Journal of the Nigerian Mathematical Society
Vol. 37, Issue 3, pp. 203-215, 2018

PURSUIT DIFFERENTIAL GAME PROBLEM WITH INTEGRAL AND GEOMETRIC CONSTRAINTS IN A HILBERT SPACE

J. RILWAN [1] AND A.J. BADAKAYA

ABSTRACT. We study pursuit differential game problem with countable number of pursuers and one evader. Control functions of some finite number of pursuers are subject to integral constraints while that of the remaining pursuers and evader are subject to geometric constraint. Sufficient conditions for completion of pursuit in two different theorems are presented. Moreover, attainability domains and strategies of the players are also constructed. Furthermore, illustrative examples are given.

Keywords and phrases: Pursuit, Integral constraint, Geometric constraint, Hilbert space.
2010 Mathematical Subject Classification: 49N70, 49N90 and 91A23

1. INTRODUCTION

Differential game, due to its importance, has been an area of great interest to many researchers. Thus a lot of publications have been devoted to this area of Applied Mathematics and fundamental results were published in books like Avner [2], Isaacs [13] and Ramachandran et al. [15].

Pursuit-evasion differential game is a differential game involving two players namely pursuer and evader. It's application to solving real life problems motivated lot of researchers to study such type of differential game problem (see, for example Alias et al. [11]; Ibragimov [3, 4, 5]; Ibragimov and Rikhsiev [6]; Ibragimov and Salimi [7]; Ibragimov and Hussin [8]; Ibragimov and Satimov [9]; Ibragimov and Kuchkarov [10]; Ivanov and Ledyeav [14]; Levchenko and Pashkov [1]). In many of these researches, the motion of the pursuer and evader were described by the differential equations, in respective order

Received by the editors December 02, 2017; Revised: July 30, 2018; Accepted: September 13, 2018
www.nigerianmathematicalsociety.org; Journal available online at www.ojs.ictp.it
[1]Corresponding author

$$\begin{aligned} \dot{x}(t) &= a(t)u(t), \quad x(0) = x_0, \\ \dot{y}(t) &= b(t)v(t), \quad y(0) = y_0 \end{aligned} \tag{1}$$

where $u(\cdot)$ and $v(\cdot)$ are control functions of the pursuer and evader respectively, which are either subject to geometric or integral constraints. The functions $a(\cdot)$, $b(\cdot)$ are scalar and measurable. Different spaces were considered in different researches. Some of the questions answered include but not limited to finding value of the game, conditions for completion of pursuit, conditions for evasion and construction of optimal strategies of the players.

In the problems considered by the authors in [11], [3], [6] and [1], motion of the players obeyed the differential equations in (1), a case where $a(t) = b(t) = 1$. For example, Levchenko and Pashkov [1] considered such case and developed an algorithm for determining the payoff functions for all possible positions. They also present an analytical description of the regions where the pursuit is one to one and collective.

Pursuit differential game of many pursuers and single evader was investigated on a closed bounded convex subset of $\mathbb{R}^n$ by Alias et al. [11] where the players motion is described by (1) with $a(t) = b(t) = 1$. In the game, all players are confined within the given set and their control parameters are subjected to geometric constraints. They obtained estimate for guaranteed pursuit time and proved that indeed pursuit can be completed at this time.

In the space $\mathbb{R}^n$, a differential game in which m dynamical objects pursuing a single object was investigated by Ibragimov [3]. The motion of the players obey (1) with $a(t) = b(t) = 1$ and mixed constraints on the players control functions are considered. Under some certain conditions in the paper, they constructed an optimal strategy for the pursuers and also obtained value of the game. Mehdi et al [12] studied the same problem considered in [3] with infinite number of dynamical objects (pursuers) in the space l_2. Optimal strategy of the players were constructed and value of the game was obtained.

Ibragimov and Rikhsiev [6] considered a game with players motion described by (1) with geometric constraints imposed on the players control functions. Sufficient conditions for optimality of the pursuit time were obtained. Optimal strategy was also constructed.

