Journal of the Nigerian Mathematical Society
Vol. 37, Issue 3, pp. 139-153, 2018

THE CONCEPT OF RHOTRIX QUASIGROUPS AND RHOTRIX LOOPS

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ABSTRACT. This work chronicles the existing articles on rhotrices as associative rhotrix theory, and introduces non-associative binary multiplications as alternative methods of multiplication of rhotrices. The concept of "rhotrix quasigroups and rhotrix loops" is investigated and presented. This new concept is referred to in this work, as non- associative rhotrix theory.

Keywords and phrases: Rhotrix, non-associative multiplications, quasigroups, and loops
2010 Mathematical Subject Classification: primary 15B99; secondary 08A05.

1. INTRODUCTION

In 1998, Atanassov and Shannon[16] discussed mathematical arrays that are in some way, between two-dimensional vectors and (2×2)-dimensional matrices in their paper denoted matrix-tertions and noitrets. Extending this idea Ajibade[6] in 2003 introduced objects which are in some ways, between (2×2)-dimensional and (3×3) dimensional matrices. Such an object is called a rhotrix in [6], and went further to define a real rhotrix by

$$R = \{\left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle : a, b, c, d, e \ \in \mathfrak{R}\}$$

where $c = h(R)$ is called the heart of any rhotrix R and $\mathfrak{R}$ is the set of real numbers, and noted that R is the set of all 3-dimensional rhotrices. However, the paper observed that an extension of this set was possible in various ways, and also noted that the name rhotrix was a result of the rhomboid nature of the arrangement of its entries. The stage was then set to examine the properties of these objects which are geometric in appearance but algebraic in

Received by the editors January 13, 2017; Revised: November 21, 2018; Accepted: December 14, 2018
www.nigerianmathematicalsociety.org; Journal available online at www.ojs.ictp.it
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content. Interestingly, the later was examined in [6]. Again, the paper observed that the set R of all rhotrices is a commutative algebra under the method of multiplication defined. Noteworthily, a far-reaching observation was made in [6] that multiplication on rhotrices can be defined in many ways. The following year, in 2004, Sani[40] introduced an alternative method of rhotrix multiplication. This new method was not commutative unlike the former. These two definitions set a bearing for researches in rhotrix theory. Therefore, researchers referred to the former definition as commutative method and the later as non-commutative method of rhotrix multiplication. Consequently, the first review of articles on rhotrix theory between 2003 and 2013 by [36] classifies over forty articles in the literature of rhotrix theory, in the years under review as commutative rhotrix theory and non-commutative rhotrix theory depending on the methods of rhotrix multiplication employed in the research. The review further revealed that 45.24 per cent of articles in the literature of rhotrix theory belonged to the class of commutative rhotrix theory while 54.46 per cent was of non-commutative rhotrix theory.

It is worthy to note that only these two methods of multiplication of rhotrices are currently available in literature. These two binary multiplication methods are associative. Thus, all articles so far existing in rhotrix theory can further be classified into associative rhotrix theory. Therefore, the objective of this work is not to highlight the dichotomy between the two already existing classes of rhotrix theory. Rather, this work chronicles the existing articles as associative rhotrix theory, and introduces non-associative binary operations as alternative rhotrix multiplications.

2. First Chronicle of Associative Rhotrix Theory

This section presents a survey of articles in rhotrix theory available to us and categorize them by years as below:

2.1. The Formative Years in Rhotrix Theory (1998-2003)

As discussed in the preceding section, these years are considered as formative years in rhotrix theory.

2.2. The First Generation Years of Rhotrix Theory (2004-2009)

The question of how can a rhotrix be converted to a matrix and vice versa posed in the concluding section in [6] was answered in

2004 when Sani in [40] proposed the first alternative rhotrix multiplication, and was later generalized in 2007 in [41]. These works motivated a lot of researches in this new field, as it was now possible to investigate many properties of matrices in rhotrices. This new method of multiplication was originally called the row-column multiplication method. In the same year, the heart-based multiplication by Ajibade was also generalized in [28, 29] extending rhotrices of dimension three to dimension n. A milestone was reached in 2008 when a rhotrix was first converted to a special matrix in [42] called a coupled matrix. Using this special matrix of dimension n, a two different systems of linear equations was solved simultaneously in [43] where one is a $(n \times n)$ system $AX = b$ while the other is a $(n-1) \times (n-1)$ system, $CY = d$. Then, in 2009, a remark on classification of rhotrices as abstract structures was proposed in [30]. A system of linear equations arising from the rhotrix equation $A \circ X = C$ was investigated in [7] and the conditions for their solvability were determined in the article. That climaxed the first generation years in rhotrix theory.

