A GENERALIZATION OF A STARLIKENESS CONDITION

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In this paper, we prove a generalization of a well known starlike-
ness condition in the unit disk using a method of linear combi-
nations of certain geometric expressions. Some interesting corol-
laries are mentioned.

1. INTRODUCTION

Let A denote the class of functions:

$$f(z) = z + a_2 z^2 + \cdots$$

which are analytic in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$. A function $f \in A$ is called starlike if and only if f maps E onto a starlike domain and it is well known that such f satisfy the necessary and sufficient condition:

$$\operatorname{Re} \frac{zf'(z)}{f(z)} > 0. \tag{1}$$

In [2] the author introduced subclasses S_n^σ of starlike functions satisfying

$$\operatorname{Re} \frac{L_{n+1}^\sigma f(z)}{L_n^\sigma f(z)} > \frac{\sigma - (n+1)}{\sigma - n}, \quad \sigma \geq n+1, \ n \in N \tag{2}$$

using the convolution operators $L_n^\sigma : A \to A$ defined as follows:

$$L_n^\sigma f(z) = (\tau_\sigma * \tau_{\sigma,n}^{(-1)} * f)(z)$$

where $*$ denote the convolution of $f * g$ defined as

$$(f * g)(z) = z + \sum_{k=2}^\infty a_k b_k z^k$$

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for $g(z) = z + b_2 z^2 + \cdots \in A$ and

$$\tau_{\sigma,n}(z) = \frac{z}{(1-z)^{\sigma-(n-1)}}, \quad \sigma - (n-1) > 0,$$

$\tau_\sigma = \tau_{\sigma,0}$ and $\tau_{\sigma,n}^{(-1)}$ such that

$$(\tau_{\sigma,n} * \tau_{\sigma,n}^{(-1)})(z) = \frac{z}{1-z}$$

for a fixed real number σ and $n \in N$. From inclusion relations for the subclass S_n^σ, the author in [2] deduced the following important univalence condition:

Theorem 1:([2]) Let $f \in A$ satisfy

$$\mathrm{Re}\, \frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} > 0. \tag{3}$$

Then f is starlike univalent in E.

The following were given as examples of those functions:

Examples: The functions $f_j(z)$ $j = 1, 2, 3, 4.$, given by

$$f_1(z) = \frac{2[1 - (1-z)e^z]}{z}, \quad f_2(z) = \frac{2[1 - (1+z)e^{-z}]}{z},$$

$$f_3(z) = \frac{-2[z + \log(1-z)]}{z}, \quad f_4(z) = \frac{2[z - \log(1+z)]}{z}.$$

satisfy the inequality (3) and are thus starlike univalent in the open unit disk.

Motivated by the above result, we prove in this paper a generalization of it using a method of linear combinations of geometric expressions. First we prove a new sufficient condition for starlikeness of analytic functions in the disk similar to the above. Then we define linear combinations of the associated geometric expressions and derive the conditions for the starlikeness of the linear sums. As it is well known that starlikeness of analytic functions in the unit disk is sufficient for their univalence there, it will follow that the resulting linear sums, with appropriate constraints, also provide a wide range of sufficient conditions for univalence in the unit disk. Some interesting corollaries are mentioned.

In the next section we state the preliminary result we shall use in the work. In Section 3, the main results are proved.

2. PRELIMINARY LEMMA

Let P denote the class of Caratheodory functions

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots$$

which are regular in E and satisfy Re $p(z) > 0$, $z \in E$.

Following Lewandowski et al [3], let $b > 0$ be a real number and let Q consist of functions $q(z) = b + c_1 z + c_2 z^2 + \cdots$ which are regular in E.

Definition 1: Let $u = u_1 + u_2 i$, $v = v_1 + v_2 i$ and let $\Psi_b(u, v)$ be the set of complex-valued functions $\psi(u, v)$ satisfying:

  (a) $\psi(u, v)$ is continuous in a domain Ω of $\mathbb{C}^2$,
  (b) $(b, 0) \in \Omega$ and Re $\psi(b, 0) > 0$,
  (c) Re $\psi(u_2 i, v_1) \leq 0$ when $(u_2 i, v_1) \in \Omega$ and $2v_1 \leq -(b^2 + u_2^2)/b$.