Ibragimov and Salimi [7] and also Ibragimov and Hussin [8] studied pursuit-evasion differential game of fixed time duration with countably many pursuers and one evader in a Hilbert space l_2 , where $a(t) = b(t) = \theta - t$. Integral constraints and geometric constraints are considered respectively in both problems. Under certain conditions, they obtained value of the game and constructed optimal strategies of players in both research works.

In this research work, we consider pursuit differential game problems of fixed duration in which equation of motion of the players is described by (1), where $a(t) = b(t)$, in a Hilbert space l_2. Mixed constraints (that is, integral and geometric constraints) are imposed on the players' control functions. Termination time of the game is fixed. This work is motivated by the work of Ibragimov [3].

2. STATEMENT OF THE PROBLEM

Consider the space $l_2 = \{\alpha = (\alpha_1, \alpha_2, \dots) : \sum_{n=1}^{\infty} \alpha_n^2 < \infty\}$, with inner product $\langle \cdot, \cdot \rangle : l_2 \times l_2 \to \mathbb{R}$ given by $\langle \alpha, \beta \rangle = \sum_{n=1}^{\infty} \alpha_n \beta_n$ and norm $|| \cdot || : l_2 \to [0, +\infty)$ given by $||\alpha|| = \left(\sum_{n=1}^{\infty} \alpha_n^2 \right)^{1/2}$, $\alpha,\ \beta \in l_2$. Let the motions of countably many pursuers P_j , $j = 1, 2, 3, ..., k, k+1, \dots$ and an evader E be described by the equations

$$\begin{cases} P_j : \quad \dot{x}_j(t) = a(t)u_j(t), \quad x_j(0) = x_{j0}, \\ E : \quad \dot{y}(t) = a(t)v(t), \qquad y(0) = y_0, \quad 0 \le t \le \theta, \end{cases} \tag{2}$$

where $x_j(t)$, x_{j0}, $u_j(t)$, $y(t)$, y_0, $v(t) \in l_2$; $u_j(t) = (u_{j1}(t), u_{j2}(t), u_{j3}(t), \dots)$ is the control functions of the pursuer P_j and $v(t) = (v_1(t), v_2(t), v_3(t), \dots)$ is that of the evader E; $a(\cdot)$ is a positive scalar and measurable function defined on some closed interval $[0, \theta]$. Throughout the game, $I = \{1, 2, 3, \dots, k, k + 1, \dots\}$ while θ is a fixed positive constant which denotes the duration of the game.

The j^{th} pursuer's control function $u_j(t)$ is subjected to the constraints below

$$\begin{aligned} \int_0^{\theta} \| u_j(t) \|^2 dt \leq {\rho_j}^2 \quad for \quad j \in \{1,2,3,\dots,k\} = I_0, \\ \|u_j(t)\| \leq \rho_j \quad for \quad j \in \{k+1, k+2, \dots\} = I_1, \end{aligned} \tag{3}$$

while that of the evader satisfies the inequality

$$\| v(t) \| \leq \sigma, \qquad 0 \leq t \leq \theta, \tag{4}$$

where $\rho_j, j \in I$ and σ are given positive real numbers .
Once the admissible control functions $u_j(\cdot)$ and $v(\cdot)$ are chosen by the players, the corresponding motions $x_j(\cdot)$ and $y(\cdot)$ of the players are defined by

$$x_j(t) = x_{j0} + \int_0^t a(s)u_j(s)ds \quad and \quad y(t) = y_0 + \int_0^t a(s)v(s)ds, \tag{5}$$

respectively, $0 \leq t \leq \theta$.
Denote $H(x_0, r)$ as a closed ball in l_2 with radius r and center x_0.

Definition 1: A measurable function $u_j = u_j(t)$, $0 \leq t \leq \theta$ satisfying the following inequalities;

$$\int_0^{\theta} \|u_j(t)\|^2 dt \leq \rho_j^2, \quad j \in \{1,2,3,\dots,k\} = I_0, \tag{6}$$

$$\|u_j(t)\| \leq \rho_j, \quad j \in \{k+1, k+2, \dots\} = I_1, \tag{7}$$

is called an **admissible control of the pursuer** P_j.

