Journal of the Nigerian Mathematical Society
Vol. 35, pp. 408-423, 2016

A DOUBLE EXPONENTIAL SINC COLLOCATION METHOD FOR VOLTERRA-FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

ENO D. JOHN AND NKEM OGBONNA[1]

ABSTRACT. A Sinc collocation method for numerical solution of Volterra-Fredholm integral equations of the second kind is developed by incorporating a variable transformation of double exponential order into the Sinc function expansion technique. The derived Sinc collocation formula is used to convert a Volterra-Fredholm integral equation defined on a finite interval into a set of algebraic equations. Numerical examples are presented to show the rapid convergence and exceptional accuracy of the method.

Keywords and phrases: Sinc collocation, Volterra-Fredholm equation, modified variable transformation, double exponential.
2010 Mathematical Subject Classification: 45B05, 65L60

1. INTRODUCTION

We consider integral equations of the form:

$$u(x) = \lambda_1 \int_a^x k_1(x,t)u(t)dt + \lambda_2 \int_a^b k_2(x,t)u(t)dt + g(x), \tag{1}$$

where $(a \leq x \leq b)$, λ_1 and λ_2 are constants, $k_1(x,t)$, $k_2(x,t)$ and $g(x)$ are given analytic functions, and $u(x)$ is the solution to be determined. Equation (1) is called a Volterra-Fredholm integral equation of the second kind [1]. Integral equations of this type arise from mathematical modelling of some physical and biological situations, such as the spatio-temporal development of an epidemic [2]. The existence of solution for equation (1) has been established [3] but, in many cases, the equation defies analytical methods for exact solution, making approximate solutions desirable. Some of the well-known techniques for its solution include the Adomian decomposition method ([2], [4]), the Taylor series method ([2], [5])

Received by the editors June 18, 2015; Revised: September 22, 2016; Accepted: October 31, 2016
[1]Corresponding author

and the Trapezoidal-Nystrom method [6]. Generally, the convergence rate of these methods is of polynomial order with respect to the number, N, of terms of the series approximation.

An issue of considerable interest concerns the development of highly accurate methods for numerical solution of integral equations of type (1). It is well known that polynomials or associated splines are traditionally used as basis functions in many numerical methods such as multistep methods, finite difference method (FDM), boundary element method (BEM) and projection methods (e.g. Galerkin, collocation, finite element methods). Such polynomial-based approximation methods have convergence rates of polynomial order and are plagued by difficulty in handling problems that involve semi-infinite or infinite domains and singularities [7]. Extensive studies by Stenger ([8] - [10]) have drawn sustained attention to the use of Sinc functions, rather than polynomials, in numerical approximation techniques. Unlike polynomial-based approximation methods, Sinc-based numerical methods are characterized by rapid convergence rates of exponential order. Also, they are effective in handling problems with singularities and they are applicable over finite, semi-infinite as well as infinite domains. These features distinguish Sinc methods from conventional polynomial-based numerical methods.

On account of their high efficiency, Sinc numerical methods have become valuable tools for numerical solution of integral equations. In particular, the Sinc collocation method has been used to solve Volterra integral equations ([11] - [13]), Hammerstein integral equations [14], Fredholm integral equations ([15] - [17]) and Volterra-Fredholm integral equations ([18], [19]). Single exponential(SE) Sinc approximations have been widely used in Sinc numerical methods for solution of integral equations (see, for example, ([12]-[16])). Typically, the error in SE-Sinc method is $\mathrm{O}\left(\sqrt{N}\exp\left[-c\sqrt{N}\right]\right)$, where $c > 0$ and N represents the number of terms of the Sinc expansion. For further improvement of the convergence rate of Sinc approximations, Muhammad and Mori [20] have proposed the use of double exponential (DE) approximations, so called because they have convergence rates of double exponential order. DE-Sinc methods, characterized by convergence rates of $\mathrm{O}\left(\exp\left[\frac{-c_1 N}{\log c_2 N}\right]\right)$, have been successfully employed to solve Volterra integral equations of first and second kind (see, for example, [21], [22]).

In this paper, we extend the application of the DE-Sinc numerical method to the solution of Volterra-Fredholm integral equations of the second kind defined on a finite interval. We shall construct a DE-transformation that maps the real line onto a finite interval, and use it in conjunction with the Sinc function expansion technique to derive a collocation formula for conversion of integral equations of type (1) into a system of algebraic equations. We shall, also, illustrate the efficiency and accuracy of the derived Sinc collocation formula with some numerical examples.

