|x_n - \pi u_n\|,$$

which implies

$$\lim_{n \rightarrow \infty} \|u_n - \pi u_n\| = 0.$$

Again,

$$\|w_n - u_n\| \leq \|w_n - \pi u_n\| + \|\pi u_n - u_n\|,$$

which implies

$$\lim_{n \rightarrow \infty} \|w_n - u_n\| = 0.$$

We also have,

$$\|w_n - x_n\| \leq \|w_n - u_n\| + \|u_n - x_n\|,$$

which implies

$$\lim_{n \rightarrow \infty} \|w_n - x_n\| = 0.$$

Since $\{x_n\}$ is bounded and E is reflexive, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p \in C$. From (13) and the assumption $(I - T)$ is demiclosed at the origin, we have $p \in F(T)$, also from (19) and the fact that $(I - \pi)$ is demiclosed at the origin, we have $p \in \cap_{i=1}^k (N_i + M_i)^{-1}(0)$.

Next we show that $p \in EP(G)$, from (6) we have

$$G(z_n, y) + \frac{1}{r_n} \langle y - z_n, \nabla f(x_n) - \nabla f(u_n) \rangle \geq 0 \quad \forall y \in C.$$

By applying (A2) we have for each $n \geq 1$

$$\frac{1}{r_n} \langle y - x_n, \nabla f(z_n) - \nabla f(u_n) \rangle \geq -G(z_n, y) \geq G(y, z_n) \quad \forall y \in C.$$

By (A4), (16) and $x_{n_i} \rightharpoonup p$ as $n \rightarrow \infty$, we have

$$G(y, z_{n_i}) \leq 0 \quad \forall y \in C, \text{ which implies}$$

$$G(y, p) \leq 0 \quad \forall y \in C.$$

Let $v_t = tv + (1 - t)p$ for $t \in (0, 1)$ and $v \in C$. This yields $G(v, p) \leq 0$. It follows from (A1) and (A4) that

$$0 = G(v_t, v_t) \leq tG(v_t, v) + (1 - t)G(v_t, p) \leq tG(v_t, v).$$

This implies

$$0 \leq G(v_t, v).$$

From condition (A3), we obtain

$$G(p, v) \geq 0, \quad \forall v \in C.$$

This implies that $p \in EP(G)$. Hence we have $p \in \Omega := F(T) \cap EP(G) \cap (\cap_{i=1}^k (N_i + M_i)^{-1}(0))$.

2. NUMERICAL EXAMPLE

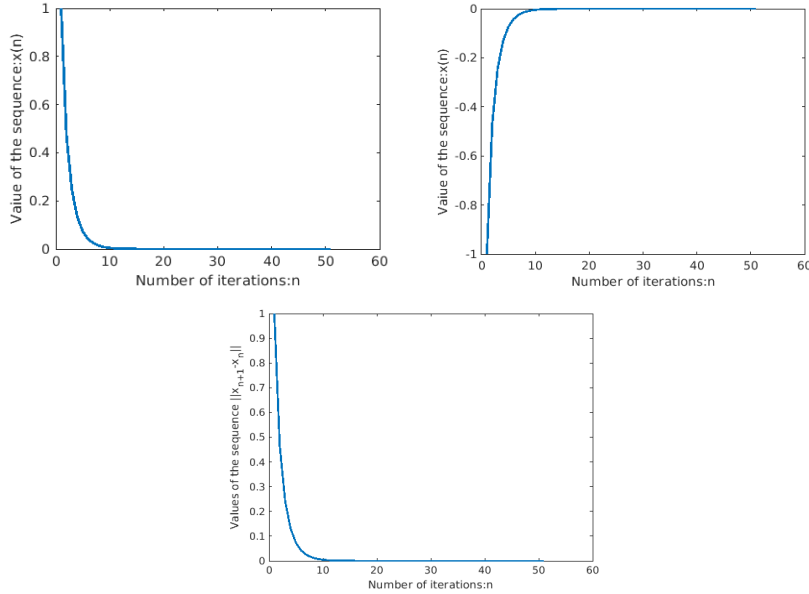
We give the following nemerical example to justify Theorem 3

Example : Let $E = \mathbb{R}$, $C = [-1, 1]$ and $G : C \times C \rightarrow \mathbb{R}$ be a bifunction defined by $G(x, y) = y^2 + yx - 2x^2$, $\forall x, y \in C$. Let $M_i : \mathbb{R} \rightarrow \mathbb{R}$ $i = 1, 2, 3$ defined by $M_1x = 2x$, $M_2x = 4x$, $M_3x = 6x$ $\forall x \in \mathbb{R}$. Let the mapping $N_i : C \rightarrow \mathbb{R}$ $i = 1, 2, 3$ be defined by $N_1x = \frac{x}{2}$, $N_2x = \frac{x}{4}$, $N_3x = \frac{x}{6}$ $\forall x \in \mathbb{R}$. Let $T : C \rightarrow C$ be defined by $Tx = x$. let $f(x) = \frac{2}{3}x^2$, $\nabla f(x) = \frac{4}{3}x$, since $f^*(x^*) = \sup\{\langle x^*, x \rangle - f(x) : x \in \mathbb{R}\}$, then $f^*(z) = \frac{3}{8}z^2$ and $\nabla f^*(z) = \frac{3}{4}z$. Clearly G is a bifunction satisfying A1 – A4, M_i is m-accretive operator, N_i is $\frac{1}{2}$ –inverse strongly accretive and T is quasi bregman nonexpansive mapping. From the scheme we obtain the following

$$\begin{cases} z_n = x_n \\ u_n = z_n \\ w_n = \frac{n}{2n+1}x_n + (1 - \frac{n}{2n+1})\frac{77}{384}u_n \\ x_{n+1} = \begin{cases} -1, & \text{if } x < -1 \\ 1, & \text{if } x > 1 \\ w_n, & \text{otherwise} \end{cases} \end{cases} \quad (20)$$

where $\beta_n = \frac{n+1}{4n}$, $\alpha_n = \frac{n}{2n+1}$, $r_n = 1$ and $\lambda = 1$. Then $\{x_n\}$ converges to $0 \in \Omega = \{0\}$.

Next, using Matlab software we have the following figures which shows that the sequence $\{x_n\}$ converges to 0.



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