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# NUMERICAL COMPUTATION OF FRACTIONAL PARTIAL DIFFERENTIAL EQUATIONS ARISING IN PHYSICS

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ABSTRACT. In this article, we aim to propose a reliable numerical algorithm based on homotopy analysis transform method for solving various kinds of linear and nonlinear time-fractional partial differential equations arising in physics. The method is exemplified by linear and nonlinear time- fractional heat-like, invicid Burgers and fifth orders KdV equations arising in the study of thermodynamics, fluid mechanics and quantum mechanics respectively. We investigate the influence of the convergence-control parameter $\hbar$ that provides us, a simple way to guarantee the convergence of series solution of linear and nonlinear problems. The proposed method may give better approximations which are uniformly valid for either small and large parameters or variables with highly accurate numerical solutions.

**Keywords and phrases:** Fractional heat-like equations, Fractional invicid Burgers equation, Fifth orders KdV equations, Laplace transform, Homotopy analysis transform method
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## 1. INTRODUCTION

Fractional differential equations have garnered a lot of attention and appreciation due to their ability to facilitate an exact description of different linear and nonlinear phenomena's. There are many books and definitions that develop and investigate about the fractional order integrations and differentiations [1-4]. During the last decades, number of powerful computational techniques were introduced and investigated by many research workers for obtaining exact and approximate solutions of fractional equations [5-11].
This paper adopts homotopy analysis transform method (HATM) to solve higher dimensional initial value problems of constant and variable coefficients, linear and nonlinear, partial fractional differential equations occurring in scientific and technological fields.

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The HATM is in fact a coupling of well-known Laplace transform method and homotopy analysis method (HAM) [12]. The HAM is a general analytical approach to solve many types of nonlinear differential equations of integer and fractional order [13-18].
The HATM contains the homotopy perturbation method [19-21] and homotopy perturbation transform method [22-24] as special cases. HATM provides us with a simple way to adjust and control the convergence region of the series solutions by introducing the auxiliary parameter $\hbar$, auxiliary function H(x,t) and the initial guess $u_0(x,t)$. In order to assess the advantages and the accuracy of the homotopy analysis transform method (HATM), here we used only fourth order approximate solution. Even then, however, the proposed technique has been successfully applied in a realistic and efficient way to several fractional partial differential equations, which are rapidly converging to the exact solutions. We investigate the validity of the auxiliary parameter $\hbar$ on the convergence of the approximate series solution by plotting $\hbar$-curves. We observed that the properly chosen auxiliary parameter $\hbar$ can ensure the convergence of series solution.

## 2. PRELIMINARY

Fractional Calculus: Fractional calculus deals with generalizations of integer order derivatives and integrals to arbitrary order. There exists literature on different definitions; the most popular once are the Riemann-Liouville and the Caputo derivatives.
**Definition 1**. The Laplace transform of continuous (or an almost piecewise continuous) function $f(t)\, in\, [0,\infty)$ is defined and represented as follows

$$F(s) = L[f(t)] \; = \; \int_0^t e^{-st} f(t)dt, \tag{1}$$

where s is indicating a real or complex number.
**Definition 2**. The Riemann- Liouville fractional integral operator of order $\alpha > 0$, of a function $f(x) \in C_\mu, \mu \geq -1$ is defined as

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau, (\alpha > 0) \tag{2}$$

$$J^0 f(x) = f(x), \tag{3}$$

for the Riemann- Liouville fractional integral we have the following result:

$$J^{\alpha}x^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+\alpha+1)}x^{\alpha+\gamma}. \tag{4}$$

**Definition 3**. The fractional derivative of $f(x)$in the Caputo sense is defined by [25] which is illustrated as follows

$$D^{\alpha}f(x) = J^{n-\alpha}D^{n}$$

$$= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{x}(x-\tau)^{n-\alpha-1}f^{n}(\tau)d\tau, \tag{5}$$

for $n-1<\alpha\leq n, n\in N, x>0$.

**Definition 4**. The Laplace transform of the Caputo derivative [25-26] is given in the following form

$$L\left[D^{\alpha}f(t)\right] = s^{\alpha}L\left[f(t)\right] - \sum_{k=0}^{n-1}s^{\alpha-k-1}f^{(k)}(0^{+}), n-1<\alpha\leq n. \tag{6}$$

## 3. BASIC IDEA OF HATM

We take a general fractional nonlinear non-homogeneous partial differential equation of the form:

$$D_{t}^{\alpha}\,u(x,t) + R\,u(x,t) + \mathrm{N}u(x,t) = g(x,t), \quad n-1<\alpha\leq n. \tag{7}$$

In the above Eq. (7) $D_{t}^{\alpha}u(x,t)$ indicates the Caputo fractional derivative of the function $u(x,t)$, $R$ denotes the linear differential operator, N represents the general nonlinear differential operator and $g(x,t)$ is the source term.