Some examples (see [3]) of such $\psi(u, v)$ include:

  (i) $\psi_1(u, v) = u + \alpha v / u$, α real and $\Omega = [\mathbb{C} - \{0\}] \times \mathbb{C}$.
  (ii) $\psi_2(u, v) = u + \alpha v$ with $\alpha \geq 0$ and $\Omega = \mathbb{C}^2$.
  (iii) $\psi_3(u, v) = u - v / u^2$ with $\Omega = [\mathbb{C} - \{0\}] \times \mathbb{C}$.

More examples for the particular case of $b = 1$ can also be found in literatures [1,3,4].

We also claim that $\psi_4(u, v) = (u^2 + v)/(1 + u)$ with $\Omega = \{\mathbb{C} - \{-1\}\} \times \mathbb{C}$ belongs to $\Psi_b(u, v)$ for ψ_4 is continuous in Ω; $(b, 0) \in \Omega$ and Re $\psi_4(b, 0) > 0$; and for $(u_2 i, v_1) \in \Omega$, $\psi_4(u_2 i, v_1) = (v_1 - u_2^2)(1 - u_2 i)/(1 + u_2^2)$ so that Re $\psi_4(u_2 i, v_1) = (v_1 - u_2^2)/(1 + u_2^2)$. When $v_1 \leq -\frac{b^2 + u_2^2}{2b}$, we have

$$\text{Re } \psi_4(u_2 i, v_1) \leq -\frac{b^2 + u_2^2 + 2bu_2^2}{2b(1 + u_2^2)}$$

which is nonpositive as required.

Definition 2: Let $\psi \in \Psi_b$ with corresponding domain Ω. Define $Q_b(\psi)$ as consisting of functions $q(z) = b + c_1 z + c_2 z^2 + \cdots$ which are regular in E and satisfy:

  (a) $(q(z), zq'(z)) \in \Omega$,
  (b) Re $\psi(q(z), zq'(z)) > 0$.

The class $Q_b(\psi)$ is not empty since for sufficiently small $|q_1|$ (depending on ψ), the function $q(z) = b + q_1 z$ belong to $Q_b(\psi)$. The above definition is more inclusive than that given by Lewandowski et al [3] where the positive real number b is restricted by $b = e^{i\sigma}$, $|\sigma| < \pi/2$.

Next we prove the following fundamental lemma.

Lemma 1: Let $q(z) = b + c_1 z + c_2 z^2 + \cdots \in Q_b(\psi)$. Then Re $q(z) > 0$.

Proof: Since $q(z) = b + c_1 z + c_2 z^2 + \cdots$ is regular in E, define

$$q(z) = \frac{1 + w(z)}{1 - w(z)}b. \tag{4}$$

Then $w(z)$ is a meromorphic function with $w(0) = 0$ and $w(z) \neq 1$ in E. Suppose there exists a point $z_0 \in E$ such that $|z| \leq |z_0|$, $\max |w(z)| = |w(z_0)| = 1$. Then by the well-known Jack's lemma

$$z_0 w'(z_0) = m w(z_0), \quad m \geq 1.$$

Since $|w(z_0)| = 1$ and $w(z_0) \neq 1$, then $(1 + w(z_0))/(1 - w(z_0))$ must be purely imaginary, that is,

$$\frac{1 + w(z_0)}{1 - w(z_0)} = Ai, \quad A \text{ real}, \tag{5}$$

so that from (4) we have

$$q(z_0) = bAi.$$

Also from (4) we have

$$zq'(z) = \frac{2bzw'(z)}{(1 - w(z))^2}$$

which implies that

$$z_0 q'(z_0) = \frac{2bz_0 w'(z_0)}{(1 - w(z_0))^2} = \frac{2bmw(z_0)}{(1 - w(z_0))^2}.$$