Definition 2: A measurable function $v = v(t)$, $0 \leq t \leq \theta$ satisfying the inequality

$$\|v(t)\| \leq \sigma, \; 0 \leq t \leq \theta, \tag{8}$$

is called an **admissible control of the evader** E.

Definition 3: A function $U_j(x_j, y, v(t))$,
(i) $U_j : l_2 \times l_2 \times l_2 \to l_2, \qquad j \in I_0$
(ii) $U_j : l_2 \times l_2 \times H(0,\sigma) \to l_2, \quad j \in I_1$
for which the system

$$\begin{aligned} \dot{x}_j(t) &= a(t)U_j(x_j, y, v(t)), \qquad & x_j(0) &= x_{j0} \\ \dot{y}(t) &= a(t)v(t), & y(0) &= y_0 \end{aligned}$$

has a unique absolutely continous solution for any admissible control $v(\cdot)$ of the evader E is called **strategy of the pursuer** P_j. A strategy U_j is said to be admissible if every control generated by U_j is admissible.

Definition 4: Pursuit is said to be completed at time t^*, $0 \leq t^* \leq \theta$ in the game described by (2), (3) and (4), if $x_j(t^*) = y(t^*)$ for some $j \in I$.

Definition 5: A closed ball $H(x_0, r)$ is called **attainability domain** of a player if the following conditions are satisfied:
i. $\|x_0 - x(\theta)\| \leq r$.
ii. If $\bar{x} \in H(x_0, r)$, then there exists an admissible control of the player such that $x(t) = \bar{x}$ for some $t \in [0, \theta]$.

Research Question:
What are the conditions for completion of pursuit in the game described by (2), (3) and (4)?

3. MAIN RESULTS

Considering the differential game problem described by (2), (3) and (4);
let

$$A(t) = \int_0^t a^2(s)ds, \quad and \quad B(t) = \int_0^t a(s)ds.$$

Lemma 1: (Attainability domain) Let x_{i0}, x_{j0} and y_0 be the initial positions of the i^{th} pusuer, j^{th} pursuer and the evader, respectively. The closed balls

$$H_1(x_{i0}, \rho_i A^{\frac{1}{2}}(\theta)), \quad H_2(x_{j0}, \rho_j B(\theta)) \quad and \quad H_3(y_0, \sigma B(\theta)),$$

are the attainability domains of the pursuer $P_i, i \in I_0; P_j, j \in I_1$ and of the evader E, respectively.

To prove this lemma, we need to show

i. $x_i(\theta) \in H_1(x_{i0}, \rho_i A^{\frac{1}{2}}(\theta))$ for $i \in I_0$; $x_j(\theta) \in H_2(x_{j0}, \rho_j B(\theta))$ for $j \in I_1$ and $y(\theta) \in H_3(y_0, \sigma B(\theta))$.

ii. If $\bar{x} \in H_1(x_{i0}, \rho_i A^{\frac{1}{2}}(\theta))$ or $\bar{x} \in H_2(x_{j0}, \rho_j B(\theta))$ $(\bar{y} \in H_3(y_0, \sigma B(\theta)))$, then there exists an admissible control of the pursuer $P_i, i \in I_0$ or $P_j, j \in I_1$ (of the evader E) such that $x_i(t) = \bar{x}$ or $x_j(t) = \bar{x}$ $(y(t) = \bar{y}$, respectively) for some time $t \in [0, \theta]$.

To prove (i), we have for $i \in I_0$

$$\|x_i(\theta) - x_{i0}\| = \left\| x_{i0} + \int_0^\theta a(s)u_i(s)ds - x_{i0} \right\| \le \int_0^\theta |a(s)| \|u_i(s)ds\| \\ \le \rho_i A^{\frac{1}{2}}(\theta).$$

The case of the pursuers $P_j, j \in I_1$ and evader E can be proven in a similar way.