2.3. The Second Generation Years of Rhotrix Theory (2010-2015)

The second generation saw a prolific research in the field even across the continent. Starting from 2010, an algebraic properties of singleton, coiled and modulo rhotrices were investigated in [49], and later in the year, a note was given on rhotrices and the construction of finite fields in [50]. A one-sided system of the form $R_nX = b$ was presented in [9]. Also of interest was the concept of rhotrix vector spaces developed in [10]. Finally, in that year, an example of linear mappings was extended to rhotrices by [11]. In 2011, the first Ph.D thesis in rhotrix theory was successfully supervised in Ahmadu Bello University, Zaria[27]. In that year, the concept of tree in graph theory was extended to rhotrix by [33]. Later, construction of certain field of fractions over rhotrices was presented in [55]. Great Progress was made in the direction of further generalizing the commutative multiplication of rhotrices of dimension n by [31] and was also expanded by [1]. Construction and analysis of metric topological spaces using rhotrix set as the underlying set were considered in [32], and various constructions of finite fields over rhotrices were carried out by [57]. That same year, an algorithm was designed for the non-commutative method of rhotrix multiplication by [2]. Finally to be considered in that year 2011 are a note on relationship between invertible rhotrices and associated invertible matrices by [45], and the parallel multiplication of rhotrices using a systolic array architecture by [3]. In 2012, more examples on certain field of fractions as a note were presented in [56]. A study of the structure of rhotrices having entries from the set of integers modulo P and their properties were conducted by [51]. Later, the rhotrix quadratic polynomial was extended by [52]. The local classification of rhotrices into rhotrix sets and rhotrix spaces was carried out in [34]. The work opened up more avenues of studying associative rhotrices that are yet to be harnessed fully. In that year also, a note was further made on rhotrix system equations and representation in [8][17]. The equivalence of Cayley-Hamilton theorem for matrix in rhotrices was proposed in [44][12]. Another milestone was the investigation of the concepts of inner product and Bilinear forms over real rhotrices and determinant method of solving rhotrix system of equations considered in [47][13], and a study of adjoint of a rhotrix and its basic properties was presented in [46]. Self- invertible rhotrices were constructed as paradigms of involutory matrices with enormous importance in matrix theory and algebraic cryptography by [58], and by [48] in the following year. Next on review, are the rhotrix multiplication of two-dimensional process grid topologies in [4], the rhotrix linear transformation in [35] and rhotrices and elementary row-operation in [59]. Still in 2013, rhotrix polynomial and polynomial rhotrices were investigated by [53].

In 2014, there was an introduction to the concept of paraletrix as a generalization of rhotrix by [15], and the time was right for the first review of articles on rhotrix theory since its inception presented in [36]. Also in that year, some constructions of rhotrix semigroup were given in [37]. A new technique for expressing rhotrices in a generalized form was examined in [38]. 2015 saw a lot of applications of rhotrices into computer science, Statistics, coding theorem and cryptography in [54], [24],[25] and [26]. This set a pace for the third generation rhotrices.

2.4. Third Generation Years of Rhotrix Theory (2016 Till Date)

The third generation years in rhotrix theory is approaching rhotrix theory from diverse areas of research, employing the inherent applicability of rhotrices into other areas of mathematical sciences, Environmental Sciences, computer sciences Engineering, coding theory and cryptography[25]. The year also saw the emergence of a beautiful rhotrix called natural rhotrix with the concept of minor rhotrices and index of a natural rhotrix presented in [19]. Also of interest, is the algebraic presentation of rhotrix group of roots of unity given in [5]. Several other works are on-going such as: Rhotrices with even dimension in [20] born from the desire to answer popular question of whether or not there can be rhotrices of even dimension. Later, a representation of higher rhotrices with even dimension is also investigated in [22]. A note on the classical and non-classical rhotrices is also made in [21].

Interestingly, more algebraic properties of rhotrices would be revealed when a non-associative binary multiplication of rhotrices that removes the condition of associativity from the binary systems of rhotrices is introduced, just as we had much results in rhotrix theory when a binary multiplication that removed commutativity was introduced in 2004. Therefore, next section focuses on a new dimension of studying rhotrix-the non-associative rhotrix theory.