By applying the Laplace transform on both sides of Eq. (7), we get

$$L\left[D_{t}^{\alpha}\,u\right] + L\left[R\,u\right] + L\left[\mathrm{N}\,u\right] = L\left[g(x,t)\right]. \tag{8}$$

Employing the differentiation property of the Laplace transform, we have

$$s^{\alpha}L[u] - \sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(x,0) + L[R\,u] + L[\mathrm{N}\,u] = L[g(x,t)]. \tag{9}$$

If we simplify the above equation, we have the following result

$$L\,[u]-\frac{1}{s^\alpha}\sum_{k=0}^{n-1}s^{\alpha-k-1}u^{(k)}(x,0)+\frac{1}{s^\alpha}\left[L\,[Ru]+L[\mathrm{N}\,u]-L[g(x,t)]\right]=0. \tag{10}$$

We define the nonlinear operator as

$$\mathrm{N}[\phi(x,t;q)]=L\,[\phi(x,t;q)]-\frac{1}{s^\alpha}\sum_{k=0}^{n-1}s^{\alpha-k-1}\phi^{(k)}(x,t;q)(0^+)$$
$$+\frac{1}{s^\alpha}\left[L\,[R\phi(x,t;q)]+L[\mathrm{N}\phi(x,t;q)]-L[g(x,t)]\right]. \tag{11}$$

In the above expression $q\in[0,\,1]$ and $\phi(x,t\,;q)$ is a real function of x, t and q. We construct a homotopy as follows

$$(1-q)\,L\,[\phi(x,t\,;q)-u_0(x,t)]=\hbar\,qH(x,t)\,\mathrm{N}\,[u(x,t\,)], \tag{12}$$

where L denotes the Laplace transform operator, $H(x,t)$ denotes a nonzero auxiliary function, $\hbar\neq 0$ is an auxiliary parameter, $u_0(x,t)$ is an initial guess of $u(x,t)$ and $\phi(x,t\,;q)$ is a unknown function. Obviously, when the embedding parameter $q=0$ and $q=1$, it holds

$$\phi(x,t\,;0)=u_0(x,t),\qquad \phi(x,t\,;1)=u(x,t), \tag{13}$$

respectively. Thus, as q increases form 0 to 1, the solution $\phi(x,t\,;q)$ varies from the initial guess $u_0(x,t)$ to the solution $u(x,t)$. Expanding $\phi(x,t\,;q)$ in Taylor series with respect to q, we have

$$\phi(x,t\,;q)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t)\,q^m, \tag{14}$$

where

$$u_m(x,t)=\frac{1}{m\,!}\frac{\partial^m\phi(x,t\,;q)}{\partial q^m}\,|_{q=0}. \tag{15}$$

If the initial guess, the auxiliary parameter $\hbar$, and the auxiliary function are properly selected, the series (14) converges at q = 1, then we have

$$u(x,t)=u_0(x,t)+\sum_{m=1}^{\infty}u_m(x,t), \tag{16}$$

which must be one of the solutions of the original nonlinear fractional order differential equations. According to the definition (16), the governing equation can be deduced from the zero-order deformation (12).
Define the vectors

$$\vec{u}_m = \{u_0,\ u_1, ...,\ u_m\}. \tag{17}$$

Differentiating the zeroth-order deformation Eq. (12) m-times with respect to q and then dividing them by m! and finally setting $q = 0$, we get the following mth-order deformation equation:

$$L\left[u_m(x,t) - \chi_m u_{m-1}(x,t)\right] = \hbar\, H(x,t)\Re_m(\vec{u}_{m-1}). \tag{18}$$

Applying the inverse Laplace transform, we have

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + \hbar\, L^{-1}[H(x,t)\Re_m(\vec{u}_{m-1})], \tag{19}$$

where

$$\Re_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\,\mathrm{N}[\phi(x,t\,;q)]}{\partial q^{m-1}}\,|_{q=0}, \tag{20}$$

and the value of $\chi_m$ is

$$\chi_m = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases} \tag{21}$$

## 4. FRACTIONAL HEAT-LIKE EQUATIONS

Here we apply the HATM to study the two-dimensional and three-dimensional fractional heat-like equations arising in electromagnetic waves and thermodynamics studies. The fractional heat-like equation describes the heat in a given region over time. The number of methods were used to obtain the approximate and exact solutions of heat-like equations [27-30].
**Example 4.1.** Consider the following three-dimensional fractional heat-like equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = x^4y^4z^4 + \frac{1}{36}(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2}), 0 < \alpha \leq 1, \tag{22}$$

$0 <\ x, y, z < 1,\ t > 0$
with the initial conditions

$$u(x,y,z,0) = u_0 = 0. \tag{23}$$

In the above Eq. (22) $u(x,y,z,t)$ indicates the temperature.
Applying the Laplace transform subject to the initial condition, we have

$$L\left[u(x,y,t)\right] - \frac{1}{s}u_0 - \frac{1}{s^\alpha}L\left[x^4y^4z^4 + \frac{1}{36}(x^2\frac{\partial^2 u}{\partial x^2} + y^2\frac{\partial^2 u}{\partial y^2} + z^2\frac{\partial^2 u}{\partial z^2})\right] = 0. \tag{24}$$

