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STRONG CONVERGENCE THEOREM OF AN M-STEP HALPERN-TYPE ITERATION PROCESS FOR FINITE FAMILIES OF TOTAL ASYMPTOTICALLY NONEXPANSIVE MAPS

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ABSTRACT. In this paper we introduce an m-step Halpern-type iteration process and prove strong convergence of the scheme in a real Hilbert space H to the common fixed point of a finite family of Total asymptotically nonexpansive mappings. Our results improve previously known ones obtained for the class of nonexpansive and asymptotically nonexpansive mappings. As application iterative methods for approximation of: solution of variational inequality problems, fixed point of finite family of continuous pseudocontractive mappings, solutions of classical equilibrium problems and solutions of convex minimization problems are proposed. Our theorems unify and complement many recently announced results.

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1. INTRODUCTION

Let K be a nonempty subset of a normed linear space E. K is said to be (sequentially) compact if every bounded sequence in K has a subsequence that converges in K. K is said to be boundedly compact if every bounded subset of K is relatively compact. Finite dimensional spaces are boundedly compact. Given a subset S of K, we shall denote by $co(S)$ and $ccl(S)$ the convex hull and the closed convex hull of S respectively. If K is boundedly compact convex and S is bounded, then $co(S)$ and hence $ccl(S)$ are relatively compact convex and compact convex subsets of K respectively.
A map $T : K \to K$ is said to be semi-compact if for any bounded

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sequence $\{x_n\} \subset K$ such that $\|x_n - Tx_n\| \to 0$ as $n \to \infty$ there exists a subsequence $\{x_{n_j}\} \subset \{x_n\}$ such that x_{n_j} converges strongly to some $x^* \in K$ as $j \to \infty$. The map T is said to be demi-compact at $z \in H$ if for any bounded sequence $\{x_n\} \subset K$ such that $\|x_n - Tx_n\| \to z$ as $n \to \infty$ there exist a subsequence $\{x_{n_j}\} \subset \{x_n\}$ and a point $p \in K$ such that x_{n_j} converges strongly to p as $j \to \infty$. (Observe that if T is additionally continuous, then $p - Tp = z$). A nonlinear map $T : K \to K$ is said to be completely continuous if it maps bounded sets into relatively compact sets. A mapping $T : K \longrightarrow K$ is called *nonexpansive* if and only if for all $x, y \in K$, we have that

$$\|Tx - Ty\| \leq \|x - y\|. \tag{1}$$

A mapping T is called *asymptotically nonexpansive mapping* if and only if there exists a sequence $\{\mu_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0$ such that for all $x.y \in K$,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| \quad \forall\, n \in \mathbb{N} \tag{2}$$

T is *asymptotically nonexpansive in the intermediate sense* if it is continuous and the following holds:

$$\limsup_{n\to\infty} sup_{x,y\in D(T)}\big(\|T^n x - T^n y\| - \|x - y\|\big) \leq 0 \tag{3}$$

Observe that if we define: $\sigma_n := sup_{x,y\in D(T)}\big(\|T^n x - T^n y\| - \|x - y\|\big), \delta_n := max\{0, \sigma_n\}$ then $\delta_n \to 0$ as $n \to \infty$ and (3) reduces to

$$\|T^n x - T^n y\| \leq \|x - y\| + \delta_n \quad \forall\, x, y \in D(T), n \in \mathbb{N} \tag{4}$$

A mapping T is called *generalised asymptotically nonexpansive mapping* if and only if there exist a sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ such that for all $x, y \in D(T)$,

$$\|T^n x - T^n y\| \leq (1 + \mu_n)\|x - y\| + \eta_n \quad \forall\, n \in \mathbb{N} \tag{5}$$

The class of asymptotically nonexpansive mappings was introduced by Goebel and Kirk [6] as a generalisation of nonexpansive mappings. As further generalisation of class of nonexpansive mappings, generalised asymptotically nonexpansive map was introduced and later in 2006, Alber, Chidume and Zegeye [2] introduced the class of total asymptotically nonexpansive mappings, where a mapping $T : K \longrightarrow K$ is called *total asymptotically nonexpansive* if and only if there exist two sequences $\{\mu_n\}_{n\geq 1}, \{\eta_n\}_{n\geq 1} \subset [0, +\infty)$, with $\lim_{n\to\infty} \mu_n = 0 = \lim_{n\to\infty} \eta_n$ and nondecreasing continous function

$\phi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\phi(0) = 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \leq \|x - y\| + \mu_n \phi(\|x - y\|) + \eta_n \quad n \geq 1 \qquad (6)$$

Ofoedu and Nnubia [16] gave an example to show that the class of asymptotically nonexpansive mappings is a proper subset of the class of total asymptotically nonexpansive mappings.

The class of total asymptotically nonexpansive type mappings includes the class of mappings which are asymptotically nonexpansive in the intermediate sense. These classes of mappings have been studied by several authors (see e.g.[6], [9], [22]).

Let H be a Hilbert space and K a nonempty closed convex subset of H, the metric projection $P_K : H \longrightarrow K$ is the function which assigns to each $x \in H$ its nearest point denoted by $P_K x$. Thus $P_K x$ is the unique point on K such that

$$\|x - P_K x\| \leq \|x - y\| \quad \forall\, y \in K. \qquad (7)$$

It is also well known that P_K satisfies

$$\langle P_K x - P_K y, x - y \rangle \geq \|P_K x - P_K y\|^2, \quad \forall\, x, y \in H \qquad (8)$$

Moreover, $P_K x$ is characterized by the properties that $P_K x \in K$ and $\langle x - P_K x, P_K x - y \rangle \geq 0, \quad \forall\, y \in K$.

2. PRELIMINARY

We shall make use of the following in the sequel.

A Banach space E is said to satisfy *Opial's condition* if for each sequence $\{x_n\}_{n \geq 1} \subset E$ which converges weakly to a point $z \in E$, we have that $\liminf_{n\to\infty} \|x_n - z\| < \liminf_{n\to\infty} \|x_n - y\|$, $\forall y \ \in E,\ y \neq z$. It is well known that every Hilbert space satisfies Opial's condition(see eg [17]).