2. PRELIMINARIES

2.1. **Sinc function and approximation on the real line.** The Sinc function is defined for all real numbers, t, by

$$\mathrm{Sinc}(t) = \begin{cases} \frac{\sin \pi t}{\pi t}, & (t \neq 0) \\ 1, & (t = 0) \end{cases}. \tag{2}$$

The Sinc approximation of a function $f(t)$ on the real line $\mathbb{R}$ is given by

$$f(t) \approx \sum_{j=-N}^{N} f(jh) S(j,h)(t), \quad t \in \mathbb{R}, \tag{3}$$

where $S(j,h)(t)$ is the Sinc basis function defined by

$$S(j,h)(t) = \frac{\sin\left[\pi\left(\frac{t}{h} - j\right)\right]}{\pi\left(\frac{t}{h} - j\right)}, \quad (j = 0,\ \pm 1,\ \pm 2,\ \cdots) \tag{4}$$

and $h > 0$ is a step size suitably chosen for a given positive integer N. At interpolating points $t_k = kh$, the Sinc basis function takes the form:

$$S(j,h)(kh) = \begin{cases} 0, & k \neq j \\ 1, & k = j \end{cases}. \tag{5}$$

By integrating both sides of equation (3) over the entire real line, the Sinc quadrature is obtained as

$$\int_{-\infty}^{\infty} f(t)dt \approx \sum_{j=-N}^{N} f(jh) \int_{-\infty}^{\infty} S(j,h)(t)dt = h \sum_{j=-N}^{N} f(jh). \tag{6}$$

The general Sinc indefinite integration associated with equation (3) is as follows: [23]

$$\int_{-\infty}^{\infty} f(s)ds \approx \sum_{j=-N}^{N} f(jh) \int_{-\infty}^{t} S(j,h)(t)dt = h \sum_{j=-N}^{N} f(jh) J(j,h)(t), \tag{7}$$

where

$$J(j,h)(t) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}\left(\frac{\pi t}{h} - j\pi\right), \tag{8}$$

and $\mathrm{Si}(t)$ is the sine integral function defined by

$$\mathrm{Si}(t) = \int_{-\infty}^{t} \frac{sinu}{u} du. \tag{9}$$

2.2. **DE-Sinc approximation on a finite interval** $\Gamma = [a, b]$**.** Whereas the Sinc function (2) is defined for functions with domain on the entire real line, the domain of the integral equation (1) which we need to solve is the finite interval Γ. For approximation of a function $f(t)$ on a finite interval, an appropriate variable transformation which maps the real line onto the finite interval must be incorporated into the Sinc series approximation (3). For our purpose, we require a variable transformation of double exponential (DE) order.

Tanaka et al. [24] have shown that a variable transformation for successful double exponential (DE) Sinc approximation on an interval can be constructed by appropriately modifying a compatible single exponential (SE) transformation defined on that interval. This procedure has been employed in the study of Volterra integral equations ([22], [25]) and will be adopted here. For this purpose, we consider the SE-transformation

$$x = \sigma_{SE}(t) = \frac{a + be^t}{1 + e^t}, \quad t \in \mathbb{R}, \tag{10}$$

with the inverse transform as

$$t = \sigma_{SE}^{-1}(x) = \log\left(\frac{x-a}{b-x}\right), \quad x \in \Gamma.$$

The transformation (10) is frequently used in Sinc numerical methods for solution of integral equations (see, for example, ([12] - [16]). We construct a DE-transformation which is compatible with the SE-transformation (10) by replacing e^t with $\exp\left(\frac{\pi}{2}\sinh t\right)$, in accordance with the function classes proposed by Tanaka et al. [24]. Then, our modified variable transformation of double exponential order which maps the infinite interval $(-\infty, \infty)$ onto the finite interval $[a, b]$ is

$$x = \sigma_{DE}(t) = \frac{a + b\exp\left(\frac{\pi}{2}\sinh t\right)}{1 + \exp\left(\frac{\pi}{2}\sinh t\right)}, \quad t \in \mathbb{R}, \tag{11}$$

with the inverse transform as

$$t = \sigma_{DE}^{-1}(x) = \sinh^{-1}\left\{\frac{2}{\pi}\log\left(\frac{x-a}{b-x}\right)\right\}, \quad x \in \Gamma. \tag{12}$$

$$= \log\left[\frac{2}{\pi}\log\left(\frac{x-a}{b-x}\right) + \sqrt{\frac{2}{\pi}\log\left(\frac{x-a}{b-x}\right)^2 + 1}\right], \quad x \in \Gamma. \tag{13}$$

The map σ_{DE} carries $\mathbb{R}$ onto Γ, such that $\sigma_{DE}(-\infty) = a$, $\sigma_{DE}(\infty) = b$ and, for $h > 0$, the collocation points in Γ are $x_j = \sigma_{DE}(jh)$, $(j = 0, \pm 1, \pm 2, \cdots)$. When incorporated with the DE transformation (11), the Sinc approximation (3) can be applied to a function $f(x)$ defined on the interval Γ to obtain the DE-Sinc approximation as follows:

$$f(x) \approx \sum_{j=-N}^{N} f(\sigma_{DE}(jh))\, S(j,h)\left(\sigma_{DE}^{-1}(x)\right), \quad x \in \Gamma. \tag{14}$$

Then, the DE-Sinc indefinite integration on the finite interval Γ is given by

$$\int_a^x f(s)ds \approx \sum_{j=-N}^{N} f(\sigma_{DE}(jh))\left(\sigma_{DE}^{'}(x)\right) J(j,h)\left(\sigma_{DE}^{-1}(x)\right). \tag{15}$$

2.3. **Convergence Theorems for DE-Sinc Approximation**. The convergence theorems for the DE-Sinc approximation (14) require the specification of appropriate function spaces which are defined below with reference to the complex plane.