To show (ii),
we construct the controls of the pursuer P_i and P_j as follows

$$\begin{cases} u_i(t) = \frac{a(t)}{A(\theta)}(\bar{x} - x_{i0}), & 0 \le t \le \theta, \quad i \in I_0 \\ u_j(t) = \frac{\bar{x} - x_{j0}}{B(\theta)}, & \quad j \in I_1. \end{cases} \tag{9}$$

For the case of evader, let

$$v(t) = \frac{\bar{y} - y_0}{B(\theta)}, \quad 0 \le t \le \theta. \tag{10}$$

Now for $i \in I_0$, we have

$$\begin{aligned} x_i(\theta) = x_{i0} + \int_0^\theta a(t)u_i(t)dt &= x_{i0} + \int_0^\theta a(t) \left(\frac{a(t)}{A(\theta)}(\bar{x} - x_{i0}) \right) dt \\ &= x_{i0} + \frac{\bar{x} - x_{i0}}{A(\theta)} \int_0^\theta a^2(t)dt \\ &= \bar{x} \end{aligned}$$

The proof for the cases of pursuers $P_j; j \in I_1$ and evader E is similar. Moreover, it is not difficult to show that the controls defined by (9) and (10) are admissible.

We now present two theorems with their proofs. Each of the theorem provides answer to the research question, that is, sufficient conditions for completion of pursuit in the game described by (2), (3) and (4).

Consider the set X_j defined as follows :

For the pursuers $P_j,\ j \in I_0$, if $\rho_j \geq \sigma\sqrt{\theta}$, we set
$X_j = \{m \in l_2 : 2\langle x_{j0}, m\rangle \geq\| m \|^2 + \| x_{j0} \|^2 - \left(\frac{B(\theta)}{\sqrt{\theta}}(\rho_j - \sigma\sqrt{\theta})\right)^2\}$
while for $j \in I_1$, we set
$X_j = \{m \in l_2 : 2\langle y_0 - x_{j0}, m\rangle \leq (\rho_j^2 - \sigma^2)B^2(\theta) + \|y_0\|^2 - \|x_{j0}\|^2\}$,
where $x_{j0} \neq y_0$ for all $j \in I = I_0 \cup I_1$.

Theorem 1: If $y_0 \in X_j$, then there exists a strategy for the pursuer's $P_j, j \in I_0$ that guarantees completion of pursuit in the game described by (2), (3) and (4) for the time θ .
Proof: From the inclusion $y_0 \in X_j$, we can deduce that

$$\|y_0 - x_{j0}\|^2 \leq \left(\frac{B(\theta)}{\sqrt{\theta}}(\rho_j - \sigma\sqrt{\theta}\ )\right)^2. \tag{11}$$

We construct the stategy for the pursuer's P_j as follows :

$$U_j(t) = \frac{y_0 - x_{j0}}{B(\theta)} + v(t). \tag{12}$$

Using the inequality (11) and Cauchy-Schwartz inequality, we can show the admissibilty of the strategy (12) as follows:

$$\begin{aligned}
\int_0^\theta \|U_j(s)\|^2 ds &= \int_0^\theta \left\|\frac{y_0 - x_{j0}}{B(\theta)} + v(t)\right\|^2 ds \\
&\leq \int_0^\theta \frac{\|y_0 - x_{j0}\|^2}{B^2(\theta)} + \frac{2}{B(\theta)}\langle y_0 - x_{j0}, v(t)\rangle + \|v(t)\|^2 dt \\
&\leq \frac{\|y_0 - x_{j0}\|^2}{B^2(\theta)}\theta + \frac{2}{B(\theta)}\langle y_0 - x_{j0}, \int_0^\theta v(t)dt\rangle + \sigma^2\theta \\
&\leq \frac{\|y_0 - x_{j0}\|^2}{B^2(\theta)}\theta + \frac{2\|y_0 - x_{j0}\|\sigma\theta}{B(\theta)} + \sigma^2\theta \\
&\leq \frac{\left(\frac{B(\theta)}{\sqrt{\theta}}(\rho_j - \sigma\sqrt{\theta}\ )\right)^2}{B^2(\theta)}\theta + \frac{2\left(\frac{B(\theta)}{\sqrt{\theta}}(\rho_j - \sigma\sqrt{\theta}\ )\right)\sigma\theta}{B(\theta)} \\
&+ \sigma^2\theta \\
&= \rho_j^2.
\end{aligned}$$

Now if the pursuer uses the strategy (12) , then completion of pursuit follows from

$$x_j(\theta) = x_{j0} + \int_0^\theta a(s)\left(\frac{y_0 - x_{j0}}{B(\theta)} + v(s)\right) ds = y(\theta)$$

For the case $x_0 = y_0$, it is trivial to show that $x_j(\theta) = y(\theta)$.