A map T is said to satisfies *condition B* if there exists a continous, strictly inccreasing function $f : [0, \infty) \to [0, \infty)$,with $f(0) = 0$ such that for all $x \in D(T), \|x - Tx\| \geq f(d(x, F))$ where $F = F(T) = \{x \in D(T) : x = Tx\}$ and $d(x, F) = \inf\{\|x - y\| : y \in F\}$.

Lemma 1:[19] Let $\{a_n\}$ be sequence of nonegative real numbers satisfying the following relation:

$$a_{n+1} \leq (1 - \alpha_n) a_n + \delta_n,\ \ n \geq n_0,$$

where $\{\alpha_n\}_{n \geq 1} \subset (0, 1)$ and $\{\delta_n\}_{n \geq 1} \subset \mathbb{R}$ satisfying the following conditions:$\sum_{n=0}^{\infty} \alpha_n = \infty$, $\lim_{n\to\infty} \alpha_n = 0$ and $\limsup_{n\to\infty} \delta_n \leq 0$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2:[10] Let $\{\mu_n\}, \{\nu_n\}, \{\eta_n\}$ be nonnegative sequences such that $\sum_{n\geq 0} \nu_n < \infty$, $\sum_{n\geq 0} \eta_n < \infty$ and $\mu_{n+1} \leq (1+\nu_n)\mu_n + \eta_n$. Then $\lim_{n\to\infty} \mu_n$ exists.

Lemma 3:[3] Let E be a reflexive Banach space with weakly continuous normalized duality mapping. Let K be a closed convex subset of E and $T : K \to K$ a uniformly continuous total asymptotically nonexpansive mapping with bounded orbits. Then $I - T$ is demiclosed at zero.

Lemma 4:[5] For any x, y, z in a Hilbert space H and a real number $\lambda \in [0, 1]$,

$$\|\lambda x+(1-\lambda)y-z\|^2 = \lambda\|x-z\|^2+(1-\lambda)\|y-z\|^2-\lambda(1-\lambda)\|x-y\|^2. \tag{9}$$

Lemma 5:[13] Let K be a closed convex nonempty subset of a Banach space E and let $T_i : K \longrightarrow K$ where $i \in I = \{1, 2, ..., m\}$ be a finite family of continous nonlinear maps in K such that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence in K satisfying

(1) $\lim_{n\to\infty} \|x_n - T_i x_n\| = 0 \ \ \forall i \in I$
(2) $\exists \ n_* \in \mathbb{N}$ such that $\|x_{n+1} - x^*\| \leq (1 + \tau_n)\|x_n - x^*\| + \nu_n; \ \forall \ n > n_*$

where $\sum_{n\geq 0} \nu_n < \infty$ and $\sum_{n\geq 0} \tau_n < \infty$. Then,

(1) $\lim_{n\to\infty} \|x_n - x^*\|$ exist and $\{x_n\}$ is bounded.
(2) $\{x_n\}$ converges strongly to a common fixed point of T_i's if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.
(3) $\{x_n\}$ converges strongly to a common fixed point of T_i's if one of the T_i's satisfies any of the following conditions:
   (a) condition B,
   (b) semi-compact or demi compact at 0,
   (c) completely continuous.
(4) $\{x_n\}$ converges strongly to a point of F if any of the following conditions holds:
   (a) K is compact.
   (b) K is boundedly compact.

3. Main Results

Proposition 1: Suppose that there exist $c > 0, k > 0$ constants such that $\phi(t) \leq ct \; \forall \; t \; \geq k$, then T is total asymptotically nonexpansive if and only if T is generalised asymptotically nonexpansive map.
Proof: It is known that every generalised asymptotically nonexpansive map is total asymptotically nonexpansive, so it suffices to show that that every total asymptotically nonexpansive with the condition of our hypothesis is generalised asymptotically nonexpansive map. Now, let T be such that

$$\|T^n x - T^n y\| \;\; \leq \;\; \|x - y\| + \mu_n \phi(\|x - y\|) + \eta_n \quad n \geq 1$$

Since ϕ is continuous, it follows that ϕ attains its maximum (say c_0) on the interval $[0, k]$; moreover, $\phi(t) \leq ct$ whenever $t > k$. Thus,

$$\phi(t) \;\; \leq \;\; c_0 + ct \;\; \forall \, t \in [0, +\infty).$$

So, we have,

$$\begin{aligned} \|T^n x - T^n y\| \;\; &\leq \;\; \|x - y\| + \mu_n (c_0 + c\|x - y\|) + \eta_n \quad n \geq 1 \\ &= \;\; (1 + \mu_n c)\|x - y\| + \mu_n c_0 + \eta_n \\ &= \;\; (1 + \nu_n)\|x - y\| + \gamma_n \end{aligned}$$

where $\nu_n = \mu_n c$ and $\gamma_n = \mu_n c_0 + \eta_n$. Thus completing the proof.
Proposition 2: Let H be a real Hilbert space, K a closed convex nonempty subset of H and $T : K \longrightarrow H$ a continuous Total asymptotically nonexpansive mapping satisfying condition of Proposition 1 with sequences $\{\mu_n\}_{n \geq 1}, \{\eta_n\}_{n \geq 1} \subset [0, +\infty)$ with $\sum_{n=0}^{\infty} \mu_n < \infty$, $\sum_{n=0}^{\infty} \eta_n < \infty$. Suppose that the fixed piont set, $F(T) \neq \emptyset$, then $F(T)$ is closed and convex.
Proof: Let $\{x_n\}$ be a sequence in $F(T)$ converging to $x^* \in K$ then $x_n = Tx_n \forall \; n \geq 0$. By continuity of $T, x^* = \lim_{n\to\infty} x_n = \lim_{n\to\infty} Tx_n = T(\lim_{n\to\infty} x_n) = Tx^*$ thus, $x^* \in F(T)$ and $F(T)$ is closed. Next, we show that $F(T)$ is convex. For $t \in (0, 1)$ and $x, y \in F(T)$, put