3. The Concept of Non-Associative Rhotrix Theory

This section presents the non-associative rhotrix multiplication. It begins with a particular example of a multiplication method that is non-associative and then prepares the stage for many definitions of rhotrix multiplications as observed earlier in [6] but with emphasis on the non-associative binary multiplications which could be commutative or non-commutative.

Definition 3.1: Let

$$\hat{R} = \{R : R = \left\langle \begin{array}{ccc} & a & \\ b & c & d \\ & e & \end{array} \right\rangle, a, b, c, d, e \ \in \mathfrak{R}\} \tag{1}$$

be a set of all three dimensional rhotrices where $h(R) = c$ is called the heart of R.

Definition 3.2: Let $R, Q \in \hat{R}$, then, we call the operation below a left conjugate multiplication:

$$R \odot Q = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(R) & d_1 \\ & e_1 & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(Q) & d_2 \\ & e_2 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & \overline{a_1}a_2 & \\ \overline{b_1}b_2 & \overline{h(R)}h(Q) & \overline{d_1}d_2 \\ & \overline{e_1}e_2 & \end{array} \right\rangle \tag{2}$$

Definition 3.3: Let $R, Q \in \hat{R}$, then the right conjugate multiplication is given as

$$R \odot Q = \left\langle \begin{array}{ccc} & a_1 & \\ b_1 & h(R) & d_1 \\ & e_1 & \end{array} \right\rangle \odot \left\langle \begin{array}{ccc} & a_2 & \\ b_2 & h(Q) & d_2 \\ & e_2 & \end{array} \right\rangle$$

$$= \left\langle \begin{array}{ccc} & a_1\overline{a_2} & \\ b_1\overline{b_2} & h(Q)\overline{h(Q)} & d_1\overline{d_2} \\ & e_1\overline{e_2} & \end{array} \right\rangle \tag{3}$$

Remark 3.1: (i) where $\overline{a_1}$ is the left conjugate of a_2 and $\overline{a_2}$ is the right conjugate of a_1 such that $a\overline{a} = \overline{a}a = 1$ and $a1 = a = 1a$. Then, $(\hat{R}, \odot)$ is a groupoid, called a rhotrix groupoid. (ii) Also, $\overline{a} = a^{-1}$ where 'a' is a real number, and the juxtaposition is a direct multiplication of real numbers.

Behold, The conjugacy operation has no effect on the heart of a rhotrix. In other words, the heart of a rhotrix is self-conjugate i.e. $\overline{h(R)} = h(R)$ for every rhotrix R. Thus, a conjugate operation is strictly a left conjugate or a right conjugate.

Definition 3.4: Let $R, I \in \hat{R}$ and $\odot$ a conjugate operation. If

$$R \odot I = R = I \odot R$$

then, we called I a two sided conjugate identity, under $\odot$ defined in (2) and (3). This implies that

$$I = \left\langle \begin{array}{ccc} & 1 & \\ 1 & 1 & 1 \\ & 1 & \end{array} \right\rangle$$

The concept of identity rhotrix makes the inverse rhotrix meaningful.

Definition 3.5: Let $R, X \in \hat{R}$ and $\odot$ a conjugate operation. If

$$R \odot X = I$$

or

$$X \odot R = I$$

then, we called X the conjugate inverse of R. That is

$$R = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

implies that

$$R^{-1} = \left\langle \begin{array}{ccc} & a & \\ b & h(R) & d \\ & e & \end{array} \right\rangle$$

Then R is self-invertible or self-conjugate under $\odot$.

Remark 3.2: All conjugacy rhotrices are self-invertible or self-conjugate. Strictly, one can also refer to X as either a left - conjugate inverse or a right-conjugate inverse.

We can easily verify that conjugacy operation on rhotrices as defined above is non-commutative and non-associative, except at trivial cases where the rhotrices are identical or when having the value of the heart repeated at every other point etc.

Lemma 3.1: Let A and B be distinct rhotrices of the same dimension in $\hat{R}$. Then, $A \odot X = B$ and $Y \odot A = B$ have unique solutions in $\hat{R}$ (unique solvability)

Proof: $A \odot X = B$ implies that $X = A^{-1} \odot B$ and $Y \odot A = B$ implies that $Y = B \odot A^{-1}$. Since, A is invertible under $\odot$, then $A^{-1} \odot B \in \hat{R}$ and also $B \odot A^{-1} \in \hat{R}$. The solution exists in $\hat{R}$

To show uniqueness, we prove by contradiction. Suppose, X and Y are not unique. Then, it means that there are more than one rhotrix X satisfying the above definition, say

$$A \odot X' = B \tag{4}$$

$$A \odot X'' = B \tag{5}$$

combining (4) and (5) implies that

$$A \odot X' = A \odot X''$$

The equality of the conjugacy operation implies that $X' = X''$.
Similarly,

$$Y' = Y''$$

Hence, the above equations has unique solutions in $\hat{R}$.