From (5) we have

$$1 + A^2 = \frac{-4w(z_0)}{(1 - w(z_0))^2}$$

so that

$$z_0 q'(z_0) = -\frac{mb(1 + A^2)}{2}$$

$$= -\frac{m}{2}b(1 + A^2) = d \text{ real}.$$

Hence at $z = z_0$ in E we have Re $\psi(q(z_0), z_0 q'(z_0)) = $ Re $\psi(bAi, d)$ where bA and d are real and $d \leq -\frac{1}{2b}(b^2 + u_2^2)$ (since $m \geq 1$) where $u_2 = bA$. It follows that Re $q(z) > 0$ with $b > 0$ as required.

The above lemma shows that in the special case $b = 1$, the class $Q_1(\psi)$ is a subset of the class P of Caratheodory functions. In the next, we prove the main results.

3. MAIN RESULTS

First we prove:

Theorem 2: Let $f \in A$ satisfy:

$$\text{Re } \frac{zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} > 0.$$

Then $f(z)$ is starlike univalent in E.

Proof: Observe that

$$\frac{zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = \frac{1}{2} + A_1 z + \cdots$$

so define

$$\frac{zf'(z)}{f(z)} = q(z)$$

for some $q \in Q_{\frac{1}{2}}$. Then

$$\frac{zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = \frac{zq(z)f'(z) + zq'f(z)}{f(z) + zf'(z)}$$

so that

$$\frac{zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = \frac{q(z)\frac{zf'(z)}{f(z)} + zq'}{1 + \frac{zf'(z)}{f(z)}}$$

$$= \frac{q(z)^2 + zq'(z)}{1 + q(z)}.$$

Now using ψ_4 defined as $\psi_4(q(z), zq'(z)) = \frac{q(z)^2 + zq'(z)}{1+q(z)}$, then it follows by Lemma 1 that

$$\text{Re } \frac{zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} > 0$$

implies

$$\text{Re } \frac{zf'(z)}{f(z)} > 0$$

and the proof is complete.

Now we consider linear combinations of the geometric expressions in Theorems 1 and 2 above. Suppose $G(z) = \frac{zf'(z)+z^2 f''(z)}{f(z)+zf'(z)}$ and $H(z) = \frac{2zf'(z)+z^2 f''(z)}{f(z)+zf'(z)}$, define the linear combinations $(1-\alpha)G(z) + \alpha H(z)$.

Obviously this yields

$$L(\alpha : f) = (1 - \alpha)G(z) + \alpha H(z) = \frac{(1 + \alpha)zf'(z) + z^2 f''(z)}{f(z) + zf'(z)}.$$

For simplicity of analysis, we define $\beta = 1 + \alpha$ so that the linear combinations become

$$L(\beta : f) = \frac{\beta zf'(z) + z^2 f''(z)}{f(z) + zf'(z)},$$

then we examine the conditions on the real number β which ensures the starlikeness of the linear combinations.

Theorem 3: Let $f \in A$. If $f(z)$ satisfies:

$$\text{Re } \frac{\beta zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} > 0. \tag{3}$$

Then $f(z)$ is starlike univalent in E whenever $\frac{2}{3} < \beta \leq 2$.

Proof: We note that the linear combinations $L(\beta : f)$ present series expansion of the form:

$$\frac{\beta zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = \frac{\beta}{2} + B_1 z + \cdots .$$