$z_t = (1-t)x + ty$, we show that $z_t = Tz_t$. Set $u_n = 1 + \mu_n$, we have

$$\begin{aligned}
\|z_t - T^n z_t\|^2 &= \|(1-t)[x - T^n z_t] + t[y - T^n z_t]\|^2 \\
&= (1-t)\|x - T^n z_t\|^2 + t\|y - T^n z_t\|^2 \\
&\quad -t(1-t)\|x-y\|^2 \\
&\leq (1-t)(u_n\|x - z_t\| + \eta_n)^2 + t(u_n\|y - z_t\| + \eta_n)^2 \\
&\quad -t(1-t)\|x-y\|^2 \\
&= (1-t)(u_n t\|x-y\| + \eta_n)^2 \\
&\quad +t(u_n(1-t)\|x-y\| + \eta_n)^2 - t(1-t)\|x-y\|^2 \\
&= (1-t)(u_n^2 t^2\|x-y\|^2 + 2tu_n\eta_n\|x-y\| + \eta_n^2) \\
&\quad +t(u_n^2(1-t)^2\|x-y\|^2 + 2\eta_n u_n(1-t)\|x-y\| \\
&\quad +\eta_n^2) - t(1-t)\|x-y\|^2 \\
&= t(1-t)[u_n^2(t+1-t) - 1]\|x-y\|^2 \\
&\quad +4t(1-t)u_n\eta_n\|x-y\| + \eta^2 \\
&= t(1-t)(u_n^2 - 1)\|x-y\|^2 + 4t(1-t)u_n\eta_n\|x-y\| \\
&\quad +\eta^2.
\end{aligned}$$

Thus, $\lim_{n\to\infty} \|z_t - T^n z_t\| = 0$, which implies that $\lim_{n\to\infty} T^n z_t = z_t$ and hence $z_t = \lim_{n\to\infty} T^n z_t = T(\lim_{n\to\infty} T^{n-1} z_t) = Tz_t$. Thus, $z_t \in F(T)$. This completes the proof.

Proposition 3: Let H be a real Hilbert space, let K be a closed convex nonempty subset of H and let $T_i : K \longrightarrow K$ (i = 1,...,m) be a finite family of uniformly continuous total asymptotically nonexpansive maps with associated sequences $\{\mu_{n,i}\}_{n\geq 1}, \{\eta_{n,i}\}_{n\geq 1} \subset [0, +\infty)$ with $\sum_{n=0}^{\infty} \mu_{n,i} < \infty \ \ \sum_{n=0}^{\infty} \eta_{n,i} < \infty$. Suppose that there exist $c > 0, k > 0$ constants such that $\phi(t) \leq ct \forall\ t \geq k$, and that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ then F is closed and convex.

Proof: By Proposition 2 , we have that $F(T_i)$ is closed for each i. Now,$F = \bigcap_{i=1}^m F(T_i)$ is a finite intersection of closed sets, hence closed. Also, we have that $F(T_i)$ is convex for each i. Since $F = \bigcap_{i=1}^m F(T_i)$ is a finite intersection of convex set, we have that F is convex.

Proposition 4: Let H be a real Hilbert space, let K be a closed convex nonempty subset of H and let $T_i : K \longrightarrow K$ ($i \in I = \{1, ..., m\}$) be a finite family of uniformly continuous Total asymptotically nonexpansive map with sequences $\{\mu_{n,i}\}_{n\geq 1}, \{\eta_{n,i}\}_{n\geq 1} \subset$

$[0, +\infty)$ such that $\sum_{n=0}^{\infty} \mu_{n,i} < \infty$,$\sum_{n=0}^{\infty} \eta_{n,i} < \infty$. Suppose that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively from an arbitrary $x_0 \in K$ by

$$\begin{aligned} y_n &= P_K[(1-\beta_n)x_n + \beta_n u], \\ z_{n,0} = y_n;\ \ z_{n,i} &= (1-\alpha_n)x_n + \alpha_n T_i^n z_{n,i-1};\ \ i = 1, ..., m \\ z_{n,m} &= x_{n+1} \quad n \geq 0. \end{aligned} \tag{10}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$,$0 < a < \alpha_n < b < 1\ \forall\ n \geq 1$(for some constants a,b). Let $x^* \in F$, then $\lim_{n\to\infty} \|x_n - x^*\|$ exist.

Proof: Let $x^* \in F, i \in I = \{1, ..., m\},\ u_n = 1 + \mu_n$, then from (10) we have that

$$\begin{aligned}
\|y_n - x^*\| &= \|P_K[(1-\beta_n)x_n + \beta_n u] - x^*\|, \\
&\leq (1-\beta_n)\|x_n - x^*\| + \beta_n\|u - x^*\|, \\
\|z_{n,1} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T_1^n z_{n,0} - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n(u_{n,1}\|y_{n,} - x^*\| + \eta_{n,1}). \\
&= (1-\alpha_n)\|x_n - x^*\| + \alpha_n u_{n,1}\|y_n - x^*\| + \alpha_n\eta_{n,1} \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n u_{n,1}(1-\beta_n)\|x_n - x^*\| \\
&\quad +\alpha_n u_{n,1}\beta_n\|u - x^*\| + \alpha_n\eta_{n,1} \\
&= [1 + \alpha_n(u_{n,1} - 1) - \beta_n\alpha_n u_{n,1}]\|x_n - x^*\| \\
&\quad +\alpha_n u_{n,1}\beta_n\|u - x^*\| + \alpha_n\eta_{n,1} \\
\|z_{n,2} - x^*\| &\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|T_2^n z_{n,1} - x^*\| \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n u_{n,2}\|z_{n,1} - x^*\| + \alpha_n\eta_{n,2} \\
&\leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n u_{n,2}\big[[1 + \alpha_n(u_{n,1} - 1) \\
&\quad -\beta_n\alpha_n u_{n,1}]\|x_n - x^*\| \\
&\quad +\alpha_n u_{n,1}\beta_n\|u - x^*\| + \alpha_n\eta_{n,1}\|\big] + \alpha_n\eta_{n,2} \\
&= [1 + \alpha_n(u_{n,2} - 1) + \alpha_n^2 u_{n,2}(u_{n,1} - 1) \\
&\quad -\beta_n\alpha_n^2 u_{n,1}u_{n,2}]\|x_n - x^*\| + \beta_n\alpha_n^2 u_{n,2}u_{n,1}\|u - x^*\| \\
&\quad +\alpha_n^2 u_{n,2}\eta_{n,1} + \alpha_n\eta_{n,2}
\end{aligned}$$