Theorem 2: If $\sigma \leq \rho_j$ and $y(\theta) \in X_j$, then the pursuer $P_j, j \in I_1$ can complete pursuit in the game described by (2), (3) and (4) for the time θ.

Proof

Let $\sigma \leq \rho_j$ and $y(\theta) \in X_j$. We construct the pursuer's strategy given below ;

For $x_{j0} = y_0$, we set

$$U_j(t) = v(t), \quad 0 \leq t \leq \theta \tag{13}$$

and if $x_{j0} \neq y_0$, we set

$$U_j(t) = \begin{cases} v(t) - \langle v(t), e_j\rangle e_j + e_j((\rho_j^2 - \sigma^2) + \langle v(t), e_j\rangle^2)^{\frac{1}{2}}, & 0 \leq t \leq \tau \\ v(t), & \tau < t \leq \theta, \end{cases} \tag{14}$$

where

$$e_j = \frac{y_0 - x_{j0}}{\|y_0 - x_{j0}\|}. \tag{15}$$

The admissibility of the strategy (13) and (14) is shown as follows;

If $x_{j0} = y_0$, then

$$\|U_j(t)\| = \|v(t)\| \leq \sigma \leq \rho_j.$$

Now if $x_{j0} \neq y_0$,

$$\begin{aligned}
\|U_j(t)\|^2 &= \left\|v(t) - \langle v(t), e_j\rangle e_j + e_j\sqrt{\rho_j^2 - \sigma^2 + \langle v(t), e_j\rangle^2}\right\|^2 \\
&\leq \|v(t) - \langle v(t), e_j\rangle e_j\|^2 + \left\|e_j\sqrt{\rho_j^2 - \sigma^2 + \langle v(t), e_j\rangle^2}\right\|^2 \\
&= \langle v(t) - \langle v(t), e_j\rangle e_j, v(t) - \langle v(t), e_j\rangle e_j\rangle + \rho_j^2 - \sigma^2 + \langle v(t), e_j\rangle^2 \\
&= \langle v(t) - \langle v(t), e_j\rangle e_j, v(t)\rangle - \langle v(t) - \langle v(t), e_j\rangle e_j, \langle v(t), e_j\rangle e_j\rangle \\
&+ \rho_j^2 - \sigma^2 + \langle v(t), e_j\rangle^2 \\
&= \langle v(t), v(t)\rangle - 2\langle v(t), \langle v(t), e_j\rangle e_j\rangle + \langle\langle v(t), e_j\rangle, \langle v(t), e_j\rangle\rangle \\
&+ \rho_j^2 - \sigma^2 + \langle v(t), e_j\rangle^2 \\
&= \|v(t)\|^2 + \rho_j^2 - \sigma^2 \\
&\leq \rho_j^2.
\end{aligned}$$

If the pursuers uses the strategies defined by (13) and (14), for the case $x_{j0} = y_0$, it is not difficult to show that $x_j(\theta) = y(\theta)$, that is, pursuit can be completed at time θ.