Definition 1: [24] Let D be a bounded and simply connected domain. Then, $H^\infty(D)$ denotes the family of functions $f \in Hol(D)$ such that $\|.\|_{H^\infty(D)}$ is finite, where

$$\|f\|_{H^\infty(D)} = \sup_{z \in D} |f(z)|. \tag{16}$$

Definition 2: [24] Let $\alpha > 0$ be a constant, and let D be a bounded and simply-connected domain which satisfies $(a,b) \subset D$. Then, $L_\alpha(D)$ denotes the family of functions $f \in H^\infty(D)$ for which there exists a constant C such that, for all $z \in D$,

$$|f(z)| \le C|Q(z)|^\alpha, \tag{17}$$

where $Q(z) = (z-a)(b-z)$.

Definition 3: [17] Let D be a bounded and simply connected domain, and let $HC(D)$ denote the family of all functions $f \in$

$Lip_\alpha\left(\overline{D}\right) \cap Hol(D)$. Then, the function space $HC(D)$ is complete with norm $\|.\|_{HC(D)}$ defined by

$$\|f\|_{HC(D)} = \max_{z\in\overline{D}} |f(z)| . \tag{18}$$

Definition 4: [26] Let D be a bounded and simply connected domain that satisfies $(a, b) \subset D$ and let $\alpha \in (0, 1]$ be a constant. Then, $M_\alpha(D)$ denotes the class of functions $f \in HC(D)$ which have finite limits, $f(a)$ and $f(b)$, at the endpoints of (a, b) such that

$$f(x) - f(a) = \mathrm{O}\left(|\rho\left(x\right)|^{\alpha}\right) \text{ as } x \to a$$

and

$$f(x) - f(b) = \mathrm{O}\left(|\rho(x)|^{-\alpha}\right) \text{ as } x \to b, \tag{19}$$

where

$$\rho(x) = \exp\left(\sigma^{-1}(x)\right).$$

For the purpose of DE-Sinc approximation addressed in this paper, the function σ and domain D in Definitions 1 - 4 are replaced by σ_{DE} and $D_{DE}(d)$, respectively, where for $d > 0$,

$$D_{DE}(d) = \{z \in \sigma_{DE}(w) : w \in D_d\} \tag{20}$$

is the image of the region

$$D_d = \{z = x + iy \in \mathbb{C} : |y| < d\} . \tag{21}$$

in the complex plane $\mathbb{C}$, under the DE-transformation σ_{DE} given by equation (11). Then, similar to Stenger [9] and Tanaka et al. [24], we state the convergence theorem for the DE-Sinc approximation (14) as follows:

Theorem 1: Let $f \in L_\alpha(D_d)$, $0 < d < \frac{\pi}{2}$, and let h be given by

$$h = \frac{\log\left(4dN/\alpha\right)}{N}, \tag{22}$$

where N is a positive integer. Then, there exists a constant C which is independent of N, such that

$$\begin{aligned}\max_{x\in[a,b]} &\left| f\left(t\right) - \sum_{j=-N}^{N} f\left(\sigma_{DE}\left(jh\right)\right) S\left(j,h\right)\left(\sigma_{DE}^{-1}(x)\right)\right| \\ &\leq \quad C \exp\left\{\frac{-\pi d N}{\log\left(\frac{4dN}{\alpha}\right)}\right\}.\end{aligned} \tag{23}$$

By Definition 2 of the function space $L(D)$, Theorem 1 holds only for functions $f(x)$ which vanish at the end points $x = a$ and $x = b$. The generalized DE-Sinc approximation that achieves exponential convergence without this restriction on $f(x)$ is obtained by considering the function space $M_\alpha(D)$ specified by Definitions 3 and 4. This is facilitated by defining a translated function

$$T[f](x) = f(x) - \frac{f(a) + \rho(x) f(b)}{1 + \rho(x)}. \tag{24}$$

Then, $f \in M_\alpha(D)$ implies that $T[f] \in L_\alpha(D)$. Therefore, we can apply DE-Sinc approximation (14) to $T[f]$ and obtain

$$T\,[f]\,(x) \approx \sum_{j=-N}^{N} T\,[f]\,(\sigma_{DE}(jh))\, S(j,h)\left(\sigma_{DE}^{-1}(x_j)\right). \tag{25}$$