$$\begin{aligned}
\|z_{n,3}-x^*\| &\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n\|T_3^n z_{n,2}-x^*\| \\
&\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n u_{n,3}\|z_{n,2}-x^*\|+\alpha_n\eta_{n,3} \\
&\le (1-\alpha_n)\|x_n-x^*\|+\alpha_n u_{n,3}\big[[1+\alpha_n(u_{n,2}-1) \\
&\quad +\alpha_n^2 u_{n,2}(u_{n,1}-1)-\beta_n\alpha_n^2 u_{n,1}u_{n,2}]\|x_n-x^*\| \\
&\quad +\beta_n\alpha_n^2 u_{n,2}u_{n,1}\|u-x^*\|+\alpha_n^2 u_{n,2}\eta_{n,1}+\alpha_n\eta_{n,2}\big] \\
&\quad +\alpha_n\eta_{n,3} \\
&= [1+\alpha_n(u_{n,3}-1)+\alpha_n^2 u_{n,3}(u_{n,2}-1) \\
&\quad +\alpha_n^3 u_{n,3}u_{n,2}(u_{n,1}-1)-\beta_n\alpha_n^3 u_{n,1}u_{n,2}u_{n,3}]\|x_n \\
&\quad -x^*\|+\beta_n\alpha_n^3 u_{n,3}u_{n,2}u_{n,1}\|u-x^*\|+\alpha_n^3 u_{n,2}\eta_{n,1} \\
&\quad +\alpha_n^2 u_{n,2}\eta_{n,2}+\alpha_n\eta_{n,3}.
\end{aligned}$$

So,

$$\begin{aligned}
\|z_{n,j}-x^*\| &\le \Big[1+\sum_{t=0}^{j-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}u_{n,j-s}(u_{n,j-t}-1) \\
&\quad -\beta_n\alpha_n^{j}\Pi_{s=0}^{j-1}u_{n,j-s}\Big]\|x_n-x^*\| \\
&\quad +\beta_n\alpha_n^{j}\Pi_{s=0}^{j-1}u_{n,j-s}\|u-x^*\| \\
&\quad +\sum_{t=0}^{j-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}u_{n,j-s}\eta_{n,j-t}
\end{aligned}$$

and hence,

$$\begin{aligned}
\|x_{n+1}-x^*\| &\le \Big[1+\sum_{t=0}^{m-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}u_{n,m-s}(u_{n,m-t}-1) \\
&\quad -\beta_n\alpha_n^{m}\Pi_{s=0}^{m-1}u_{n,m-s}\Big]\|x_n-x^*\| \\
&\quad +\beta_n\alpha_n^{m}\Pi_{s=0}^{m-1}u_{n,m-s}\|u-x^*\| \\
&\quad +\sum_{t=0}^{m-1}\alpha_n^{t+1}\Pi_{s=0}^{t-1}u_{n,m-s}\eta_{n,m-t}
\end{aligned}$$

(since $\exists\, n_0$ such that $u_{n,i} \leq q \;\forall\, n \geq n_0, \forall\, i \in I$ for some positive constant q). Hence,

$$\begin{aligned}
\|x_{n+1}-x^*\| &\leq \Big[1+bq^m\sum_{j=1}^{m}(u_{n,j}-1)\Big]\|x_n-x^*\| \\
&\quad -\beta_n\alpha_n^m\Pi_{j=1}^m u_{n,j}(\|x_n-x^*\|-\|u-x^*\|) \\
&\quad +bq^m\sum_{j=1}^{m}\eta_{n,j} \\
&\leq \Big[1+d\sum_{t=0}^{m-1}(u_{n,m-t}-1)\Big]\|x_n-x^*\| \\
&\quad +D(\beta_n+\sum_{s=0}^{m-1}\eta_{n,m-s}) \\
&= \Big[1+d\sum_{j=1}^{m}(u_{n,j}-1)\Big]\|x_n-x^*\| \\
&\quad +D(\beta_n+\sum_{j=1}^{m}\eta_{n,j}) \\
&\quad \text{for some constants } d, D>0.
\end{aligned}$$

So that, $\lim_{n\to\infty}\|x_n-x^*\|$ exist; and hence $\{x_n\}$ is bounded.

Theorem 1: Let H be a real Hilbert space, let K be a closed convex nonempty subset of H and let $T_i : K \longrightarrow K$ $(i \in I = \{1, ..., m\})$ be a finite family of uniformly continuous total asymptotically nonexpansive map with sequences $\{\mu_{n,i}\}_{n\geq 1}, \{\eta_{n,i}\}_{n\geq 1} \subset [0, +\infty)$ such that $\sum_{n=0}^{\infty}\mu_{n,i} < \infty$, $\sum_{n=0}^{\infty}\eta_{n,i} < \infty$ and with function $\phi : [0, +\infty) \longrightarrow [0, +\infty)$ satisfying $\phi(t) \leq ct \;\forall\, t \geq k$, for some constants $c, k > 0$. Suppose that $F = \bigcap_{i=1}^m F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by starting with an arbitrary $x_0 \in K$, define by (10) where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0, 1)$ such that $\sum_{n=0}^{\infty}\beta_n < \infty$,$0 < a < \alpha_n < b < 1 \;\forall\, n \geq 1$. Then $\forall j \in \{1, 2, ..., m\}$, $\lim_{n\to\infty}\|x_n - T_jx_n\| = 0$ and $\{x_n\}_{n\geq 1}$ converges weakly to a point of F.

Proof: Let $x^* \in F, u_n = 1 + \mu_n$,

$$\begin{aligned}
\|y_n - x^*\|^2 &\leq \|(1-\beta_n)x_n + \beta_n u - x^*\|^2, \\
&= (1-\beta_n)\|x_n - x^*\|^2 + \beta_n\|u - x^*\|^2 \\
&\quad -\beta_n(1-\beta_n)\|x_n - u\|^2
\end{aligned}$$

