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BOUND STATES OF PSEUDO-HARMONIC OSCILLATOR IN THE PRESENCE OF MAGNETIC FIELD

K. J. OYEWUMI, E. O. TITILOYE, A. B. ALABI AND B. J. FALAYE[1]

ABSTRACT. In this paper, we study the effect of external magnetic field on the bound state solution of the Schrödinger equation with the pseudoharmonic oscillator potential. The formula method has been applied in our calculations. The results obtained by using different Larmor frequencies and potential parameters are compared with the results of the absence of magnetic field case. We find that the energy spectrum given is mainly depending on dissociation energy and the magnetic quantum numbers $m = 0, \pm 1, \pm 2, \ldots$, which are influenced by the magnetic field pointing along z-axis, split energy to maximum and minimum levels.

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1. INTRODUCTION

Potential represents a field with which many important physical properties may be derived and understood. One of these potential field is called the pseudo-harmonic oscillator (PHO) potential, which has been extensively used to describe the bound state of the interaction systems and has been applied in both classical and modern physics. It plays a basic role in chemical and molecular physics by using it to calculate the molecular vibration-rotation energy spectrum of linear and non-linear systems [11]. This potential is considered as an intermediate between harmonic oscillator (HO) and Morse-type potentials which are more realistic anharmonic potentials. In the non-relativistic quantum mechanics, the PHO is one of the exactly solvable potentials which has been studied in one-dimensional (1D), two-dimensional (2D), three dimensional (3D) and even in D-dimensional space. The potential can be

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[1]Corresponding author

written in the standard form as [2]

$$V(r) = D_e \left(\frac{r}{r_e} - \frac{r_e}{r} \right)^2. \tag{1}$$

The pseudoharmonic oscillator behaves asymptotically as a harmonic oscillator, but has a minimum at $r = r_e$ and exhibits a repulsive inverse-square-type singularity at $r = 0$. The energy eigenvalues and the eigenfunctions of the pseudoharmonic oscillator can be found exactly for any angular momentum. These wavefunctions have reasonable behavior at the origin, near the equilibrium, and at the infinity [3]. Its characteristics make it useful to model various physical systems, including some molecular physical ones.

From the mathematical point of view, it resembles the harmonic oscillator, from which it deviates by two correction terms depending on the potential depth and the equilibrium distance parameter r_e: the first one is an energy shift and the second one is a modified centrifugal term. The latter can also be viewed as originating formally from a non-integer orbital angular momentum [4]. The eigenfunctions and energy eigenvalues are similar to those of the harmonic oscillator, which can be obtained exactly in the limit $r_e \to 0$.

In the recent years, there has been several worthy attempt in solving both the relativistic and non-relativistic wave equations in the presence of PHO. For arbitrary values n and ℓ quantum numbers, Oyewumi and Sen present the solutions of the 3-dimensional Schrödinger wave equation with the PHO via the $SU(1,1)$ Spectrum Generating Algebra (SGA) approach. They also obtained the matrix elements r^2 and $r\frac{d}{dr}$ directly from the creation and annihilation operators [5].

The discrete (bound) and continuous (scattering) energy spectra of the PHO have been investigated by the $SU(1,1)$ spectrum generating algebra [6]. The exact bound-state solutions of the KG and the Dirac equations with equal scalar and vector PHO potential have been obtained using the supersymmetric quantum mechanics, shape invariance and other alternative methods [7]. The bound-state solutions of the Dirac equation with PHO have been obtained in the presence of spin and pseudospin symmetries [8].

The spectral properties in a 2D charged particle (electron or hole) confined by a PHO potential in the presence of external strong uniform magnetic field along the z direction and Aharonov-Bohm (AB) flux field created by a solenoid have been studied. The Schröodinger equation is solved exactly for its bound states (energy spectrum and wave functions) [9]. Ikhdair and Hamzavi study the effects of the perpendicular magnetic and Aharonov-Bohm (AB) flux fields on the energy levels of a two-dimensional (2D) Klein-Gordon (KG) particle subjects to equal scalar and vector pseudo-harmonic oscillator (PHO) [10]. They have also obtained the exact energy spectra and corresponding wave functions of the spherical quantum dots for any (n, ℓ) state in the presence of a combination of pseudo-harmonic, Coulomb and linear confining potential terms within the exact analytical iteration method (EAIM) [11].