The nonlinear operator is defined in the following manner

$$\begin{aligned} N\left[\phi(x,y,z,t;q)\right] = L\left[\phi(x,y,z,t;q)\right] - \tfrac{1}{s}u_0 - \tfrac{1}{s^\alpha}L\Big[x^4y^4z^4 \\ + \tfrac{1}{36}(x^2\tfrac{\partial^2\phi(x,y,z,t;q)}{\partial x^2} + y^2\tfrac{\partial^2\phi(x,y,z,t;q)}{\partial y^2} + z^2\tfrac{\partial^2\phi(x,y,z,t;q)}{\partial z^2})\Big], \end{aligned} \tag{25}$$

and thus

$$\begin{aligned} \Re(\vec{u}_{m-1}) = L(u_{m-1}) - (1-\chi_m)\tfrac{1}{s}u_0 \\ -\tfrac{1}{s^\alpha}L\left[(1-\chi_m)x^4y^4z^4 + \tfrac{1}{36}(x^2\tfrac{\partial^2 u_{m-1}}{\partial x^2} + y^2\tfrac{\partial^2 u_{m-1}}{\partial y^2} + z^2\tfrac{\partial^2 u_{m-1}}{\partial z^2})\right] \end{aligned} \tag{26}$$

The $m^{th}$-order deformation equation is given by

$$L\left[u_m(x,y,z,t) - \chi_m u_{m-1}(x,y,z,t)\right] = \hbar\Re_m(\vec{u}_{m-1}). \tag{27}$$

Applying the inverse Laplace transform, we have

$$u_m(x,y,z,t) = \chi_m u_{m-1}(x,y,z,t) + \hbar L^{-1}\left[\Re_m(\vec{u}_{m-1})\right]. \tag{28}$$

Solving the above equation (7), for m = 1, 2, 3, ..., we get

$$\begin{aligned} u_1 &= -\hbar x^4y^4z^4\frac{t^\alpha}{\Gamma(\alpha+1)}, \\ u_2 &= (\hbar+1)\,u_1 + \hbar^2x^4y^4z^4\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}, \\ u_3 &= (\hbar+1)\,u_2 + x^4y^4z^4\left[\hbar^2(\hbar+1)\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \hbar^3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}\right], \\ u_4 &= (\hbar+1)\,u_3 + x^4y^4z^4\Big[\hbar^2(\hbar+1)^2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ &\quad -2\hbar^3(\hbar+1)\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \hbar^4\frac{t^{4\alpha}}{\Gamma(4\alpha+1)}\Big], \\ &\ \ \vdots \end{aligned} \tag{29}$$

Proceeding in this manner, the rest of the components $u_n(x, y, z, t)$ for $n > 4$ can be completely obtained and the series solution is thus entirely determined. Therefore, the approximate solution is

$$u(x, y, z, t) = u_0(x, y, z, t) + \sum_{n=1}^{\infty} u_n(x, y, z, t). \tag{30}$$

If we select $\alpha = 1$ and $\hbar = -1$ then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x, y, z, t)$ converges to the exact solution $u = x^4y^4z^4(e^t - 1)$, which is same as obtained by HDM [27]. We observe that, properly chosen auxiliary parameter $\hbar$ can provide more exact result, compared to HDM and perturbation method for same term iterations as depicted by absolute error. From Figs. 1-2, we can observe that as the value of space variable $x$ and time variable $t$ increase, the value of temperature also increases. From Fig. 3, we can observe that the results obtained with the aid of HATM are very accurate. Fig. 4 shows the effect of the order of fractional derivative on the temperature $u(x, y, z, t)$. Fig. 5 presents the $\hbar$-curves that shows that the valid range of $\hbar$ is $-2.0082 \leq \hbar < 0$ and the absolute convergence range of $\hbar$ is the horizontal line segments.

**Example 4.2.** In this example we consider the following two-dimensional fractional heat-like equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \;\; 0 < x, y < 2\pi, t > 0, \; 0 < \alpha \leq 1 \tag{31}$$

with the initial conditions

$$u(x, y, 0) = \sin(x)\, \sin(y)\,. \tag{32}$$

In the above Eq. (31) $u(x, y, t)$ represents the temperature.
We apply HATM for solving Eq. (31) subject to the initial condition (32) and get the following components of the series solution

$$u_0(x, y, t) \;=\; \sin(x)\sin(y), u_1(x, t) = 2\hbar\sin(x)\, \sin(y)\frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x, y, t) = 2\hbar(\hbar+1)\sin(x)\, \sin(y)\frac{t^\alpha}{\Gamma(\alpha+1)} + 4\hbar^2\sin(x)\, \sin(y)\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)},$$