Now suppose that $x_{j0} \neq y_0$, then

$$y(t) - x_j(t) = e_j f(t),$$

where

$$f(t) = \|y_0 - x_{j0}\| + \int_0^t a(s)\langle v(s), e_j\rangle ds - \int_0^t a(s)(\rho_j^2 - \sigma^2 + \langle v(s), e_j\rangle^2)^{\frac{1}{2}} ds. \tag{16}$$

Obviously,

$$f(0) = \|y_0 - x_{j0}\| > 0.$$

We now show that $f(\theta) \leq 0$, which will imply $f(\tau) = 0$, for some $\tau \in [0, \theta]$.
To this end, we consider the vector-valued function below

$$g(t) = \left(a(t)\sqrt{\rho_j^2 - \sigma^2}, \quad a(t)\langle v(t), e_j\rangle\right), \quad 0 \leq t \leq \theta. \tag{17}$$

Using the last integral in (16) and (17), we obtain
$f(\theta) \leq \|y_0 - x_{j0}\|$

$$+ \int_0^\theta a(s)\langle v(s), e_j\rangle ds - \left(B^2(\theta)(\rho_j^2 - \sigma^2) + \left(\int_0^\theta a(s)\langle v(s), e_j\rangle ds\right)^2\right)^{\frac{1}{2}}. \tag{18}$$

By assumption, $y(\theta) \in X_j$, therefore

$$2\langle y_0 - x_{j0}, y(\theta)\rangle \leq B^2(\theta)(\rho_j^2 - \sigma^2) + \|y_0\|^2 - \|x_{j0}\|^2. \tag{19}$$

This implies that

$$\int_0^\theta a(s)\langle v(s), e_j\rangle ds \leq d - \langle y_0, e_j\rangle, \tag{20}$$

where

$$d = \frac{B^2(\theta)(\rho_j^2 - \sigma^2) + \|y_0\|^2 - \|x_{j0}\|^2}{2\|y_0 - x_{j0}\|}.$$

On the other hand, $\psi(t) = \|y_0 - x_{j0}\| + t - (B^2(\theta)(\rho^2 - \sigma^2) + t^2)^{\frac{1}{2}}$ is an increasing function on $(-\infty, \infty)$ since

$$\begin{aligned}\psi'(t) &= 1 - \frac{t}{\sqrt{B^2(\theta)(\rho^2-\sigma^2)+t^2}}\\ &\geq 1 - \frac{t}{\sqrt{t^2}}\\ &= 0.\end{aligned}$$

Then it follows from (19) and (20) that

$$f(\theta) \leq \|y_0 - x_{j0}\| + d - \langle y_0, e_j\rangle - \left(B^2(\theta)(\rho_j^2 - \sigma^2) + (d - \langle y_0, e_j\rangle)^2\right)^{\frac{1}{2}}. \tag{21}$$

Lastly, we show that the right-hand side of (21) is equal to zero, that is

$$\|y_0 - x_{j0}\| + d - \langle y_0, e_j\rangle = \left(B^2(\theta)(\rho_j^2 - \sigma^2) + (d - \langle y_0, e_j\rangle)^2\right)^{\frac{1}{2}}. \tag{22}$$

Note that:

$$\|y_0 - x_{j0}\|^2 = \|y_0\|^2 + \|x_{j0}\|^2 - 2\langle y_0, x_{j0}\rangle, \tag{23}$$

$$2\|y_0 - x_{j0}\|d = B^2(\theta)(\rho_j^2 - \sigma^2) + \|y_0\|^2 - \|x_{j0}\|^2 \tag{24}$$

and

$$2\|y_0 - x_{j0}\|\langle y_0, e_j\rangle = 2\|y_0 - x_{j0}\|\langle y_0, \frac{y_0 - x_{j0}}{\|y_0 - x_{j0}\|}\rangle = 2\|y_0\|^2 - 2\langle y_0, x_{j0}\rangle. \tag{25}$$

Also,

$$\begin{aligned}(\|y_0 - x_{j0}\| + d - \langle y_0, e_j\rangle)^2 &= \|y_0 - x_{j0}\|^2 + 2\|y_0 - x_{j0}\|(d - \langle y_0, e_j\rangle)\\ &\quad + (d - \langle y_0, e_j\rangle)^2\\ &= \|y_0 - x_{j0}\|^2 + 2\|y_0 - x_{j0}\|d\\ &\quad - 2\|y_0 - x_{j0}\|\langle y_0, e_j\rangle + (d - \langle y_0, e_j\rangle)^2.\end{aligned} \tag{26}$$