2. CALCULATION AND RESULT

In this paper, we study the Schrödinger equation with the pseudo-harmonic potential in the presence and absence of a constant magnetic field within the framework of the formula method [12]. The results obtained by using different Larmor frequencies and potential parameters are compared with the results of the absence of the magnetic field case. To begin, we write the two dimensional radial Schrödinger equation for a charged particle moving in a constant magnetic field as [13]

$$\frac{d^2R(r)}{dr^2} + 2\left[E - U_{\text{eff.}}(r)\right]R(r) = 0, \tag{2}$$

with the effective potential as

$$U_{\text{eff.}} = m\omega_L + \frac{1}{2}\omega_L^2 r^2 + \frac{m^2 - \frac{1}{4}}{2r^2} + V(r), \tag{3}$$

where $\omega_L = B/2c$, m and E denotes the Larmor frequency, the eigenvalue of angular momentum and the energy eigenvalue of the particle respectively. by taking the $V(r)$ as the PHO, the effective potential depending on the magnetic field strength can be written as

$$U_{\text{eff.}} = m\omega_L - 2D_e + \frac{\frac{m^2-\frac{1}{4}}{2} + D_e r_e^2}{r^2} + \left(\frac{\omega_L^2}{2} + \frac{D_e}{r_e^2}\right) r^2. \tag{4}$$

In Figure , we plot the effective potential for the case of low vibrational ($n = 0, 1, 2, 3$) and rotational (m = 1) levels. It can be seen from the Figure that when the magnetic field strength increases, there is a corresponding increment in potential energy function as well as the bound state energy eigenvalues. Now, by introducing a new variable of the form $s = r^2$, equation (2) can be easily transformed to the following form

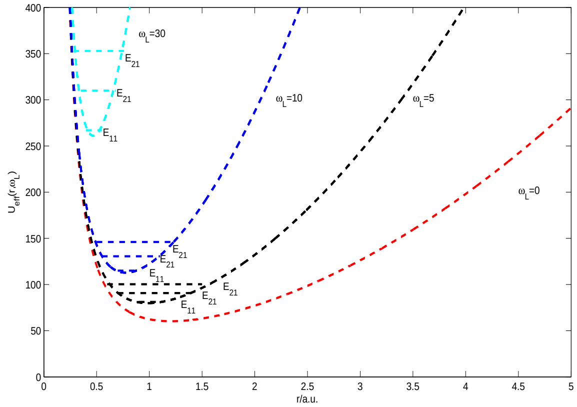

Fig. 1. The effective potential energy function and corresponding bound state energy levels (E_{nm}) in case of the low vibrational ($n = 1, 2, 3$) and rotational ($m = 1$) levels With $D_e = 8$ and $r_e = 1.2$.

$$s\frac{d^2R(s)}{ds^2} + \frac{1}{2}\frac{dR(s)}{ds} + \frac{1}{4}\left[2(E - m\omega_L + 2D_e) - \left(\omega_L^2 + 2\frac{D_e}{r_e^2}\right)s + \frac{\frac{1}{4} - m^2 - 2D_e r_e^2}{s}\right]R(s) = 0. \tag{5}$$

We solve equation (5) using the formula method. By following the procedure describe in the appendix, it is straightforward to obtain an explicit expression for the energy equation in the presence of the external magnetic field as:

$$E_{nm} = m\omega_L - 2D_e + \sqrt{\omega_L^2 + 2\frac{D_e}{r_e^2}}\left[2n + 1 + \sqrt{m^2 + 2D_e r_e^2}\right]. \tag{6}$$

Furthermore, we obtain the radial wave function as

$$\begin{aligned} R_{nm}(r) &= N_n r^\eta \exp\left(\sqrt{\frac{m{\omega_L}^2}{4} + \frac{D_e}{2r_e^2}}\right) \\ &\quad \times {}_1F_1\left(-n, 2\eta + \frac{1}{2}, s^2\sqrt{{\omega_L}^2 + 2\frac{D_e}{r_e^2}}\right), \\ &\quad \eta = \frac{1}{4} + \frac{1}{2}\sqrt{m^2 + 2D_e r_e^2}. \end{aligned} \tag{7}$$

N_n is the normalization constant. In Tables 1 & 2, we show the effect of a varying magnetic field on the low vibrational and rotational energy levels of the pseudoharmonic potential. It has been shown that as the magnetic field strength increases, the energy levels increases.

TABLE 1. For various Lamor frequency ω_L, the energy eigenvalues E_{nm} of a particle subjected to pseudoharmonic potential field with different D_e and r_e and for $m = 0$

	$D_e = 2.05$, $r_e = 1.781$			$D_e = 2.13$, $r_e = 1.694$		
n	$\omega_L = 0$	$\omega_L = 5$	$\omega_L = 10$	$\omega_L = 0$	$\omega_L = 5$	$\omega_L = 10$
0	1.136915	19.51914	42.25924	1.218404	18.87975	41.03628
1	3.451075	29.77440	62.38808	3.655213	29.17237	61.18142
2	5.684575	40.02965	82.51693	6.092020	39.46499	81.33201
3	7.958410	50.28491	102.6458	8.528830	49.75761	101.4800
	$D_e = 2.25$, $r_e = 1.776$			$D_e = 4.56$, $r_e = 1.768$		
0	1.194437	20.00805	43.51353	1.708110	24.37474	55.19056
1	3.583311	30.28943	63.65569	5.124320	34.94217	75.48022
2	5.972190	40.57081	83.79785	8.540540	45.50960	95.76990
3	8.361060	50.85219	103.9400	11.95675	56.07703	116.0596