$$\begin{aligned}
\|z_{n,1} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n\|T_1^n y_n - x^*\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_1^n y_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(u_{n,1}\|y_n - x^*\|^2 + \eta_{n,1})^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_1^n y_n\|^2 \\
&= (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n u_{n,1}^2\|y_n \\
&\quad -x^*\|^2 + \alpha_n(2u_{n,1}\|y_n - x^*\| + \eta_{n,1})\eta_{n,1} \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_1^n y_n\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n u_{n,1}^2\big[(1-\beta_n)\|x_n - x^*\|^2 \\
&\quad +\beta_n\|u - x^*\|^2 - \beta_n(1-\beta_n)\|x_n - u\|^2\big] \\
&\quad +\alpha_n(2u_{n,1}\|y_n - x^*\| + \eta_{n,1})\eta_{n,1} \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_1^n y_n\|^2 \\
&= \left[1 + \alpha_n(u_{n,1} - 1) - \beta_n\alpha_n u_{n,1}^2\right]\|x_n - x^*\|^2 \\
&\quad +\beta_n\alpha_n u_n\|u - x^*\|^2 - \beta_n(1-\beta_n)\alpha_n u_{n,1}^2\|u - x_n\|^2 \\
&\quad +\alpha_n(2u_{n,1}\|y_n - x^*\| + \eta_{n,1})\eta_{n,1} \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_1^n y_n\|^2. \\
\|z_{n,2} - x^*\|^2 &\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n(u_{n,2}\|z_{n,1} - x^*\| + \eta_{n,2})^2 \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_2^n z_{n,1}\|^2 \\
&\leq (1-\alpha_n)\|x_n - x^*\|^2 + \alpha_n u_{n,2}^2\|z_{n,1} - x^*\|^2 \\
&\quad +\alpha_n(2u_{n,2}\|z_{n,1} - x^*\| + \eta_{n,2})\eta_{n,2} \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_2^n z_{n,1}\|^2.
\end{aligned}$$

So,

$$\begin{aligned}
\|z_{n,2} - x^*\|^2 &\leq \left[1 + \alpha_n(u_{n,2}^2 - 1) + \alpha_n^2 u_{n,2}^2(u_{n,1}^2 - 1) - \beta_n\alpha_n^2 u_{n,1}^2 u_{n,2}^2\right]\|x_n - x^*\|^2 \\
&\quad +\beta_n\alpha_n^2 u_{n,1}^2 u_{n,2}^2\|u - x^*\|^2 - \beta_n(1-\beta_n)\alpha_n^2 u_{n,1}^2 u_{n,2}^2\|u - x_n\|^2 \\
&\quad +\alpha_n(2u_{n,2}\|z_{n,1} - x^*\| + \eta_{n,2})\eta_{n,2} + \alpha_n^2 u_{n,2}(2u_{n,1}\|y_n - x^*\| + \eta_{n,1})\eta_{n,1} \\
&\quad -\alpha_n(1-\alpha_n)\|x_n - T_2^n z_{n,1}\|^2 - \alpha_n^2 u_{n,2}^2(1-\alpha_n)\|x_n - T_1^n y_n\|^2.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|z_{n,j}-x^*\|^2 \quad \leq \quad & \left[1+\sum_{t=1}^{j}\alpha_n^t\Pi_{s=0}^{t-2}u_{n,j-s}^2(u_{n,j-t}^2-1)-\beta_n\alpha_n^j\Pi_{t=0}^{j-1}u_{n,j-t}^2\right]\|x_n-x^*\|^2\\
&+\beta_n\alpha_n^j\Pi_{t=0}^{j-1}u_{n,j-t}\|u-x^*\|-\beta_n(1-\beta_n)\alpha_n^j\Pi_{t=0}^{j-1}u_{n,j-t}\|u-x_n\|^2\\
&+\sum_{t=1}^{j}\alpha_n^t\Pi_{s=0}^{t-2}u_{n,j-s}^2(2u_{n,j-t+1}\|z_{n,j-t}-x^*\|+\eta_{n,j-t+1})\eta_{n,j-t+1}\\
&-\sum_{t=1}^{j}\alpha_n^t(1-\alpha_n)\Pi_{s=0}^{t-2}u_{n,j-s}\|x_n-T_{j-s+1}^nz_{n,j-s}\|^2.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 \quad \leq \quad & \left[1+\sum_{t=1}^{m}\alpha_n^t\Pi_{s=0}^{t-2}u_{n,m-s}^2(u_{n,m-t}^2-1)-\beta_n\alpha_n^m\Pi_{t=0}^{m-1}u_{n,m-t}^2\right]\|x_n-x^*\|^2\\
&+\beta_n\alpha_n^m\Pi_{t=0}^{m-1}u_{n,m-t}\|u-x^*\|-\beta_n(1-\beta_n)\alpha_n^m\Pi_{t=0}^{m-1}u_{n,m-t}\|u-x_n\|^2\\
&+\sum_{t=1}^{m}\alpha_n^t\Pi_{s=0}^{t-2}u_{n,m-s}^2(2u_{n,m-t+1}\|z_{n,m-t}-x^*\|+\eta_{n,m-t+1})\eta_{n,m-t+1}\\
&-(1-\alpha_n)\sum_{t=1}^{m}\alpha_n^t\Pi_{s=0}^{t-2}u_{n,m-s}\|x_n-T_{m-s+1}^nz_{n,m-s}\|^2.
\end{aligned}$$

So, $\lim_{n\to\infty}\|x_n-T_{m-s+1}^nz_{n,m-s}\|=0\ \forall\ s=1,...,m$. ie $\lim_{n\to\infty}\|x_n-T_j^nz_{n,j-1}\|=0\ \forall\ j=1,...,m.\ \forall\ j\in I$.

$$\begin{aligned}
\|x_n-T_j^nx_n\| \quad &\leq \quad \|x_n-T_j^nz_{n,j-1}\|+\|T_j^nz_{n,j-1}-T_j^nx_n\|\\
&\leq \quad \|x_n-T_j^nz_{n,j-1}\|+u_{n,j}\|z_{n,j-1}-x_n\|+\eta_{n,j}.
\end{aligned}$$

$\|z_{n,j-1}-x_n\|=\alpha_n\|x_n-T_{j-1}z_{n,j-2}\|, \|z_{n,j}-x_n\|=\alpha_n\|x_n-T_{j-1}z_{n,j-2}\|$, so $\lim_{n\to\infty}\|z_{n,j}-x_n\|=0\ \forall\ j=1,...,m$. Therefore, $\lim_{n\to\infty}\|x_n-T_j^nx_n\|=0\ \forall\ j$.
Now, $\forall\ j\in I$. $\|x_n-T_jx_n\|\leq\|x_n-T_j^nx_n\|+\|T_j^nx_n-T_jx_n\|$, but

$$\begin{aligned}
\|x_n-T_j^{n-1}x_n\| \quad &\leq \quad \|x_n-x_{n-1}\|+\|x_{n-1}-T_j^{n-1}x_n\|+\|T_j^{n-1}x_{n-1}-T_j^{n-1}x_n\|\\
&\leq \quad \|x_n-x_{n-1}\|+\|x_{n-1}-T_j^{n-1}x_n\|+u_{n-1,j}\|x_{n-1}-x_n\|+\eta_{n-1,j}.
\end{aligned}$$