$$u_3(x, y, t) = \sin(x)\, \sin(y)\left[2\hbar(\hbar+1)^2\frac{t^\alpha}{\Gamma(\alpha+1)}\right.$$
$$\left. +8\hbar^2(\hbar+1)\frac{t^{2\alpha}}{\Gamma\left(2\alpha+1\right)} + 8\hbar^3\frac{t^{3\alpha}}{\Gamma\left(3\alpha+1\right)}\right],$$

$$u_4(x,y,t) = \sin(x)\,\sin(y)\Big[2\hbar(\hbar+1)^3\tfrac{t^{\alpha}}{\Gamma(\alpha+1)} + 12\hbar^2(\hbar+1)^2\tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 24\hbar^3(\hbar+1)\tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 16\sin(x)\,\sin(y)\hbar^4\tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \tag{33}$$

$$\vdots$$

Proceeding in this manner, the rest of the components $u_n(x,y,t)$ for $n > 4$ can be completely obtained and the series solutions are thus entirely determined.

$$u(x,y,t) = u_0(x,y,t) + \sum_{n=1}^{\infty} u_n(x,y,t). \tag{34}$$

If we select $\alpha = 1$ and $\hbar = -1$ then obviously, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,y,t)$ rapidly converges to the exact solution $u = \sin(x)\,\sin(y)\,e^{-2t}$, which is same as obtained by HDM [27]. Figs. 6-7 show the exact and approximate solutions obtained by using HATM. From Fig. 8, we can notice that the results obtained with the help of HATM are very accurate. Fig. 9 shows the effect of the order of fractional derivative on the temperature $u(x,y,t)$. Fig. 10 depicts the $\hbar$-curves that shows that the valid range of $\hbar$ is $-1.98 \le \hbar < 0$ and the absolute convergence range of $\hbar$ is the horizontal line segments.

## 5. FRACTIONAL INVICID BURGERS & $5^{th}$ ORDER KDV EQUATIONS

In this section, we apply the HATM algorithm to solve nonlinear nonhomogeneous time-fractional invicid Burgers equation and the time-fractional fifth order KdV equation. The nonlinear nonhomogeneous time-fractional invicid Burgers equation is a conservation equation arising in gas dynamics and traffic flow. The time-fractional fifth order KdV equation is used in quantum mechanics and nonlinear optics. The time-fractional fifth order KdV equation describes the dispersive phenomena such as plasma waves when the third-order contributions are small.

**Example 5.1.** Consider the nonlinear nonhomogeneous time-fractional invicid Burgers equation [31]

$$\frac{\partial^{\alpha} u(x,t)}{\partial t^{\alpha}} + u(x,t)\frac{\partial u}{\partial x} = 1 + x + t, \quad 0 < \alpha \le 1,\ t > 0 \tag{35}$$

with the initial conditions

$$u(x,0) = x. \tag{36}$$

In the above Eq. (35) $u(x,t)$ denotes the density of the particles. Solving the above equations, we get

$$u_0(x,0) = x,$$

$$u_1(x,t) = -\hbar\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right],$$

$$u_2(x,t) = -\hbar(\hbar+1)\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] - \hbar^2\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right],$$

$$u_3(x,t) = -\hbar(\hbar+1)^2\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] - 2\hbar^2(\hbar+1)\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] - \hbar^3\left[\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right],$$

$$\begin{aligned} u_4(x,t) = &-\hbar(\hbar+1)^3\left[\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}\right] \\ &-3\hbar^2(\hbar+1)^2\left[\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right] \\ &-3\hbar^3(\hbar+1)\left[\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right] \\ &-\hbar^4\left[\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \frac{t^{4\alpha+1}}{\Gamma(4\alpha+2)}\right], \end{aligned} \tag{37}$$

$$\vdots$$

Proceeding in this manner, the rest of the components $u_n(x,t)$ for $n > 4$ can be completely obtained and the series solutions are thus entirely determined.

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t). \tag{38}$$

Therefore, the solution for the Burgers equation Eq. (35), when $\alpha \to 1$, $\hbar = -1$ is

$$\begin{aligned} u(x,t) &= \sum_{n=0}^{\infty} u_n(x,t) \\ &= x + t + \frac{t^2}{\Gamma 3} - \frac{t^2}{\Gamma 3} - \frac{t^3}{\Gamma 4} + \frac{t^3}{\Gamma 4} + \frac{t^4}{\Gamma 5} - \frac{t^4}{\Gamma 5} \cdots . \end{aligned} \tag{39}$$

It is obvious that the self-cancelling 'noise' terms appear between alternative terms. Cancelling the noise terms and keeping the non-noise terms in Eq. (39) yields the exact solution of Eq. (35) given as

$$u(x,t) = x + t, \tag{40}$$

which is same as obtained by HPSTM [31]. From Figs. 11-12, we can see that as the value of space variable $x$ and time variable $t$ increase, the value of displacement $u(x,t)$ also increases. From Fig. 13, we can observe that the results obtained with the aid of HATM are very accurate. Fig. 14 demonstrates the effect of the order of fractional derivative on the displacement $u(x,t)$.