Substituting (23), (24), (25) into (26) and simplifying, we have

$$(\|y_0 - x_{j0}\| + d - \langle y_0, e_j\rangle)^2 = B^2(\theta)(\rho_j^2 - \sigma^2) + (d - \langle y_0, e_j\rangle)^2.$$

Therefore, the equality in (22) is true. Thus, $f(\theta) \leq 0$.
Consequently, $f(\tau) = 0$ for some τ, $\quad 0 \leq \tau \leq \theta$.
Hence, $x_j(\tau) = y(\tau)$.
For the remaining period of the game, we have $U_j(t) = v(t)$,

$\tau < t \leq \theta$, then

$$x_j(\theta) = x_j(\tau) + \int_{\tau}^{\theta} a(t)U_j(t)dt = y(\tau) + \int_{\tau}^{\theta} a(t)v(t)dt = y(\theta)$$

Hence, the proof of Theorem 2 is complete.

4. SOME ILLUSTRATIVE EXAMPLES

Example 1.
Consider the pursuit Differential Game problem described by

$$\begin{cases} \dot{x}_j(t) = e^t u_j(t), & x_j(0) = x_{j0} \\ \dot{y}(t) = e^t v(t), & y(0) = y_0, \end{cases}$$

where all the variables are defined as in section 2 with u and v satisfying the inequalities

$$\int_0^{16} \| u_j(t) \|^2 dt \leq 25, \quad j \in \{1, 2, 3, ..., k\} = I_0,$$
$$\| v(t) \| \leq 1.$$

We fixed the initial positions of the pursuer P_j and evader E as follows; $x_{j0} = (0, ..., 0, 2, 0, ...)$ and $y_0 = (0, 0, 0, ...)$ respectively, where the number 2 is in the j^{th} coordinate of point x_{j0}.
Observe that at time $\theta = 16$; $A(16) = \frac{e^{32}-1}{2}$, $B(16) = e^{16} - 1$ and also $\rho_j = 5 \geq \sigma\sqrt{16} = 1\sqrt{16} = 4$. Hence the set X_j takes the form

$$X_j = \left\{ m \in l_2 : 2\langle x_{j0}, m\rangle \geq \|m\|^2 + 4 - \frac{(e^{16}-1)^2}{16} \right\}.$$

Since $2\langle x_{j0}, y_0\rangle \geq 4 - \frac{(e^{16}-1)^2}{16}$, for each $j \in I_0$.

This implies $y_0 \in X_j$. Then by theorem 1, if the player $P_j, j \in I_0$ uses the admissible strategy

$$U_j(t) = \frac{y_0 - x_{j0}}{e^{16} - 1} + v(t),$$

then $x_j(16) = y(16)$, that is, pursuit is completed at the time $\theta = 16$

Example 2.
Here, we consider a simple differential game of one pursuer - one evader by dropping the index $j \in I_1$ in the game (2), (3) and (4) as

follows

$$\begin{cases} \dot{x}(t) = t^2 u(t), & x(0) = (0,0,0,...) \\ \dot{y}(t) = t^2 v(t), & y(0) = (1,0,0,0,...), \end{cases}$$

where all the variables are defined as in section 2 but in this case, the players control functions satisfies

$$\| u(t) \| \leq 3,$$
$$\| v(t) \| \leq \frac{1}{9}.$$

Observe that at time $\theta = 2$; $A(2) = \frac{32}{5}$, $B(2) = \frac{8}{3}$, $e = \frac{y_0 - x_0}{\|y_0 - x_0\|} = (1,0,0,0,...) = y_0$. Hence the set

$$X = \left\{ m \in l_2 : 2\langle y_0 - x_0,\, m \rangle \leq \frac{47321}{729} \right\}.$$

Also, observe that

$$\begin{aligned} 2\langle y_0 - x_0,\, y(2) \rangle &\leq 2\|y_0 - x_0\| \|y(2)\| \\ &\leq 2\|y_0\| + 2\| \int_0^2 t^2 v(t) dt \| \\ &\leq 2 + \frac{16}{27} \\ &< \frac{47321}{729}, \end{aligned}$$

which implies $y(2) \in X$. Now if the pursuer uses the admissible strategy below

$$U(t) = \begin{cases} v(t) - \langle v(t), e \rangle e + e(\frac{728}{81} + \langle v(t), e \rangle^2)^{\frac{1}{2}}, & 0 \leq t \leq \tau \\ v(t), & \tau < t \leq 2, \end{cases}$$

then from the conclusion of Theorem 2, pursuit can be completed at the time $\theta = 2$, that is, $x(2) = y(2)$.