Definition 3.6: Let $(\hat{R}, \odot)$ be a rhotrix groupoid and let Q be any fixed rhotrix in $\hat{R}$. Then, C_R is a right conjugate operator if

$$PC_R(Q) = P \odot Q$$

and a left conjugate operator if

$$PC_L(Q) = Q \odot P$$

for all $P \in \hat{R}$. It follows that $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ for each $Q \in \hat{R}$.

Remark 3.3: Whenever, the operation is not a conjugate operation, the conjugate operators are simply the translation maps i.e $C_R(Q) = R_Q$ and $C_L(Q) = L_Q$ -see[18] [39]

Definition 3.7: A rhotrix groupoid $(\hat{R}, \odot)$ is commutative means that

$$C_L(Q) = C_R(Q)$$

for all $Q \in \hat{R}$

Definition 3.8: A rhotrix groupoid $(\hat{R}, \odot)$ is associative if the conjugate operator

$$C_R(Q \odot P) = C_R(Q)C_R(P)$$

for all $Q, P \in \hat{R}$

Remark 3.4: These definitions above are helpful in determining whether or not a rhotrix multiplication defined usually by a Cayley table is commutative, associative or otherwise.

Theorem 3.1: The heart of a conjugate rhotrix commutes and associates

Proof: Let R and Q be two rhotrices of the same dimension.
Consider:

$$R \odot Q = h(R)\overline{h(Q)} = \overline{h(R)}h(Q) = Q \odot R$$

Then, the heart commutes.
Next, we show associativity. Let R, Q and P be three rhotrices

$$(R \odot Q) \odot P = (h(R)\overline{h(Q)}) \odot h(P) = (h(R)\overline{h(Q)})\overline{h(P)}$$

$$= h(R)(\overline{h(Q)h(P)}) = h(R) \odot (h(Q)h(P)) = R \odot (Q \odot P)$$

The heart associates.

Remark 3.5: Since the heart is self-conjugate, then the results hold.

Definition 3.9: Let $(\hat{R}, \odot)$ be a rhotrix groupoid and let $I \in \hat{R}$. Then I is a left(right)identity rhotrix for $(\hat{R}, \odot)$ means that

$$C_L(I) : \hat{R} \mapsto \hat{R}(C_R(I) : \hat{R} \mapsto \hat{R})$$

is the identity conjugate operator on $\hat{R}$.

Definition 3.10: A rhotrix groupoid $(\hat{R}, \odot)$ is called a rhotrix quasigroup if the conjugate operators $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ are bijections. And if it possesses in addition, an identity rhotrix, then $(\hat{R}, \odot)$ becomes a rhotrix loop. The order of $\hat{R}$ is its cardinality $|\hat{R}|$.

But for the sake of definition, we define a rhotrix loop below:

Definition 3.11: A rhotrix quasigroup with a left and right identity rhotrix is called a rhotrix loop

This means that a rhotrix groupoid $(\hat{R}, \odot)$ is a rhotrix loop if $(\hat{R}, \odot)$ is a rhotrix quasigroup that has a two-sided identity rhotrix. Thus, all rhotrix groups are rhotrix loops. But all rhotrix loops are not rhotrix groups. Those that are rhotrix groups are associative rhotrix loops. Therefore, rhotrix loops generalize rhotrix groups.
It is worth noting that rhotrices as defined by Ajibade and Sani are associative rhotrix loops. These types of rhotrix loops are trivial rhotrix loops. Whereas, rhotrix loops defined by conjugate operation as considered in this work, are non-trivial rhotrix loops

Corollary 3.1: Let $(\hat{R}, \odot)$ be a quasigroup. Then, the following hold:

(i): For $R, Q, P \in \hat{R},\ R \odot Q = R \odot P$ implies $P = Q$ (left cancellation law)

(ii): For $R, Q, P \in \hat{R},\ Q \odot R = P \odot R$ implies $P = Q$ (right cancellation law)

Proof: The proof follows from Lemma 3.1

Lemma 3.2: The heart of a conjugate rhotrix corresponds to the center of a rhotrix quasigroup(loop)

Proof: The center of a quasigroup(loop) commutes and associates and following Theorem 3.1 above the proof holds.