**Example 5.2.** Consider the time-fractional fifth order KDV equation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} + u(x,t)\frac{\partial u}{\partial x} - u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0, \quad 0 < \alpha \leq 1,\ t > 0 \tag{41}$$

with the initial conditions

$$u(x,0) = e^x. \tag{42}$$

In the above Eq. (41) $u(x,t)$ represents the wave function.
Solving the above equations in same manner we get the solution in series form

$$u_0(x,0) = e^x, u_1(x,t) = \hbar e^x \frac{t^\alpha}{\Gamma(\alpha+1)},$$

$$u_2(x,t) = \hbar(\hbar+1)e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + \hbar^2 e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)},$$

$$u_3(x,t) = \hbar(\hbar+1)^2 e^x \frac{t^\alpha}{\Gamma(\alpha+1)} + 2\hbar^2(\hbar+1)e^x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \hbar^3 e^x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)},$$

$$\begin{aligned} u_4(x,t) = \hbar(\hbar+1)^3 e^x \tfrac{t^\alpha}{\Gamma(\alpha+1)} + 3\hbar^2(\hbar+1)^2 e^x \tfrac{t^{2\alpha}}{\Gamma(2\alpha+1)} \\ +3\hbar^3(\hbar+1)e^x \tfrac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \hbar^4 e^x \tfrac{t^{4\alpha}}{\Gamma(4\alpha+1)}, \end{aligned} \tag{43}$$

$$\vdots$$

Proceeding in this manner, the rest of the components $u_n(x,t)$ for $n > 4$ can be completely obtained and the series solutions are thus entirely determined.

$$u(x,t) = u_0(x,t) + \sum_{n=1}^{\infty} u_n(x,t). \tag{44}$$

If we select $\alpha = 1$ and $\hbar = -1$ then clearly, we can conclude that the obtained solution $\sum_{n=0}^{\infty} u_n(x,t)$ rapidly converges to the exact solution $u = e^{x-t}$, which is same as obtained by HPSTM [31]. Figs. 15-16 present the exact and approximate solutions derived by using HATM. From Fig. 17, we can notice that the results obtained with the aid of HATM are very accurate. Fig. 18 presents the effect of the order of fractional derivative on the displacement $u(x,t)$.

## 6. CONCLUDING REMARKS

The theme of this paper is to extend the application of homotopy analysis transform method (HATM) for solving fractional differential equations. It is observed that the HATM is capable of reducing the size of calculations and easy to apply in space of higher dimensions as well. The supremacy of the proposed method over the perturbations methods is that it provides series solutions for both small and large parameters, freedom to choose the convergence-control parameter $\hbar$ and initial guess $u_0(x,t)$. The proposed method gives more generalized series solution which rapidly convergence to the exact solution and the solutions obtained by HDM, HPSTM, HPM and VIM are special cases of HATM solution. Hence, we can conclude that the HATM is a very efficient technique for strongly nonlinear differential equations of fractional order.

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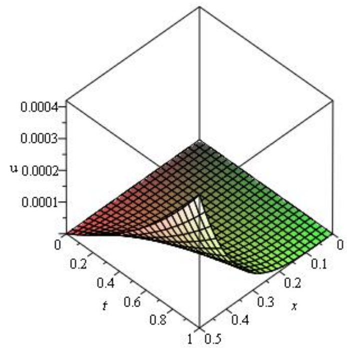


FIGURE 1. The surface shows the exact solution $u(x, y, z, t)$ at $y = z = 0.5$ and $\alpha = 1$ for Eq. (22).

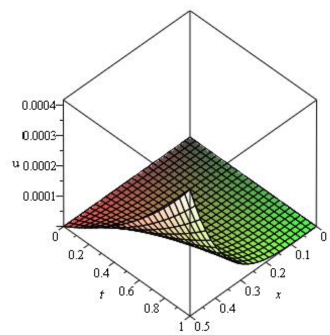


FIGURE 2. The surface shows the 4th order HATM approximate solution $u(x, y, z, t)$ at $y = z = 0.5$, $\hbar = -1$ and $\alpha = 1$ for Eq. (22).

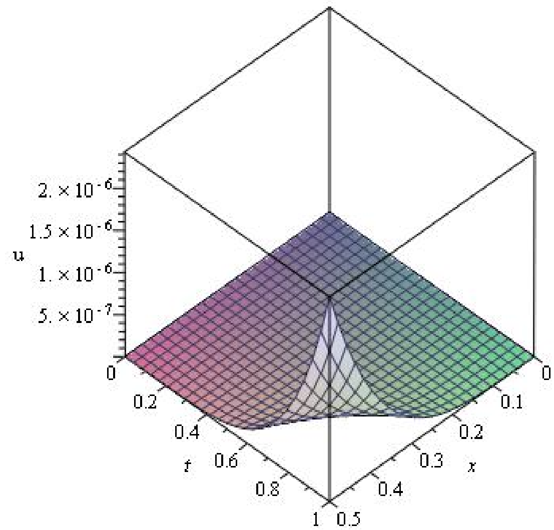


FIGURE 3. The absolute error $E_4(u) = |u_{ex} - u_{app}|$ when $y = z = 0.5$,$\hbar = -1$ and $\alpha = 1$ for Eq. (22).