By equations (24) and (25),

$$\begin{aligned} f(x) \;\approx\; P_N\,[f]\,(x) \;=\; & \sum_{j=-N}^{N} T\,[f]\,(\sigma_{DE}\,(jh))\, S\,(j,h)\left(\sigma_{DE}^{-1}\,(x_j)\right) \\ & + \quad \frac{f(a) + \rho(x) f(b)}{1 + \rho(x)}. \end{aligned} \tag{26}$$

Thus, the generalized DE-Sinc approximation to $f(x)$ may be expressed as

$$\begin{gathered} P_N[f](x) = f(a) w_a(x) + \\ \sum_{j=-N}^{N} T[f]\,(\sigma_{DE}(jh)) S(j,h)\left(\sigma_{DE}^{-1}(x_j)\right) + f(b) w_b(x), \end{gathered} \tag{27}$$

where w_a and w_b are auxiliary basis functions defined by

$$w_a(x) = \frac{1}{1 + \rho(x)}, \qquad w_b(x) = \frac{\rho(x)}{1 + \rho(x)}. \tag{28}$$

Similar to Theorem 1, we state the convergence theorem for the generalized DE-Sinc approximation (27) as follows:

Theorem 2: Let $f \in M_\alpha\left(\sigma_{DE}\left(D_d\right)\right)$, $0 < d < \frac{\pi}{2}$, and let h be given by (22), where N is a positive integer. Then, there exists a constant C which is independent of N, such that

$$\|f - P_N[f]\|_{C[a,b]} \le C \exp\left\{ \frac{-\pi d N}{\log\left(\frac{4dN}{a}\right)} \right\}. \tag{29}$$

3. THE SINC-COLLOCATION METHOD

Let $u(x) \in M_\alpha\left(\sigma_{DE}\left(D_d\right)\right)$ be the exact solution of the Volterra-Fredholm equation (1), and let $u_N(x)$ be the DE-Sinc approximation of $u(x)$. Denote $u_N(x_i)$ by u_i , where

$$x_i = \begin{cases} a, & i = -(N+1), \\ \sigma_{DE}\left(ih\right), & i = -N, \ \cdots, \ N, \\ b, & i = N+1. \end{cases} \tag{30}$$

are the collocation points. Then, by equation (27),

$$u_N(x) = u_{-N-1}w_a(x) + \sum_{j=-N}^{N} u_j S(j,h)\left(\sigma_{DE}^{-1}(x_j)\right) + u_{N+1}w_b(x). \tag{31}$$

Noting that

$$S(j,h)\left(\sigma_{DE}^{-1}(x_j)\right) = S(j,h)\left(\sigma_{DE}^{-1}\left(\sigma_{DE}(jh)\right)\right) = S(j,h)(jh) = \delta_{ij}, \tag{32}$$

the application of equation (31) to the integrals in equation (1) yields the following approximations:

$$\begin{aligned} \int_a^x k_1(x,t)u(t)\,dt \approx{}& K_v[w_a](x)u_{-N-1} \\ &+h\sum_{j=-N}^{N} k_1(x,t_j)\sigma'_{DE}(jh)u_j J(j,h)(x) \\ &+ K_v[w_b](x)u_{N+1} + \mathrm{O}\left(he^{-\frac{\pi d}{h}}\right), \end{aligned} \tag{33}$$

and

$$\begin{aligned} \int_a^b k_2(x,t)u(t)dt \approx{}& K_F[w_a](x)u_{-N-1} + h\sum_{j=-N}^{N} k_1(x,t_j)\sigma'_{DE}(jh)u_j \\ &+ K_F[w_b](x)u_{N+1} + \mathrm{O}\left(he^{-\frac{\pi d}{h}}\right). \end{aligned} \tag{34}$$

In equations (33) and (34),

$$K_v\left[F\right](x) = h\sum_{j=-N}^{N} k_1(x,t_j)f(x_j){\sigma_{DE}}'(jh)J(j,h)(x), \tag{35}$$

$$K_F\left[F\right](x) = h\sum_{j=-N}^{N} k_1\left(x,t_j\right)f\left(x_j\right)\sigma'_{DE}\left(jh\right), \tag{36}$$

$$J(j,h)(x) = \frac{1}{2} + \frac{1}{\pi}\mathrm{Si}\left(\frac{\pi\sigma_{DE}^{-1}(x)}{h} - j\pi\right), \tag{37}$$

where $\sigma'_{DE}(x)$ is the derivative of $\sigma_{DE}(x)$ and is given at the sinc points by

$$\sigma'_{DE}(jh) = \frac{\pi}{2}\cosh(jh)\,\sigma_{DE}(jh)\left[1-\sigma_{DE}(jh)\right], \tag{38}$$

while $Si(x)$ is the sine integral defined by equation (9). The substitution of equations (33) to (38) into the integral equation (1) gives its DE-Sinc collocation approximation as

$$\begin{aligned} &[w_a(x_k) - K_v[w_a](x_k) - K_F[w_a](x_k)]\,u_{-N-1} \\ &+ \sum_{j=-N}^{N} \delta_{kj} - [hk_1(x_k,t_j)J(j,h)(x_k) + hk_2(x_k,t_j)]\,\sigma'(jh)u_j \\ &+ [w_b(x_k) - K_v[w_b](x_k) - K_F[w_b](x_k)]\,u_{N+1} = g(x_k). \end{aligned} \tag{39}$$