Theorem 3.2: Let $(\hat{R}, \odot)$ be a rhotrix groupoid. the following are equivalent:

(i): $(\hat{R}, \odot)$ is a rhotrix quasigroup.

(ii): $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ are injective for all $Q \in \hat{R}$.

(iii): $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ are surjective for all $Q \in \hat{R}$.

(iv): The left and right cancellation laws hold for $(\hat{R}, \odot)$.

Proof: $(i) \Rightarrow (ii)$

That $RC_L(Q) = Q \odot R$ and $RC_R(Q) = R \odot Q$, and since these equations have unique solutions in $(\hat{R}, \odot)$ implies that $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ are injective for all $Q \in \hat{R}$.

$(ii) \Rightarrow (iii)$

Since $\hat{R}$ is a set of finite rhotrices and $C_R(Q)$ and $C_L(Q)$ are injective implies that $C_R(Q) = \hat{R}$ and $C_L(Q) = \hat{R}$. Thus, $C_R(Q) : \hat{R} \mapsto \hat{R}$ and $C_L(Q) : \hat{R} \mapsto \hat{R}$ are surjective for all $Q \in \hat{R}$.

$(iii) \Rightarrow (iv)$

If $R \odot Q = Q \odot R$, then, the left and right cancellation laws hold.
$(iv) \Rightarrow (i)$
The left and right cancellations imply that the equations $A \odot X = B$ and $Y \odot A = B$ have unique solutions in $(\hat{R}, \odot)$. Hence, $(\hat{R}, \odot)$ is rhotrix quasigroup.
Remark 3.6: Theorem 3.2 is simply a characterization of a rhotrix quasigroup.

The implication of Theorem 3.2 is that rhotrix quasigroups(loops) can be represented in Cayley Table where no rhotrix repeats itself along row or column. By this the stage is now set to investigate various methods of rhotrix multiplications using Cayley-tables.

4. Other Examples of Rhotrix Multiplications

This section confirms the observation in [6] that rhotrix multiplication can be defined in many ways. These multiplications are defined using Cayley tables. Depending on the definitions, rhotrix multiplications can be commutative, associative, non-commutative or non-associative, or even both. However, we are going to be concerned with non-associative rhotrix multiplications which could be commutative or non-commutative.
Example 4.1: Let $\hat{R} = \{I, P, Q\}$ be a set of arbitrary rhotrices of the same dimension, and a binary multiplication $(\cdot)$ defined as

$\cdot$	I	P	Q
I	I	P	Q
P	P	Q	I
Q	Q	I	P

Table 1. Associative and Commutative Rhotrix Loop

Then, $(\hat{R}, \cdot)$ is an associative and commutative rhotrix loop. A trivial loop.
Example 4.2: Let $\hat{R} = \{I, P, Q, R\}$ be a finite set of arbitrary rhotrices of the same dimension. Define multiplication $(\circ)$ as

$\circ$	I	P	Q	R
I	I	P	Q	R
P	P	I	R	Q
Q	Q	R	I	P
R	R	Q	P	I

Table 2. Associative and Commutative Rhotrix Loop

Then, $(\hat{R}, \circ)$ is also an associative and commutative rhotrix loop. This is also trivial.
Example 4.3: Let $\hat{R} = \{I, P, Q, R, S\}$ be a finite set of arbitrary rhotrices of the same dimension and $\odot$ be given by the table below:

$\odot$	I	P	Q	R	S
I	I	P	Q	R	S
P	P	I	R	S	Q
Q	Q	S	I	P	R
R	R	Q	S	I	P
S	S	R	P	Q	I

Table 3. Non-Associative and Non-Commutative Rhotrix Loop

Example 4.4: Let $\hat{R} = \{I, P, Q, R, S\}$ be a finite set of arbitrary rhotrices of the same dimension and $(\bullet)$ be given by the table below:

$\bullet$	I	P	Q	R	S
I	I	P	Q	R	S
P	P	Q	R	S	I
Q	Q	R	S	I	P
R	R	S	I	P	Q
S	S	I	P	Q	R

Table 4. A Commutative Rhotrix Group

Then, $(\hat{R}, \bullet)$ is a trivial rhotrix loop.