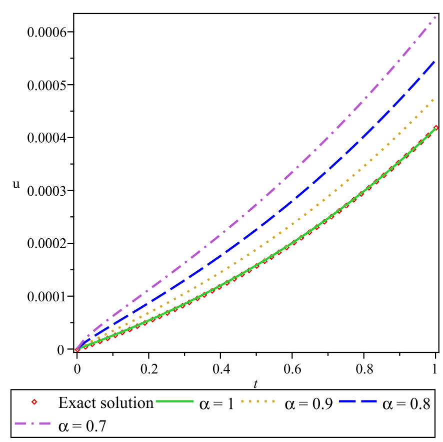


FIGURE 4. For 4th order approximate solutions versus time t for different values of $\alpha$ at $x = y = z = 0.5$ and $\hbar = -1$ for Eq. (22).

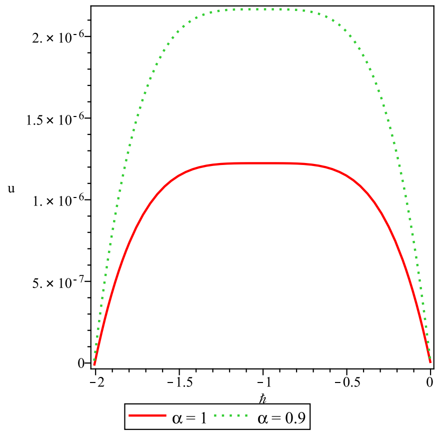


FIGURE 5. $\hbar$-curve at $x = y = z = 0.5$ and $t = 0.005$. for Eq. (22).

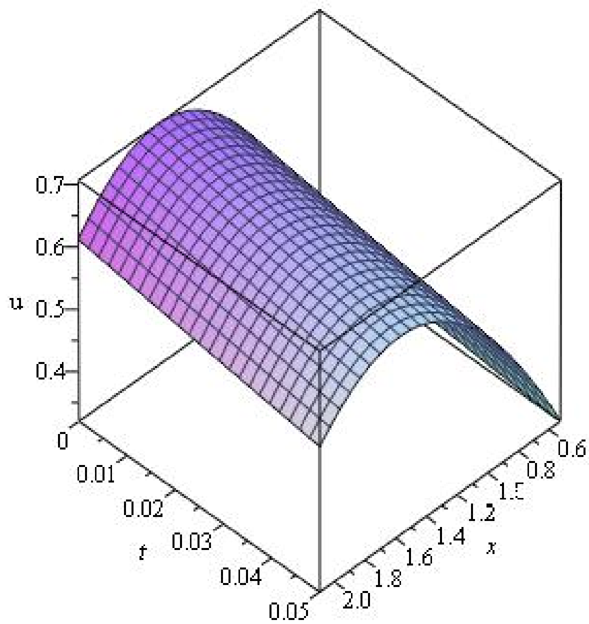


FIGURE 6. The surface shows the exact solution $u(x, y, t)$ at $y = \frac{\pi}{4}$ and $\alpha = 1$ for Eq. (31).

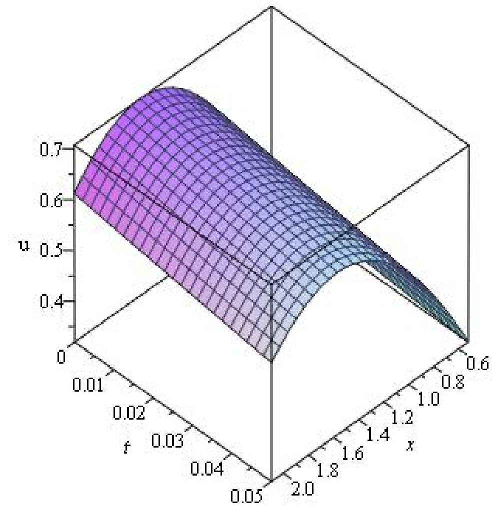


FIGURE 7. The surface shows the 4th order HATM approximate solution $u(x, y, t)$ at $y = \frac{\pi}{4}$, $\hbar = -1$ and $\alpha = 1$ for Eq. (31).