4. CONCLUDING REMARKS

In this paper, we have obtained the solution of two-dimensional radial Schrödinger equation with the pseudo-harmonic oscillator for low vibrational and rotational energy levels with varying magnetic field having arbitrary Larmor frequencies. We found that the potential energy function and corresponding energy levels are raised when the magnetic field strength increases. The details explanation for this physical behaviors are as follows. Firstly, w define the Bohr magneton as $e\hbar/2Mc$, which has the value

$$\mu_b = \frac{|e|\,\hbar}{2Mc} = 0.927 \times 10^{-20}\text{erg/gauss}. \tag{8}$$

TABLE 2. For various Lamor frequency ω_L, the energy eigenvalues E_{nm} of a particle subjected to pseudoharmonic potential field with different D_e and r_e and for $m = 1$

	$D_e = 2.05$, $r_e = 1.781$			$D_e = 2.13$, $r_e = 1.694$		
n	$\omega_L = 0$	$\omega_L = 5$	$\omega_L = 10$	$\omega_L = 0$	$\omega_L = 5$	$\omega_L = 10$
0	1.291627	25.21691	53.62882	1.389218	24.60123	52.44859
1	3.565457	35.47217	73.75766	3.826026	34.89385	72.59650
2	5.839287	45.72743	93.88650	6.262835	45.18647	92.74440
3	8.113117	55.98268	114.0153	8.699643	55.47909	112.8923
	$D_e = 2.25$, $r_e = 1.776$			$D_e = 4.56$, $r_e = 1.768$		
0	1.350259	25.67869	54.82737	29.27175	1.789920	64.46741
1	3.739134	35.96007	74.96953	39.79063	5.052863	84.73183
2	6.128008	46.24144	95.11169	50.30951	8.315805	104.9963
3	8.516882	56.52282	115.2539	60.82839	11.57875	125.2607

The relationship between Bohr magneton, magnetic field and Larmor frequency is given by [10]

$$\hbar\omega_L = \mu_b B. \tag{9}$$

Thus, For an electron, one finds the magnetic moment is directly proportional to its spin angular momentum as follows

$$\vec{\mu} = -\frac{e}{Mc}\vec{S} = -\frac{e\hbar}{2Mc}\vec{\sigma} = -\mu_b\vec{\sigma}. \tag{10}$$

Let us now consider the problem of calculating the eigenstates and eigenenergies of the present model, i.e., a spinning but otherwise fixed electron in a constant uniform magnetic field that points in the z direction. To solve this problem, we have used the Schrödinger equation. In this regard, the following can be deduced

$$\widehat{H} = -\vec{\mu}\cdot\vec{B} = \mu_b\vec{\sigma}\cdot\vec{B} = \mu_b B\sigma_z = \hbar\omega_L\widehat{\sigma}_z,\ \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ \vec{\mu} = -\mu_b\vec{\sigma}. \tag{11}$$

Now, let $|\psi\rangle = \begin{pmatrix} f \\ g \end{pmatrix}$, so that

$$\widehat{H}\,|\psi\rangle = E\,|\psi\rangle \;\to \hbar\omega_L \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} f \\ g \end{pmatrix} = E \begin{pmatrix} f \\ g \end{pmatrix}, \tag{12}$$

More explicitly, this can be written as

$$\hbar\omega_L f = Ef, \quad -\hbar\omega_L g = Eg. \tag{13}$$

If $f \neq 0$, $g = 0$, then $E = +\,\hbar\omega_L = +\mu_b B$. If $g \neq 0$, $f = 0$, then $E = -\,\hbar\omega_L = -\mu_b B$. Thus we obtain the normalized eigenstates and eigenenergies

$$\alpha = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad E = +\hbar\omega_L = +\mu_b B, \tag{14a}$$

$$\beta = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad E = -\hbar\omega_L = -\mu_b B. \tag{14b}$$

In the state of lower energy, the spin of the electron is anti-parallel to $\vec{B}$, so the magnetic moment is parallel to $\vec{B}$ and the interaction energy $-\vec{\mu}\cdot\vec{B}$ is minimum the state of higher energy, the spin of the electron is parallel to $\vec{B}$, so the magnetic moment is anti-parallel to $\vec{B}$ and the interaction energy $-\vec{\mu}\cdot\vec{B}$ is maximum. The energy spectrum given by equation (6) is mainly depending on dissociation energy and the magnetic quantum numbers $m = 0, \pm 1, \pm 2, \ldots$, which are influenced by the magnetic field pointing along z-axis, split energy to maximum and minimum levels.

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THEORETICAL PHYSICS SECTION, DEPARTMENT OF PHYSICS, UNIVERSITY OF ILORIN, P. M. B. 1515, ILORIN, NIGERIA
E-mail addresses: `kjoyewumi66@unilorin.edu.ng, remi050970@gmail.com`
DEPARTMENT OF MATHEMATICS, UNIVERSITY OF ILORIN, P. M. B. 1515, ILORIN, NIGERIA.
E-mail address: `eotitiloye@yahoo.com`
APPLIED THEORETICAL PHYSICS DIVISION, DEPARTMENT OF PHYSICS, FEDERAL UNIVERSITY LAFIA, P. M. B. 146, LAFIA, NIGERIA.
E-mail address: `babatunde.falaye@fulafia.edu.ng`