Remark 4.1: Behold, Table 3 and Table 4 are of equal order, yet the first was a rhotrix loop (Non-Associative) and the second was a rhotrix group (Associative). The difference lies on the definition of the rhotrix multiplication. An example of a rhotrix loop emerges with $|\hat{R}| \geq 5$. With $|\hat{R}| \leq 4$ one can only get rhotrix groups (trivial rhotrix loops).

Example 4.5: Let $\hat{R} = \{I, P, Q, R, S, T, U, V\}$ be a finite set of arbitrary rhotrices of the same dimension and let $\odot$ be defined by the table below:

$\odot$	I	P	Q	R	S	T	U	V
I	I	P	Q	R	S	T	U	V
P	P	I	R	Q	T	S	V	U
Q	Q	R	I	P	V	U	S	T
R	R	Q	P	I	U	V	T	S
S	S	T	U	V	I	P	Q	R
T	T	S	V	U	P	I	R	Q
U	U	V	S	T	R	Q	I	P
V	V	U	T	S	Q	R	P	I

Table 5. Non-Associative and Non-Commutative Rhotrix Loop

Then, $(\hat{R}, \odot)$ is a rhotrix loop.

Remark 4.2: The examples above are illustrative examples. Paradigms of examples in Table 1-Table 5 can be found in [39].

5. CONCLUDING REMARKS

This work opens up a large door of research to exploit the properties of rhotrices as binary systems. Though, rhotrices are geometric objects, but using algebra as a microscope, one is able to examine the scope of their properties. This is a reminiscence of the age long interplay between geometry and algebra. Therefore, there is need to investigate rhotrices through a geometric approach. This article examined the properties of the rhotrix through non-associative binary systems. Many things in nature are not linear. Thus, assuming linearity on them limits the much we can know about them. For example, in a rhotrix loop, there may be some rhotrices or a rhotrix that may commute or associate with every other rhotrix in the loop. Such a rhotrix may exist at the heart of the rhotrix loop. It is interesting to find out such a rhotrix. These are areas for future work.

ACKNOWLEDGEMENTS

The author wishes to express his profound gratitude and appreciation to the friends in Ahmadu Bello University, Zaria among whom are Dr. Ajibade A.O, Dr. Sani B. and Dr. Mohammed A for their encouragement and valuable discussions.

REFERENCES

(1) Absalom, E.A., Junaidu, S.B. and Sani, B. (2011), *The Concept of Heart-Oriented Rhotrix Multiplication*, Global Journal of Science Frontier, 11, 35-46.
(2) Absalom, E.A., Ajibade, A.O. and Sahalu, J.B (2011), *Algorithm Design for Row-Column Multiplication of N-Dimensional Rhotrices*, Global Journal of Computer Science and Technology, 11, 22-30.
(3) Absalom, E.A., Abdullahi, M. Ibrahim, K., Mohammed, A. and Junaidu, S.B (2011), *Parallel Multiplication of Rhotrices Using Systollic Array Architecture*, International Journal of Computer Information Systems, 2, 68-73.
(4) Absalom, E.A. and Sahalu, J.B (2012), *Rhotrix Multiplication of two-Dimensional Process Grid Topologies*, International Journal of Grid and High Performance Computing, 4, 21-36.
(5) Adamu, H., Babayo, A.M. (2016), *Algebraic Presentation of Rhotrix Group of Roots of Unity*, Imperial Journal of Interdisciplinary Research, 2(3), 631-634.
(6) Ajibade, A.O (2003), *The concept of Rhotrix in Mathematical Enrichment*, International Journal of Mathematical Education in Science and Technology, 34, 175-179.
(7) Aminu, A (2009), *On the Linear System over Rhotrices*, Notes on Number Theory and Discrete Mathematics, 15, 7-12.
(8) Aminu, A (2012), *A Note on the Rhotrix System of Equation*, Journal of the Nigerian Association of Mathematical Physics, 21, 289-296.
(9) Aminu, A (2010), *The Equation $R_nX = b$ over Rhotrices*, International Journal of Mathematical Education in Science and Technology, 41, 98-105.
(10) Aminu, A (2010),*Rhotrix Vector Spaces*, International Journal of Mathematical Education in Science and Technology, 41, 531-578.
(11) Aminu, A (2010), *An Example of Linear Mappings: Extension to Rhotrices*, International Journal of Mathematical Education in Science and Technology, 41, 691-698.
(12) Aminu, A (2012), *Cayley-Hamilton Theorem in Rhotrix*, Journal of the Nigerian Association of Mathematical Physics, 20, 43-48.