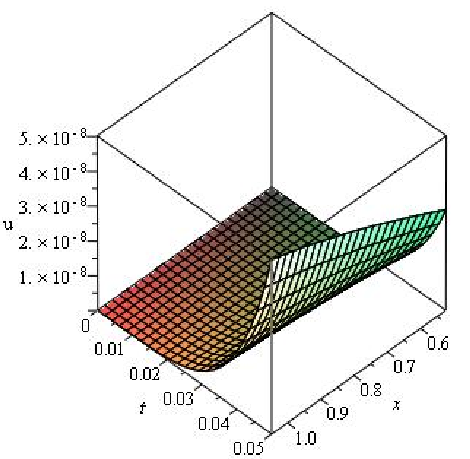


FIGURE 8. The absolute error $E_4(u) = |u_{ex} - u_{app}|$ when $y = \frac{\pi}{4}$, $\hbar = -1$ and $\alpha = 1$ for Eq. (31).

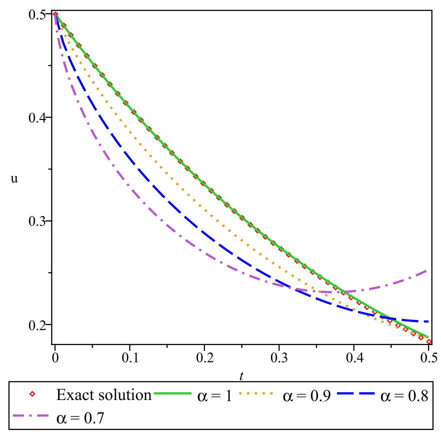


FIGURE 9. For 4th order approximate solutions versus time t for different values of $\alpha$ at $x = y = \frac{\pi}{4}$ and $\hbar = -1$ for Eq. (31).

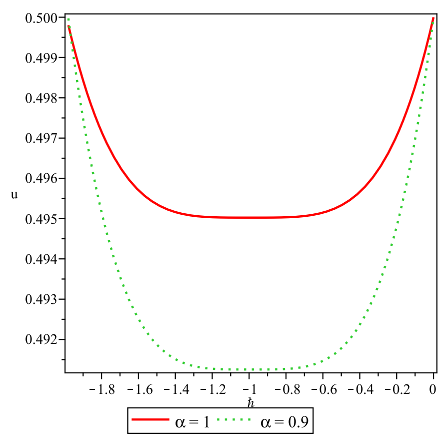


FIGURE 10. $\hbar$-curve at $x = y = \frac{\pi}{4}$ and $t = 0.005$. for Eq. (31).

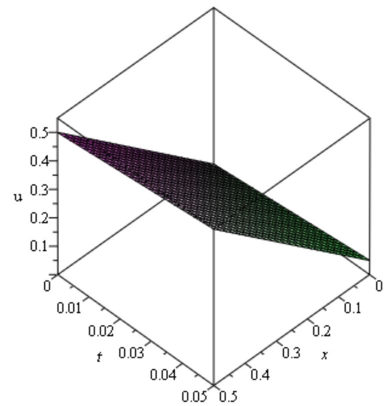


FIGURE 11. The surface shows the exact solution $u(x,t)$ at $\alpha = 1$ for Eq. (35).

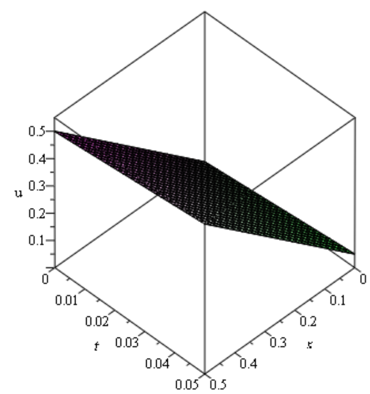


FIGURE 12. The surface shows the 4th order HATM approximate solution $u(x,t)$ at $\alpha = 1$ and $\hbar = -1$ for Eq. (35).

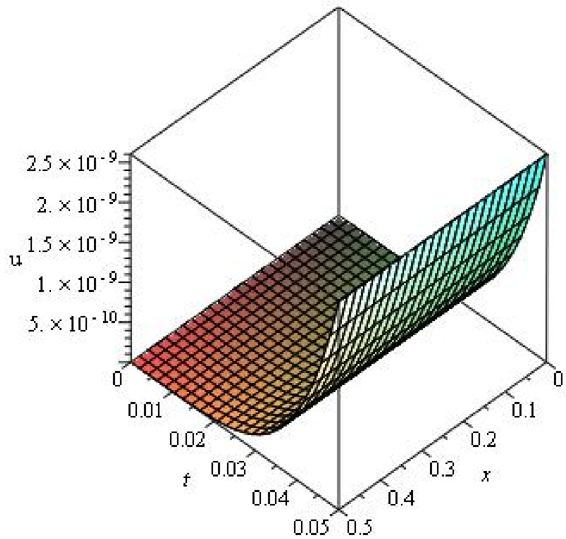


FIGURE 13. The absolute error $E_4(u) = |u_{ex} - u_{app}|$ when $\alpha = 1$ and $\hbar = -1$ for Eq. (35).