$\|x_{n+1}-x_n\|=\|z_{n,m}-x_n\|=\alpha_n\|x_n-T_mz_{n,m-1}\|$ so that $\lim_{n\to\infty}\|x_n-x_{n-1}\|=0\ \lim_{n\to\infty}\|x_n-T_j^{n-1}x_n\|\ \forall\ j$ and hence
$\lim_{n\to\infty}\|x_n-T_jx_n\|=0\ \forall\ j$. By Lemma 3 $(I-T_i)$ is demiclosed at $0\ \forall i$. Since $\{x_n\}_{n\geq1}$ is bounded and every Hilbert space satisfies Opial's condition, it follows from standard argument that $\{x_n\}_{n\geq1}$ converges weakly to a point of F. Thus completing the proof.
Theorem 2: Let $K, H, P_K, T_i's, F, \{x_n\}$ be as in Theorem Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F)=0$. (where F= F(T)).
Proof: Follows from Lemma 5 part 2.

Theorem 3: Let $K, H, P_K, T_i' s, F, \{x_n\}$ be as in Theorem 1 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if one of the T_i's satisfies any of the following conditions:

(1) condition B,
(2) semi-compact or demi compact at 0,
(3) completely continuous.

Proof: Follows from Lemma 5 part 3.
Theorem 4: Let $K, H, P_K, T_i' s, F, \{x_n\}$ be as in Theorem 1 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if K satisfies any of the following:

(1) K is compact.
(2) K is boundedly compact.

Proof: Follows from Lemma 5 part 4.
As a result of the proposition 1 we have the following theorems as corollaries.
Theorem 5: Let H be a real Hilbert space, let K be a closed convex nonempty subset of H and let $T_i : K \longrightarrow H$ $(i \in I = \{1, ..., m\})$ be a finite family of uniformly continuous generalised asymptotically nonexpansive maps with sequences $\{\mu_{n,i}\}_{n\geq 1}, \{\eta_{n,i}\}_{n\geq 1} \subset [0, +\infty)$, $\sum_{n=0}^{\infty} \mu_{n,i} < \infty$ $\sum_{n=0}^{\infty} \eta_{n,i} < \infty$. Suppose that $F = \bigcap_{i=1}^{m} F(T_i)$ is not empty and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by (10) where $\{\alpha_n\}_{n\geq 1}$ is a sequence in $(0, 1)$ satisfying the following conditions:
$0 < a < \alpha_n < b < 1$ $\forall$ $n \geq 1$, then $\forall j \in \{1, 2, ..., m\}$, $\lim_{n\to\infty} \|x_n - T_j x_n\| = 0$ and $\{x_n\}_{n\geq 1}$ converges weakly to a point of F.
Theorem 6: Let $K, H, P_K, T_i' s, F, \{x_n\}$ be as in Theorem 5 Then, $\{x_n\}$ converges strongly to a fixed point of T if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$ (where F= F(T)).
Theorem 7: Let $K, H, P_K, T_i' s, F, \{x_n\}$ be as in Theorem 5 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if one of the T_i's satisfies any of the following conditions:

(1) condition B,
(2) semi-compact or demi compact at 0,
(3) completely continuous.

Theorem 8: Let $K, H, P_K, T_i' s, F, \{x_n\}$ be as in Theorem 5 Then, $\{x_n\}$ converges strongly to a common fixed point of T_i's if any of the following condition holds:

(1) K is compact.

(2) K is boundedly compact.

. Observe that if $T_i, i \in I$ in Theorem 5 were asymptotically nonexpansive mappings, the condition there exist $c > 0$ and $k > 0$ such that $\phi(t) \leq ct \, \forall \, t > k$ is not needed. Moreso, every asymptotically nonexpansive mapping $T_i : K \longrightarrow K$ is uniformly L-Lipschitzian thus uniformly continuous. Hence we have the following theorems as an easy corollaries of Theorem 5 above:

Theorem 9: Let K be a closed convex nonempty subset of a real Hilbert space H and let $T_i : K \longrightarrow K, i \in I$, be asymptotically nonexpansive mappings such that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by (10),where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$ $, 0 < a < \alpha_n < b < 1 \, \forall \, n \geq 1$ then $\{x_n\}_{n\geq 1}$ converges strongly to $P_F u$(where $P_F u$ is a common fixed point nearest to u).

Theorem 10: Let K be a closed convex nonempty subset of a real Hilbert space H and let $T_i : K \longrightarrow K$, $i \in I$, be finite family of nonexpansive mappings from K into itself. Suppose that $F = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by (10), where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$ $, 0 < a < \alpha_n < b < 1 \, \forall \, n \geq 1$, then $\{x_n\}_{n\geq 1}$ converges strongly to $P_F u$.

Corollary 1: Suppose in our theorems the finite family is a singleton (that is if $m = 1$), our results hold. It is interesting to note that if $u = 0$ in our recursion formula, we obtain what some authors call the Minimum norm iteration process. We observe that all our theorems in this paper carry over to the so-called Minimum norm iteration process. Moreso, If $f : K \longrightarrow K$ is a contraction map and we replace u by $f(x_n)$ in our recursion formula, we obtain what some authors call viscosity iteration process. We observe also that all our theorems carry over trivially to the so-called viscosity process.