(13) Aminu, A (2012), *A Determinant Method for Solving Rhotrix System of Equation*, Journal of the Nigerian Association of Mathematical Physics, 21, 281-288.
(14) Aminu, A and Michael, O. (2013),*Adjacent Rhotrix of a Complete, Simple and Undirected Graph*, Journal of the Nigerian Association of Mathematical Physics, 25, 267-274.
(15) Aminu, A and Michael, O (2014),*An introduction to the concept of paraletrix, a generalization of rhotrix*, Journal of the African Mathematical Union and Springer-Verlag, vol. 26, no. 5-6, 871-885.
(16) Atanassov, K T and Shannon, AG (1998), *Matrix-Tertions and Matrix-Noitrets: Exercise for Mathematical Enrichment*, International Education in Science and Technology, vol.29. No. 6, pp. 898-903.
(17) Chinedu, M.P (2012), *Row-Wise Representation of Arbitrary Rhotrix*, Notes on Number Theory and Discrete Mathematics, 18, 1-27.
(18) Isere, A. O. (2014), *Construction and Classification Of Finite Non-Universal Osborn loops Of Order 4n*, Ph.D. thesis, Federal University of Agriculture, Abeokuta.
(19) Isere, A.O. (2016), *Natural rhotrix*, Cogent Mathematics, 3, 1246074.
(20) Isere, A.O.(2018), *Even Dimensional Rhotrix*, Notes on Number Theory and Discrete Mathematics, 24(2), 125-133.
(21) Isere, A.O. (2018), *A Note on Classical and Non-Classical Rhotrices*, Journal of the Mathematical Association of Nigeria (abacus) 44(2), 119-124.
(22) Isere, A.O., *Representation of Higher Even Dimensional Rhotrix*, Submitted to the Journal, Notes on Number Theory and Discrete Mathematics.
(23) Kaurangini, M.L. and Sani, B. (2007), *Hilbert Matrix and Its Relationship with a Special Rhotrix*, Journal of the Mathematical Association of Nigeria, 34, 101-106.
(24) Kailash, M.P., Singh, H.P. and Kinnari, S. (2015), *Eigen values of any Given* 3×3 *Matrix via Eigen values of its Corresponding Rhotrix*, International Journal of Computer & Mathematical Sciences, 4(11), 1-4.
(25) Khalid, H.H.A. (2015), *On Design of Binary Linear Block Code (BLBC) bases on the Coupled Matrices of Hadamard Rhotrix* , European Centre for Research Training and Development 3(4), 4-10.
(26) Khalid, H.H.A. (2015), *On Encoding and Decoding of Information By Using the Coupled Matrices of Hadamard Rhotrix*, International Journal of Mathematical Archive, 6(8), 100-103.
(27) Mohammed, A. (2011), *Theoretical Development and application of Rhotrices* , Ph.D Thesis, Ahmadu Bello University, Zaria.
(28) Mohammed, A (2007), *Enrichment Exercises through Extension to Rhotrices*, International Journal of Mathematical Education in Science and Technology, 38. 131-136.
(29) Mohammed, A (2007), *A Note on Rhotrix Exponent Rule and Its Applications to Special Rhotrices*, Notes on Number Theory and Discrete Mathematics, 18, 1-27.
(30) Mohammed, A (2009), *A remark on the classifications of rhotrices as abstract strutures*, International Journal of Physical Sciences vol. 4(9), pp. 496-499.
(31) Mohammed, A, Ezugwu, E.A, and Sani, B. (2011),*On Generalization and Algorithmatization of Heart-based Method for Multiplication of Rhotrices*, International Journal of Computer Information Systems, 2, 74-81.
(32) Mohammed, A and Tijjani, (2011), *Rhotrix Topological Spaces*, International Journal of Advances in Science and Technology, 3 .
(33) Mohammed, A and Sani, B. (2011), *On Construction of Rhomtrees as Graphical Representation of Rhotrices*, Notes on Number Theory and Discrete Mathematics, 17, 21-29.
(34) Mohammed, A and Tella, Y (2012), *Rhotrix Sets and Rhotrix Spaces Category*, International Journal of Mathematics and computational methods in Science and Technology, vol. 2, No. 5, 2012.