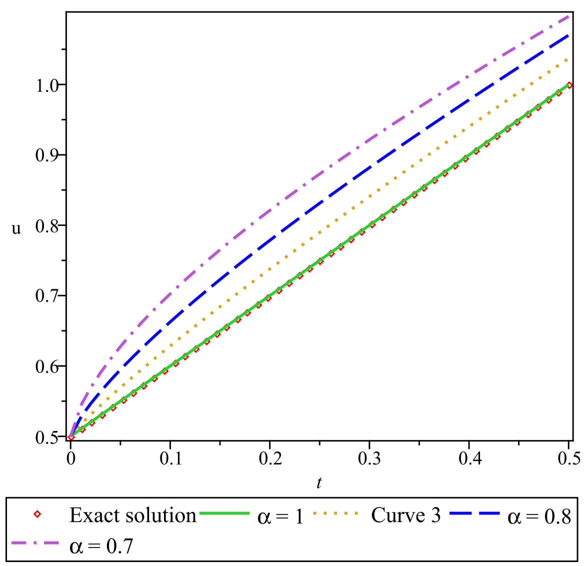


FIGURE 14. For 4th order approximate solutions $u(x,t)$ versus time t for different values of $\alpha$ at $x = 0.5$ and $\hbar = -1$ for Eq. (35).

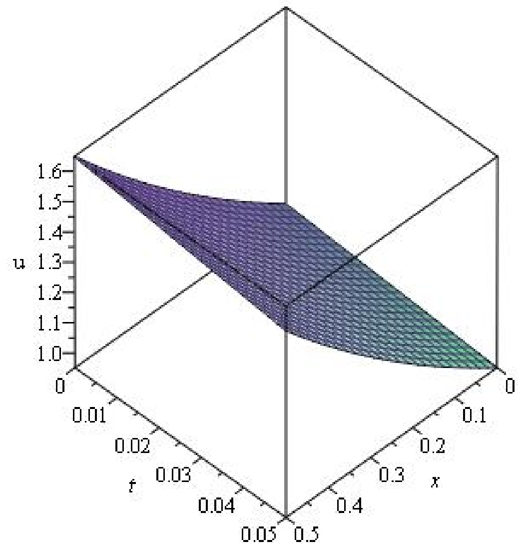


FIGURE 15. The surface shows the exact solution $u(x,t)$ at $\alpha = 1$ for Eq. (41).

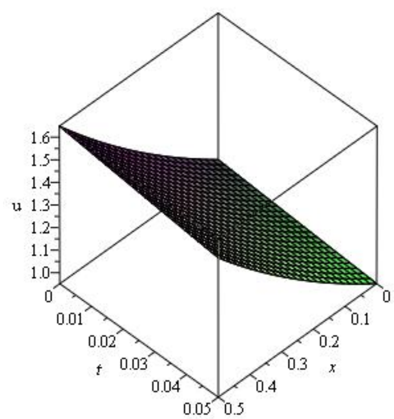


FIGURE 16. The surface shows the 4th order HATM approximate solution $u(x,t)$ at $\alpha = 1$ and $\hbar = -1$ for Eq. (41).

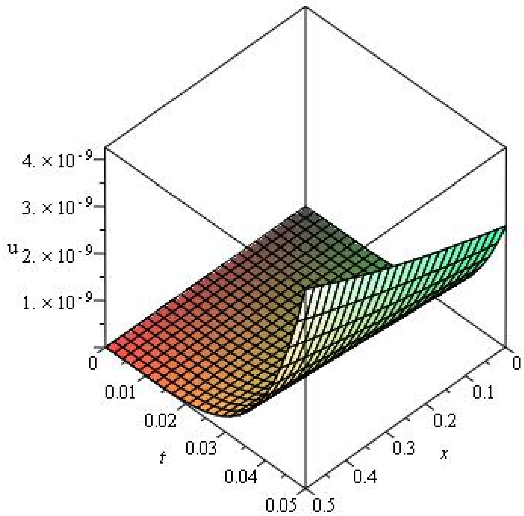


FIGURE 17. The absolute error $E_4(u) = |u_{ex} - u_{app}|$ when $\alpha = 1$ and $\hbar = -1$ for Eq. (41).

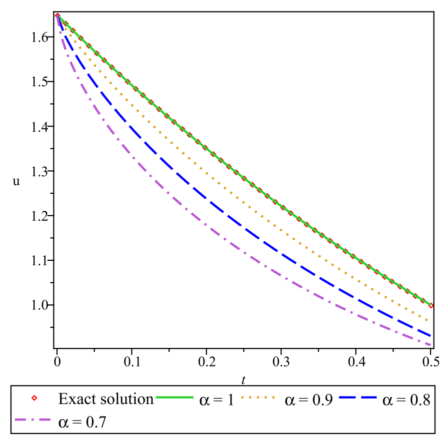


FIGURE 18. For 4th order approximate solutions $u(x,t)$ versus time t for different values of $\alpha$ at $x = 0.5$ and $\hbar = -1$ for Eq. (41).