5. CONCLUSION

We have studied a fixed duration pursuit Differential Game problem with countably many pursuers and one evader in the space l_2 with mixed constraints. We stated and proved theorems for completion of pursuit. This research work prepares ground for finding value of the game for the problem considered in this paper.

ACKNOWLEDGEMENTS

We thank the reviewers of this article for their efforts. Certainly, their suggestions and observations added quality to the paper.

REFERENCES

[1] A.Y. Levchenko, A.G. Pashkov, *A game of optimal pursuit of one non-inertial object by two inertial objects*, Journ. Appl. Math. and Mech., **49**(4), 413-422, 1985.

[2] F. Avner, *Differential Games*. John Wiley and sons, U.S.A, 1971.

[3] G.I. Ibragimov, *A game of optimal pursuit of one object by several*. Journ. Applied Math. and Mech., **62**(2), 187-192, 1998.

[4] G.I. Ibragimov, *On a multiperson pursuit problem with integral constraints on the control of the players*, Math. Notes **70**, 181-191, 2001.

[5] G.I. Ibragimov, *Optimal pursuit with countably many pursuers and one evader*, Diff. Equations, **41**(5), 627-635. 2005.

[6] G.I. Ibragimov, B.B. Rikhsiev, *On some sufficient conditions for optimality of the pursuit time in differential game with multiple pursuers*, Automation and Remote Control, **67**, 529-537, 2006.

[7] G.I. Ibragimov, M. Salimi, *Pursuit-evasion differential game with many inertial players*, Math. probl. in Eng., Article ID 653723, 2009.

[8] G.I. Ibragimov, N.A. Hussin, *A pursuit-evasion differential game with many pursuers and one evader*, Malaysian Journ. Math. Science, **4**(2), 183-194, 2010.

[9] G.I. Ibragimov, N. Satimov, *Multiplayer pursuit differential game on a convex set with integral constraints*, Abstract and Appl. Analysis, **2012**(460171), 1-12, 2012.

[10] G.I. Ibragimov, A. Sh. Kuchkarov, *Fixed duration pursuit-evasion game with integral constraints*, Journ. Physics; Conference Series, **435**(1), doi:org/10.1088/1742−6596/435/1/012017, 2013.

[11] I.A. Alias, R.N.R. Ramli, I.G. Ibragimov, A. Narzullaev, *Simple motion pursuit differential game of many pursuers and one evader on convex compact set*. Int. Journ. Pure and Applied Math. **102**(4), 733-745, 2015.

[12] M. Salimi, G.I. Ibragimov, S. Siegmund, S. Sharifi, *On a fixed duration pursuit differential game with geometric and integral constraints*, Dyn. Games and Appl., **6**(3), 409-425, 2016.

[13] R. Isaacs, *Differential Games*, John Willey and Sons, New York, 1965.

[14] R.P. Ivanov, Y.S. Ledyaev, *Time optimality for the pursuit of several objects with simple motion in a differential game*, Proc. Steklov Inst. Math., **158**, 87-97, 1981.

[15] Ramachandran,M. Kandethody, Tsokos, P. Chris, *Stochastic differential games (theory and application)*, Atlantis press, U.S.A., 2012.

DEPARTMENT OF MATHEMATICS, BAYERO UNIVERSITY, KANO, NIGERIA
E-mail addresses: `jawadrilwan3945@gmail.com, jrilwan.mth@buk.edu.ng`

DEPARTMENT OF MATHEMATICS, BAYERO UNIVERSITY, KANO, NIGERIA
E-mail address: `ajbadakaya.mth@buk.edu.ng`