Application to approximation of fixed points of continuous pseudocontractive mappings

The most important generalization of the class of nonexpansive mappings is, perhaps, the class of pseudocontractive mappings. These mappings are intimately connected with the important class of nonlinear monotone operators. For the importance of monotone operators and their connections with evolution equations, the reader may consult [5], [10], [11], [12],[15] .
Due to the above connection, fixed point theory of pseudocontractive mappings became a flourishing area of intensive research for several authors. Recently, H. Zegeye [21] established the following Lemmas:
Lemma 6: [21] Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \longrightarrow H$ be a continuous pseudocontractive mapping, then for all $r > 0$ and $x \in H$, there exists $z \in K$ such that

$$\langle y - z, Tz\rangle - \frac{1}{r}\langle y - z, (1 + r)z - x\rangle \leq 0 \quad \forall \ y \in K \tag{11}$$

Lemma 7: [21] Let K be a nonempty closed convex subset of a real Hilbert space H. Let $T : K \longrightarrow K$ be a continuous pseudocontractive mapping, then for all $r > 0$ and $x \in H$, there exists $z \in K$, define a mapping $F_r : H \longrightarrow K$ by

$$F_r(x) = \{z \in K : \langle y-z, Tz\rangle - \frac{1}{r}\langle y-z, (1+r)z-x\rangle \leq 0 \quad \forall \ y \in K\} \tag{12}$$

then the following hold:

(1) F_r is single-valued
(2) F_r is firmly nonexpansive type mapping i.e for all $x, y, z \in H$

$$\|F_r(x) - F_r(y)\|^2 \leq \langle F_r(x) - F_r(y), x - y\rangle \tag{13}$$

(3) $Fix(F_r)$ is closed and convex; and $Fix(F_r) = Fix(T)$ for all $r > 0$:

We observe that Lemmas 6 and 7 hold in particular for r = 1. Thus, if $T_i,\ i \in I = \{1, 2, ..., m\}$ is a finite family of continuous pseudocontractive mapping and we define $F_{1(i)} : H \longrightarrow K$ by

$$F_{1(i)}(x) = \{z \in K : \langle y - z, T_i z\rangle - \langle y - z, 2z - x\rangle \leq 0 \quad \forall \ y \in K\} \tag{14}$$

then $F_{1(i)}$ satisfies the conditions of Lemma 7 $\forall\, i \in I$. Hence, we easily see that $F_{1(i)}$ is nonexpansive and $Fix(F_{1(i)}) = Fix(T_i) \forall i \in I$. Thus, we have the following theorem:

Theorem 11: Let K be a closed convex nonempty subset of a real Hilbert space H and let $T_i : K \longrightarrow K$ $i \in I$ be finite family of continuous pseudocontractive mappings from K into itself. Suppose that $F' = \bigcap_{i=1}^{m} F(T_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by

$$\begin{aligned} z_{n,0} = y_n;\ \ y_n &= P_K[(1-\beta_n)x_n + \beta_n u], \\ z_{n,i} &= (1-\alpha_n)x_n + \alpha_n F_i z_{n,i-1};\ \ i = 1, ..., m \\ z_{n,m} &= x_{n+1} \quad n \geq 0 \end{aligned} \tag{15}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$ $, 0 < a < \alpha_n < b < 1\ \forall\ n \geq 1$(where a,b are constants). Let $x^* \in F'$, then $\{x_n\}_{n\geq 1}$ converges strongly to $P_{F'}u$ if any of the following hold

(1) if and only if $\liminf_{n\to\infty} d(x_n, F') = 0$.
(2) one of the F_i's satisfies any of the following conditions:
   (a) condition B,
   (b) semi-compact or demi compact at 0,
   (c) completely continuous.
(3) K satisfies any of the following:
   (a) K is compact.
   (b) K is boundedly compact.

.

. Furthermore, if $u = 0, \{x_n\}_{n\geq 1}$ converges strongly to a minimum norm fixed point of the finite family.

Application to approximation of solutions of classical equilibrium problems

Let K be a closed convex nonempty subset of a real Hilbert space H. Let $f : KXK \longrightarrow \mathbb{R}$ be a bifunction. The classical equilibrium problem (abbreviated EP) for f is to find $u^* \in K$ such that

$$f(u^*, y) \geq 0 \quad \forall\ \ y\ \in K \tag{16}$$

The set of solutions of classical equilibrium problem is denoted by $EP(f)$, where $EP(f) = \{u \in K : f(u,y) \geq 0\ \forall \quad y \quad \in K\}$. The classical equilibrium problem (EP) includes as special cases the monotone inclusion problems, saddle point problems, variational inequality problems, mini- mization problems, optimization problems, vector equilibrium problems, Nash equilibria in noncooperative games. Furthermore, there are several other problems, for example, the complementarity problems and fixed point problems, which can also be written in the form of the classical equilibrium problem. In other words, the classical equilibrium problem is a unifying model for several problems arising from engineering, physics, statistics, computer science, optimization theory, operations research, economics and countless other fields. For the past 20 years or so, many existence results have been published for various equilibrium problems (see e.g.[4], [8],[20]). In the sequel, we shall require that the bifunction $f : KXK \longrightarrow R$ satisfies the following conditions: (A1) $f(x,x) = 0 \ \forall \ \ x \ \ \in K$; (A2)$f$ is monotone, in the sense that $f(x,y) + f(y,x) \leq 0 \quad \forall \ x,y \ \ \in K$; (A3)$\limsup_{t\to 0^+} f(tz + (1-t)x, y) \leq f(x,y) \ \forall \ x,y,z \ \ \in K$;. (A4) the function $y \mapsto f(x,y)$ is convex and lower semicontinuous for all $x \in K$.

Lemma 8[(compare with Lemma 2.4 of [8])] Let K be a nonempty closed convex subset of a real Hilbert space H. Let$f_i : KXK \longrightarrow R$ be finite family of bifunction satisfying conditions (A1) - (A4) for each $i \in I = \{1, 2, ..., m\}$ then for all $r > 0$ and $x \in H$, there exists $u \in K$ such that

$$f_i(u,y) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0 \quad \forall \ \ y \ \in K \ i \in I. \tag{17}$$

moreover, if for all $x \in H$ we define $G_{ir} : H \longrightarrow 2^K$ by

$$G_{ir}(x) = \{u \in K : f_i(u,y) + \frac{1}{r}\langle y-u, u-x\rangle \geq 0 \quad \forall \ \ y \ \in K.\} \tag{18}$$

then the following hold:

(1) G_{ir} is single-valued for all $r \geq 0 \; i \in I$
(2) $Fix(G_{ir}) = EP(f_i)$ for all $r > 0$
(3) $EP(f_i)$ is closed and convex

We observe that Lemma 8 holds in particular for r = 1. Thus, if we define $G_{i1} : H \longrightarrow 2^K$ by

$$G_{i1}(x) = \{u \in K : f_i(u,y) + \langle y-u, u-x\rangle \geq 0 \quad \forall \ \ y \ \in K.\} \tag{19}$$

then G_{i1} satisfies the conditions of Lemma 8 $\ \forall \, i \in I$. Hence, we easily see that G_{i1} is nonexpansive and $Fix(G_{i1}) = EP(f_i) \, \forall \, i \in I$. Thus, we have the following theorem:

Theorem 12: Let K be a closed convex nonempty subset of a real Hilbert space H and let Let$f_i : KXK \longrightarrow R$ be finite family of bifunction satisfying conditions (A1) - (A4) for each $i \in I = \{1, 2, ..., m\}$. Suppose that $F'' = \bigcap_{i=1}^{m} EP(f_i) \neq \emptyset$ and let $\{x_n\}_{n\geq 1}$

be a sequence generated iteratively by

$$\begin{aligned} z_{n,0} = y_n;\ \ y_n &= P_K[(1-\beta_n)x_n + \beta_n u], \\ z_{n,i} &= (1-\alpha_n)x_n + \alpha_n G_{i1}^n z_{n,i-1};\ \ i = 1, ..., m \\ z_{n,m} &= x_{n+1} \quad n \geq 0 \end{aligned} \tag{20}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$ $, 0 < a < \alpha_n < b < 1\ \forall\ n \geq 1$. Let $x^* \in F''$, then $\{x_n\}_{n\geq 1}$ converges strongly to $P_{F''}u$ if any of the following hold

(1) if and only if $\liminf_{n\to\infty} d(x_n, F'') = 0$.
(2) one of the G_{1i}'s satisfies any of the following conditions:
    (a) condition B,
    (b) semi-compact or demi compact at 0,
    (c) completely continuous.
(3) K satisfies any of the following:
    (a) K is compact.
    (b) K is boundedly compact.

.

. Moreso, if $u = 0, \{x_n\}_{n\geq 1}$ converges strongly to a minimum norm fixed point of the finite family.

Several authors (see e.g.[8], [14] and references therein) have studied the following problem:Let K be a closed convex nonempty subset of a real Hilbert space H. Let $f : KXK \longrightarrow \mathbb{R}$ be a bifunction and $\Phi : K \longrightarrow \mathbb{R}$ be a proper extended real valued function, where $\mathbb{R}$ denotes the real numbers. let $\Theta : K \longrightarrow H$ be a nonlinear monotone mapping. The *generalised mixed equilibrium problem* (abbreviated GMEP) for f, Φ and Θ is to find $u^* \in K$ such that

$$f(u^*, y) + \Phi(y) - \Phi(u^*) + \langle \Theta u^*, y - u^* \rangle \geq 0 \quad \forall\ \ y\ \in K \tag{21}$$

Observe that if we define $\Gamma : KXK \longrightarrow \mathbb{R}$

$$\Gamma(x, y) = f(x, y) + \Phi(y) - \Phi(x) + \langle \Theta x, y - x \rangle \tag{22}$$

then it could be easily checked that Γ is a bi-function and satisfies properties (A1)to(A4). Thus, the so called generalized mixed equilibrium problem reduces to the classical equilibrium problem for the bifunction Γ. Thus, consideration of the so called generalized mixed equilibrium problem in place of the classical equilibrium problem studied in this section leads to no further generalization.

Applications to Convex optimization

Let us look at the problem of minimizing a continuously Frechet-differentiable convex functional with minimum norm in Hilbert spaces.
Let K be a closed convex subset of a real Hilbert space H. Consider the minimization problem given by

$$\min_{x\in K}\phi(x) \tag{23}$$

where ϕ is a Frechet-differentiable convex functional. Let Ω the solution set of (23) be nonempty. It is known that a point $z\in K$ is a solution of (23) if and only if the following optimality condition holds:

$$z\in K,\ \langle\nabla\phi(z),x-z\rangle\geq 0,x\in K, \tag{24}$$

where ∇ is the gradient of ϕ at $x\in K$. It is also known that the optimality condition (24) is equivalent to the following fixed point problem:

$$z=T_\gamma(z),\text{where}T_\gamma:=P_K(I-\gamma\nabla\phi). \tag{25}$$

for all $\gamma>0$ So we have the following corollary deduced from Theorem 1 and Lemma 5
Theorem 13: Let H be a real Hilbert space, let K be a closed convex nonempty subset of H. Let ψ be a continuously Frechet-differentiable convex functional on K such that $T_{\gamma_{(i)}}:=P_K(I-\gamma_{(i)}\nabla\psi)$ be finite family of uniformly continuous asymptotically nonexpansive in intermediate sense mapping from K into itself with sequences $\{\mu_{in}\}_{n\geq 1},\{\eta_{in}\}_{n\geq 1}\subset[0,+\infty)$ such that $\sum_{n=0}^{\infty}\mu_{n,i}<\infty$, $\sum_{n=0}^{\infty}\eta_{n,i}<\infty$. Suppose that $F=\bigcap_{i=1}^m F(T_{\gamma_{(i)}})\neq\emptyset$ and let $\{x_n\}_{n\geq 1}$ be a sequence generated iteratively by

$$\begin{aligned} y_n &= P_K[(1-\beta_n)x_n+\beta_n u],\\ z_{n,i} &= (1-\alpha_n)x_n+\alpha_n[P_K(I-\gamma_{(i)}\nabla\psi)]z_{n,i-1};\\ & i=1,...,m \end{aligned} \tag{26}$$

$$z_{n,0}=y_n;\ z_{n,m} = x_{n+1}\quad n\geq 0 \tag{27}$$

where $\{\alpha_n\}_{n\geq 1}, \{\beta_n\}_{n\geq 1}$ are sequences in $(0,1)$ such that $\sum_{n=0}^{\infty} \beta_n < \infty$, $0 < a < \alpha_n < b < 1 \; \forall \; n \geq 1$. Let $x^* \in F$, then $\{x_n\}_{n\geq 1}$ converges strongly to the minimum norm solution of the minimization problem (23) if any of the following hold

(1) if and only if $\liminf_{n\to\infty} d(x_n, F) = 0$.
(2) one of the $T_{\gamma_{(i)}}$'s satisfies any of the following conditions:
  (a) condition B,
  (b) semi-compact or demi compact at 0,
  (c) completely continuous.

4. CONCLUDING REMARKS

Our iterative process generalises some of the existing ones. Our theorems improve, generalise and extend several known results and our method of proof is of independent interest.

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