(35) Mohammed, A Balarabe, M and Imam, A.T (2012), *Rhotrix Linear Transformation*, Advances in Linear Algebra & Matrix Theory, 2, 43-47.
(36) Mohammed, A. and Balarabe, M. (2014), *First Review of Articles on Rhotrix Theory Since Its Inception*, Advances in Linear Algebra & Matrix Theory, 4, 216-224.
(37) Mohammed, A Balarabe, M and Imam, A.T (2014), *On Construction of Rhotrix Semigroup*, Journal of Nigerian Association of Mathematical Physics, 27, 69-76.
(38) Mohammed, A (2014), *A new expression for rhotrix* Advances in Linear Algebra & Matrix Theory, 4, 128-133.
(39) Pflugfelder, H. O. (1990), *Quasigroups and loops : Introduction*, Sigma series in Pure Math. 7, Heldermann Verlag, Berlin, 147pp.
(40) Sani, B (2004), *An alternative method for multiplication of rhotrices*, International Journal of Mathematical Education in Science and Technology, 35, 777-781.
(41) Sani, B (2007), *The row-column multiplication for higher dimensional rhotrices*, International Journal of Mathematical Education in Science and Technology, 38, 657-662.
(42) Sani, B (2008), *Conversion of a rhotrix to a coupled matrix*, International Journal of Mathematical Education in Science and Technology, 39, 244-249.
(43) Sani, B (2009), *Solution of Two Coupled Matrices*, Journal of the Mathematical Association of Nigeria, 32,53-57.
(44) Sharma, P.L and Kanwar, R.K (2011), *Cayley-Hamilton Theorem for Rhotrices*, Journal of Mathematics Analysis, 4, 171-178.
(45) Sharma, P.L and Kanwar, R.K (2012), *A Note on Relationship between Invertible Rhotrices and Associated Invertible Matrices*, Bulletin of Pure and Applied Sciences: Mathematics and Statistics, 30e, 333-339.
(46) Sharma, P.L and Kanwar, R.K (2012), *Adjoint of a Rhotrix and Its Basic Properties*, International Journal of Mathematical Sciences, 11, 337-343.
(47) Sharma, P.L and Kanwar, R.K (2012), *On Inner Product Spaces and Bilinear forms of Rhotrices*, Bulletin of Pure and Applied Sciences: Mathematics and Statistics, 31e, 109-118.
(48) Sharma, P.L and Kanwar, R.K (2013), *On Involutory and Pascal Rhotrices*, International Journal of Mathematical Sciences and Engineering Applications, 7, 133-146.
(49) Tudunkaya, S.M. and Manjuola S.O (2010), *Algebraic Prperties of Singleton, Coiled and Modulo Rhotrices*, African Journal of Mathematics and Computer Sciences
(50) Tudunkaya, S.M. and Manjuola S.O (2010), *Rhotrices and the Construction of Finite Fields*, Bull. Pure Appl Sci. Sect, E. Math.Stat. 29e, 2, 225-229.
(51) Tudunkaya, S.M. and Manjuola S.O (2012), *On the Structure of Rhotrices*, Journal of the Nigerian Association of Mathematical Physics, 21, 271-280.
(52) Tudunkaya, S.M. and Manjuola S.O (2012), *Certain Quadractic Extension*, Journal of the Nigerian Association of Mathematical Physics, 21, 271-280.
(53) Tudunkaya, S.M. (2013), *Rhotrix Polynomial and Polynomial Rhotrices*, Pure and Applied Mathematics Journal 2, 38-41.
(54) Tudunkaya, S.M. (2015), *Design of the Affine Ciphers of Rhotrices*, African Journal of Computing, 8(2), 79-82.
(55) Usaini, S and Tudunkaya, S.M.(2011),*Certain Fields of Fractions*,Global Journal of Science Frontier, 11, 4-8.
(56) Usaini, S and Tudunkaya, S.M.(2012),*Note on Certain Fields of Fractions*,Global Journal of Science Frontier, 12, 74-81.
(57) Usaini, S. and Tudunkaya, S.M. (2011), *Note on Rhotrices and the Construction of Finite Fields*, Bull. Pure Appl Sci. Sect, E. Math.Stat. 30e, 53-59.

(58) Usaini, S. (2012), *On the Construction of Involutory Rhotrices*, International Journal of Mathematical Education in Science and Technology, 43, 510-515.

(59) Usaini, S. (2012), *Rhotrices and Elementary Row-Operation*, Journal of the Nigerian Association of Mathematical Physics, 20, 37-42.

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