Thus we define

$$\frac{zf'(z)}{f(z)} = q(z)$$

for some $q \in Q_{\frac{\beta}{2}}$. Then

$$\frac{\beta zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = \frac{(\beta - 1)q(z)f(z) + zq(z)f'(z) + zq'f(z)}{f(z) + zf'(z)}$$

so that

$$\frac{\beta zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = \frac{(\beta - 1)q(z) + q(z)\frac{zf'(z)}{f(z)} + zq'(z)}{1 + \frac{zf'(z)}{f(z)}}$$

$$= \frac{(\beta - 1)q(z) + q(z)^2 + zq'(z)}{1 + q(z)}.$$

Now define $\psi(u, v) = \frac{(\beta-1)u + u^2 + v}{1+u}$ for $\Omega = [\mathbb{C} - \{-1\}] \times \mathbb{C}$. Then $\psi(u, v)$ is continuous in Ω. Also $(\beta/2, 0) \in \Omega$,

$$\psi(\beta/2, 0) = \frac{(\beta - 1)\beta/2 + \beta^2/4}{1 + \beta/2} = \frac{\beta(3\beta - 2)}{2(2 + \beta)}$$

and Re $\psi(\beta/2, 0) > 0$ for $3\beta - 2 > 0$ which gives the lower bound. Furthermore,

$$\psi(u_2 i, v_1) = \frac{(\beta - 1)u_2 i - u_2^2 + v_1}{1 + u_2 i}$$

so that

$$\text{Re } \psi(u_2 i, v_1) = \frac{(\beta - 2)u_2^2 + v_1}{1 + u_2^2}.$$

When $v_1 \leq -(\beta^2/4 + u_2^2)/\beta$ we have

$$\text{Re } \psi(u_2 i, v_1) \leq \frac{(\beta - 2)u_2^2 - (\beta^2/4 + u_2^2)/\beta}{1 + u_2^2} < \frac{(\beta - 2)u_2^2}{1 + u_2^2} \leq 0$$

whenever $\beta - 2 \leq 0$ as required.

Note that the special cases of $\beta = 1, 2$ in the above result correspond to Theorems 2 and 1 respectively. If we take $\beta = 3/2$ we have the following interesting corollary.

Corollary 1: Let $f \in A$ satisfy:

$$\text{Re } \frac{3zf'(z) + 2z^2 f''(z)}{f(z) + zf'(z)} > 0.$$

Then $f(z)$ is starlike univalent in E.

Furthermore if we take $\beta = \frac{\lambda}{\gamma}$ with $\frac{2}{3} \leq \frac{\lambda}{\gamma} \leq 2$, then we have

Corollary 2: Let $f \in A$ satisfy:

$$\text{Re } \frac{\lambda zf'(z) + \gamma z^2 f''(z)}{f(z) + zf'(z)} > 0, \quad \frac{2}{3} \leq \frac{\lambda}{\gamma} \leq 2.$$

Then $f(z)$ is starlike univalent in E.

In general all normalized analytic functions which are solutions of the second-order differential equations

$$\frac{\beta zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} = q(z),$$

that is

$$z^2 f''(z) + (\beta - q(z))zf'(z) - q(z)f(z) = 0 \qquad (6)$$

for $q \in Q_b$ and positive real numbers $\beta \in (\frac{2}{3}, 2]$ are the examples desired. In particular we note that the identity map $f(z) = z$ satisfies (6) for $q(z) = \beta/2$.

Finally we make the following:

Remark: The geometric condition in Theorem 3 is equivalent to

$$\text{Re}\,\frac{2zf'(z) + z^2 f''(z)}{f(z) + zf'(z)} > \text{Re}\,\frac{(2-\beta)zf'(z)}{f(z) + zf'(z)}$$

for β as already defined, which in our opinion may provide the lead to obtaining the desired nontrivial examples of functions satisfying the conditions of Theorem 3.

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REFERENCES

[1] K. O. Babalola and T. O. Opoola, *Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions*, Tamkang J. Math., **37** (4) 355-366, (2006).

[2] K. O. Babalola, *On some starlike mappings involving certain convolution opeartors*, Mathematica, Tome **51** (74) No. 2, 111-118, (2009).

[3] Z. Lewandowski, S. Miller and E. Zlotkiewicz, *Generating functions for some classes of univalent functions*, Proc. Amer. Math. Soc. **56** 111-117, (1976).

[4] S. S. Miller and P. T. Mocanu, *Second-order differential inequalities in the complex plane*, J. Math. Anal. Appl. **65** 289-305, (1978).

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