Equation (39) represents a $(2N+3)$ by $(2N+3)$ linear system of algebraic equations, which may be expressed in matrix form as

$$\mathbf{Au} = \mathbf{g}, \tag{40}$$

where

$$\mathbf{u} = [u_{-N-1}, u_{-N}, \cdots, u_N, u_{N+1}]^T,$$

$$\mathbf{g} = [g(a), g(x_{-N}), \cdots, g(x_N), g(b)]^T,$$

and the coefficients a_{kj} of the matrix $\mathbf{A}$ are given by

$$a_{kj} = e_{kj} - v_{kj} - s_{kj} \tag{41}$$

with

$$e_{kj} = w_a(x_k) + \sum_{j=-N}^{N} S(j,h)\left(\sigma_{DE}^{-1}(x_k)\right) + w_b(x_k), \tag{42}$$

$$v_{kj} = K_V[w_a](x_k) + h\sum_{j=-N}^{N} k(x_k,t_j)\sigma'_{DE}(jh)J(j,h)(x_k) + K_V[w_b](x_k), \tag{43}$$

$$s_{kj} = K_F[w_a](x_k) + h\sum_{j=-N}^{N} k(x_k,t_j)\sigma'_{DE}(jh) + K_F[w_b](x_k). \tag{44}$$

Consistent with the requirements for a well-conditioned linear system of algebraic equations, the coefficient matrix of the system (40) must be non-singular and the condition number, $\kappa(\mathbf{A}) = \|\mathbf{A}\|\|\mathbf{A}^{-1}\|$, must not be very large. These conditions ensure the consistency and stability of the system as well as the convergence of its numerical solution. The approximate solution to the integral equation (1) is obtained by solving the system (40) and substituting the result into equation (31). The error in the approximate solution given by the Sinc collocation method is obtained by bounding the difference $u(x_k)-u_N(x_k)$ in the maximum norm, where $u(x_k)$ is the exact solution and $u_N(x_k)$ is the approximate solution.

4. NUMERICAL EXAMPLES

Some examples are presented in this section to illustrate the effectiveness of the DE-Sinc collocation formula (31). Numerical calculations were carried out for $\alpha = 1$ and $d = \frac{\pi}{4}$ so that the step size is $h = \frac{\log(\pi N)}{N}$. MATLAB was used for the computations and the condition of the coefficient matrix of the system (40) was watched through an estimate of the condition number returned by the program. All the examples considered have exact solutions, making it possible to compute the error of the approximate solution at the Sinc points. The criteria for interpreting the error and accuracy of the DE-Sinc approximation are:

(1) The maximum absolute error $|E_N(h(\sigma))|$ between the exact solution $u(x)$ and the approximate solution $u_N(x)$ at the sinc points x_k. This is defined, with respect to L_∞ norm, by

$$|E_N(h(\sigma))| = \max_{k=-N-1,N,\cdots,N,N+1} |u(x_k) - u_N(x_k)|. \tag{45}$$

(2) The condition number $\kappa(\mathbf{A})$ based on infinity norm and is given by

$$\kappa(\mathbf{A}) = \|\mathbf{A}\|_\infty \|\mathbf{A}^{-1}\|_\infty, \tag{46}$$

where $\mathbf{A}$ is the coefficient matrix of the system (40).

Example 1. Consider the integral equation

$$u(x) = \int_0^x (x-t)u(t)dt + \int_0^1 xu(t)dt + 2e^x - 2x - 2, \ (0 \le x \le 1). \tag{47}$$

Its exact solution, obtained by series method [2, page 266], is $u(x) = xe^x$. The approximate solution obtained for this example by the DE-Sinc collocation formula (31) with $N = 10$ is compared with the exact solution in Figure 1. It is clearly seen that the computed results are remarkably accurate. Table 1 shows the maximum absolute error obtained at

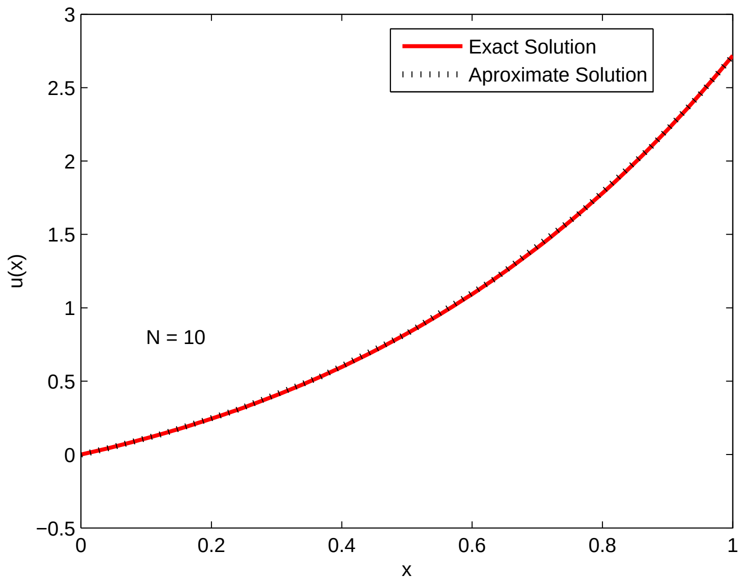

Fig. 1. Exact and approximate solutions of Example 1 when $N = 10$.

collocation points and the condition number of the coefficient matrix of the system of equations solved, for various values of N. The tabulated results show that the condition number $\kappa(\mathbf{A}) \leq 16.5$ for all values of N, confirming the stability of the system of equations given by the DE-Sinc collocation method.

Table 1. Maximum absolute error ($|E_N(h(\sigma))|$) and condition number ($\kappa(\mathbf{A})$) for Example 1

N	h	$\lvert E_N(h(\sigma))\rvert$	$\kappa(\mathbf{A})$
10	0.3447	5.1770×10^{-6}	1.62×10^{1}
15	0.2569	6.1644×10^{-8}	1.65×10^{1}
20	0.2070	1.1732×10^{-9}	1.65×10^{1}
25	0.1745	2.8807×10^{-11}	1.65×10^{1}
30	0.1515	6.7413×10^{-13}	1.65×10^{1}
35	0.1343	1.6875×10^{-14}	1.65×10^{1}
40	0.1208	1.7764×10^{-15}	1.65×10^{1}
45	0.1100	1.7764×10^{-15}	1.65×10^{1}
50	0.1101	2.2204×10^{-15}	1.65×10^{1}

Also, from Table 1 and Figure 2, it is seen that the maximum absolute error in the approximate solution decays rapidly as the number N of terms in the DE-Sinc approximation increases. For example, its value

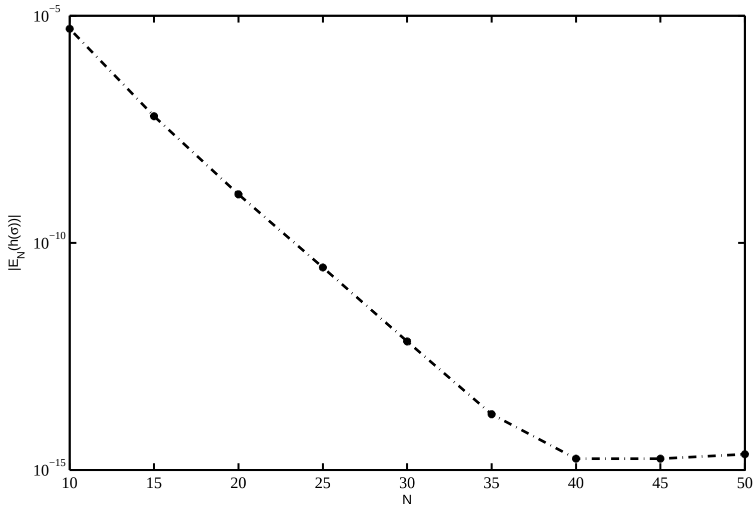

Fig. 2. Variation of maximum absolute error $|E_N(h(\sigma))|$ with N for Example 1.

drops from 5.1770×10^{-6} to 1.7764×10^{-15} as N increases from 10 to 40. Therefore, we can say that the DE-Sinc collocation method is quite stable and rapidly convergent for this example.

Example 2. Consider the integral equation

$$u(x) = \int_0^x (x-t)u(t) + \int_0^1 (x+t)u(t)dt - \frac{1}{12}x^4 + x^2 - \frac{1}{3}x - \frac{1}{4} \quad (48)$$

where $0 \leq x \leq 1$. Wazwaz [2, page 268] used the modified Adomian decomposition method to obtain the exact solution for this problem as $u(x) = x^2$. Figure 3 shows a comparison of the exact solution with the approximate solution obtained by the DE-Sinc collocation formula (31) with $N = 10$. The exceptional accuracy of the derived formula is evident. The maximum absolute error and condition number for different values of N are presented in Table 2 while the variation of the maximum absolute error with N is shown in Figure 4. Similar to results obtained for Example 1, the tabulated results show that the condition number $\kappa(\mathbf{A}) \leq 39.6$ for all values of N, confirming the stability of the system of equations given by the DE-Sinc collocation method for this example.

Also, from Table 2 and Figure 4, it is seen that the maximum absolute value of the error of the numerical solution decays rapidly as N increases. In this case, its value drops from 2.0033×10^{-6} at $N = 10$ to 9.9920×10^{-16} at $N = 40$.

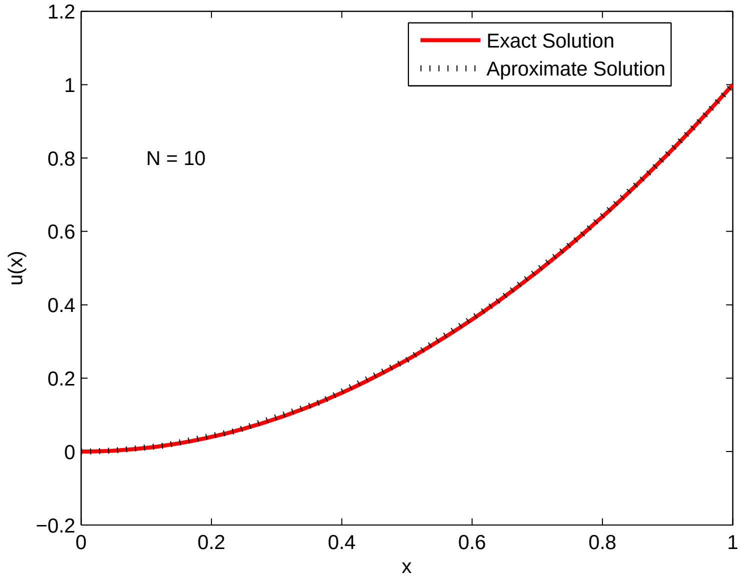

Fig. 3. Exact and approximate solutions of Example 2 when $N = 10$.

Table 2. Maximum absolute error ($|E_N(h(\sigma))|$) and condition number ($\kappa(\mathbf{A})$) for Example 2

N	h	$\|E_N(h(\sigma))\|$	$\kappa(\mathbf{A})$
10	0.3447	2.0033×10^{-6}	3.96×10^{1}
15	0.2569	1.9647×10^{-8}	3.96×10^{1}
20	0.2070	4.1832×10^{-10}	3.96×10^{1}
25	0.1745	1.0587×10^{-11}	3.96×10^{1}
30	0.1515	2.2882×10^{-13}	3.96×10^{1}
35	0.1343	0.5511×10^{-15}	3.96×10^{1}
40	0.1208	0.9920×10^{-16}	3.96×10^{1}
45	0.1100	0.8818×10^{-16}	3.96×10^{1}
50	0.1101	5.5551×10^{-16}	3.96×10^{1}

5. CONCLUDING REMARKS

In this paper, a double exponential (DE) Sinc collocation formula has been derived for numerical solution of Volterra-Fredholm integral equation of the second kind defined on a finite interval. The rapid convergence and exceptional accuracy of the derived DE-Sinc scheme were clearly illustrated with numerical examples. This paper extends the

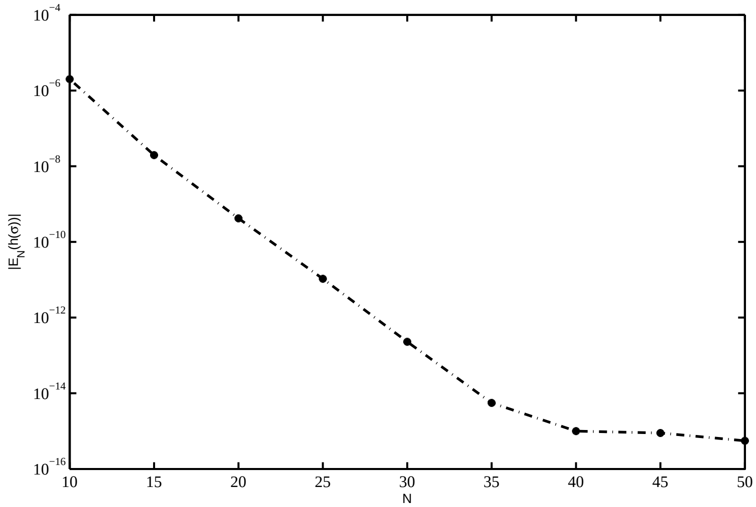

Fig. 4. Variation of maximum absolute error $|E_N(h(\sigma))|$ with N for Example 2.

work of Muhammad et. al. [21] on numerical solution of integral equations to Volterra-Fredholm equations. A rigorous theoretical convergence analysis is necessary to establish the error bounds for Sinc collocation methods for Volterra-Fredholm integral equations of the second kind. This aspect will be considered in a subsequent work.

ACKNOWLEDGEMENT

E.D. John is grateful for the support provided by Tertiary Education Trust Fund (TETFUND) of Nigeria for a study visit to Michael Okpara University of Agriculture, Umudike, to carry out this work. The authors thank the anonymous referees for their valuable comments.

References

[1] B. G. Pachppatte, *On a Certain Volterra-Fredholm Type Integral Equations*, Journal of Inequalities in Pure and Applied Mathematics, **9**, Art. 116, 12pp, (2008).

[2] A-M. Wazwaz, *Linear and Nonlinear Integral Equations, Methods and Applications*, Higher Education Press, Beijing and Springer-Verlag, Heidelberg, (2011).

[3] S. Andras, *Weakly Singular Volterra and Fredholm-Volterra Integral Equations*, Studia Univ. Babes Bolyai Mathematica, XLVIII (3), (2003).

[4] E. Deeba and S. Xie, *Numerical approximation of Integral Equations*, International Journal of Mathematics and Mathematical Sciences, **17-20** 10671076, (2004).

[5] K. Maleknejad and Y. Mehomoidi, *Taylor polynomials solution of high order non-linear Volterra-Fredholm integro-differential equations*, Journal of Applied Mathematics and Computation, **145** 641-653, (2003).

[6] G. Han and L. Zhang, *Asymptotic expansion for the trapezoidal Nystrom method of linear Volterra Fredholm equations*, Journal of Computation and Applied Mathematics, **51** 339-348,(1994).

[7] S. Narasimhan, J. Majdalani and F. Stenger, *A First Step in applying the Sinc-collocation method to the non-linear Navier-Stokes equations*, Numerical Heat Transfer Part B **41** 447-462, (2002).

[8] F. Stenger, F. *Numerical Methods Based on Sinc and Analytic Functions*, Springer-Verlag, New York, (1993).

[9] F. Stenger, *Summary of Sinc numerical methods*, Journal of Computational and Applied Mathematics, **121** 379-420, (2000).

[10] F. Stenger, *Handbook of Numerical Methods*, CRC Press, Taylor and Francis Group, Boca Raton (2011).

[11] T. S. Carlson, J. Dockery and J. Lund, *Sinc-collocation method for initial value problems*, Mathematics of Computation, **66** 215-235, (1997).

[12] J. Rashidinia M. and Zarebnia, *New Approach for numerical solution of Volterra integral equations of the second kind*, IUST International Journal of Engineering Science, **19** 59-65, (2008).

[13] M. Zarebnia and J. Rashidinia, *Convergence of the Sinc method applied to Volterra equations*, Applications and Applied Mathematics, **5** 198-216, (2010).

[14] J. Rashidinia, and M. Zarebnia, *New approach for numerical solutions of Hammerstein integral equations*, Applied Mathematics and Computation, **185** 147154, (2007).

[15] J. Rashidinia and M. Zarebnia, *Numerical solution of linear integral equations by using sinc-collocation method*, Applied Mathematics and Computation, **168** 806-822, (2005).

[16] K. Maleknejad, R. Mollapourasl and M. Alizadeh, *Convergence analysis for numerical solution of Fredholm integral equation by sinc approximation*, Communications in Nonlinear Science and Numerical Simulation, **16** 2478-2485, (2011).

[17] T. Okayama, T. Matsuo, and M. Sugihara, M. *Improvement of Sinc-collocation method for Fredholm integral equations of the second kind*, BIT Numerical Mathematics, **51** 339-366, (2011).

[18] Shamloo, A.S., Shahkar, S. And Madadi, A. *Numerical solution of Fredholm-Volterra integral Equation by the Sinc function*, American Journal of Computational Mathematics, **2** 136-142, (2012).

[19] Zarebnia, M. *A numerical solution of non-linear Volterra-Fredholm integral equations*, Journal of Applied Analysis and Computation, **3** 95-104, (2013).

[20] Muhammad, M. and Mori, M. (2005). *Double exponential formulas for numerical indefinite integration*, Journal of Computational and Applied Mathematics, **161** 431-448, (2005).

[21] Muhammad, M., Nurmuhammad, A. and Mori, M. *Numerical solution of integral equations by means of the Sinc collocation method based on Double Exponential Transformation*, Journal of Computational and Applied Mathematics, **177** 267-286, (2005).

[22] Mori, A., Nurmuhammad, A. and Murai, T. *Numerical solution of Volterra integral equations with weakly singular kernel based on the DE-Sinc method*, Japan Journal of Applied Mathematics, **25** 65-183, (2008).

[23] S. Haber, *Two formulas for indefinite numerical integration*, Mathematics of Computation, **60** 279-296, (1993).

[24] K. Tanaka, M. Sugihara and K. Murota, *Function classes for successful DE-sinc approximations*, Mathematics of Computation, **267** 1553-1571, (2009).

[25] Okayama, T. Matsuo, and M. Sugihara, *Theoretical analysis of Sinc-Nystrom methods for Volterra integral equations*, Mathematics of Computation, **84** 1189-1215, (2015).

[26] F. Stenger, *Sinc Approximation of Cauchy-type integrals over arcs*, Journal of the Australian Mathematical Society, **42** 87-97, 2000.

DEPARTMENT OF GENERAL STUDIES AKWA IBOM POLYTECHNIC, IKOT OSURUA, NIGERIA.

E-mail address: arikpo70@gmail.com

DEPARTMENT OF MATHEMATICS, MICHAEL OKPARA UNIVERSITY OF AGRICULTURE, UMUDIKE, ABIA STATE, NIGERIA.

E-mail address: ogbonna.n42@gmail.com