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A NOTE ON BIWEIGHT-PRESERVING MAPS OF PARTIAL *-ALGEBRAS

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ABSTRACT. Given two partial *-algebras $\mathcal{U}_1$ and $\mathcal{U}_2$ and a biweight φ on $\mathcal{U}_2$, if a *-linear map θ from $\mathcal{U}_1$ into $\mathcal{U}_2$ belonging to a family is given, we introduce the notion of a γ *-linear map and consider when the natural composition $\varphi_\gamma = \varphi \circ \gamma$ is a biweight on $\mathcal{U}_1$,where γ is not necessarily a *-homomorphism.

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1. INTRODUCTION

Positive invariant sesquilinear form on a partial *-algebra extend the notion of a positive functional on a *-algebra. Biweights are positive sesquilinear forms that are invariant in some sense. In developing a representation theory for partial *-algebras there is need to provide an extension of the Gel'fand-Naimark-Segal (GNS) construction to the partial *-algebraic framework. Such construction will help in the concrete representation theory for partial *-algebras,thus the need to generalized the notion of positive linear functionals on *-algebras to biweights on partial *-algebras. Biweights are sesquilinear forms that allow the GNS construction to be carried out. The difficuties encountered due to non-everywhere defined multiplication and also the lack of associativity of the partial multiplication of a partial *algebra is by-passed when biweights are used in the GNS- construction.
Biweights as sesquilinear forms exhibit unfamilar features, in particular the composition $\varphi_\Phi = \varphi \circ \Phi$ of a biweight φ on $\mathcal{U}_2$ and a *-homomorphism Φ on $\mathcal{U}_1$ into $\mathcal{U}_2$ may fail to be a biweight. The problem of biweight-preserving maps has been considered in [1]. Biweight-preserving is the case for which the natural composition $\varphi \circ \Phi$ still gives a biweight on $\mathcal{U}_1$, then we call Φ a biweight- preserving map . In [1] conditions for a *-homomorphism Φ to be

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biweight-preserving is given. This problem is important, since we know that every biweight is representable on a partial *-algebra, but if $\varphi \circ \Phi$ is not a biweight the representation may as well be trivial. Another important reason is with regard to physical applications, since biweights allows the GNS construction to be carried out in the partial *-algebraic framework which is crucial in developing a representation theory in statistical mechanics and quantum field theory [2] and [3] . In this paper,we examine the case when a composition $\varphi_\gamma = \varphi \circ \gamma$ is a biweight on $\mathcal{U}_1$ and γ is a *-linear map that is not a *-homomorphism. We relax the the condition $\Phi(x)\Phi(y) = \Phi(xy)$ of a *-homomorphism and state a similar result given in [1] for such map.
The paper is arranged as follows; In Section 1 we introduce the notion of a γ *-linear map on partial *-algebras and definitions of *-homomorphism and *-representation. In Section 2 we introduce the notion of biweights and outline the GNS construction for a biweight and we also state a definition for a representable form of a biweight. In Section 3,following[1] we state and prove the condition for the composition to be a biweight, and hence the map γ a biweight-preserving map. We start with the following definition of a partial *-agebras;

2. PRELIMINARY

We define some basic results from [2]

Definition 1: A partial *-algebra is a vector space $\mathcal{U}$ equipped with a vector space involution $*: \mathcal{U} \to \mathcal{U}: x \to x^*$ satisfying $x^{**} = x$ and partial multiplication ' .' defined by a relation $\Gamma \subset \mathcal{U} \times \mathcal{U}$ such that

(1) $(x, y) \in \Gamma$ implies $(y^*, x^*) \subset \Gamma$;
(2) $(x, y_1), (x, y_2) \in \Gamma$ and $\lambda, \mu \in \mathbb{C}$ imply $(x, \lambda y_1 + \mu y_2) \in \Gamma$
(3) for every $(x, y) \in \Gamma$, a product $xy \in \mathcal{U}$ is defined, such that xy depends linearly on y and satisfies the inequality $(xy)^* = y^* x^*$.

Whenever $(x, y) \in \Gamma$, we say that x is a left multiplier of y and y a right multiplier of x and we write $x \in L(y)$, respectively $y \in R(x)$, the product is distributive with respect to addition, that is, for $(x, v), (x, z), (y, z) \in \Gamma$ implies $(x, \alpha v + \beta z), (\alpha x + \beta z) \in \Gamma$, and then $(\alpha x + \beta y) \cdot z = \alpha(x.z) + \beta(y \cdot z)$ and $x.(\alpha v + \beta z) = \alpha(x.v) + \beta(x.z)$ for all $\alpha, \beta \in \mathbb{C}$, the complex numbers. In addition, if we asume the partial *-algebra $\mathcal{U}$ contains a unit we denote this by e, and satisfies the following, $e^* = e$, $(e, x) \in \Gamma$, and $e.x = x.e = x, \forall x \in \mathcal{U}$. Given any subset $A \subset \mathcal{U}$ we write $LA = \cap_{x \in A} L(x)$ and $RA = \cap_{x \in A} R(x)$ for the universal left and right multipliers of A respectively. The partial multiplication is not required to be associative in general, it said to be semi-associative if $y \in R(x)$ implies that $y.z \in R(x)$ for every $z \in R(\mathcal{U})$.

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A concrete partial*-algebras arises in the following way. Let $\mathcal{H}$ be a complex Hilbert space and $\mathcal{D}$ a dense subspace of $\mathcal{H}$. Denote $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ the set of all closable linear operators X such that the domain $\mathcal{D}(X) = \mathcal{D}$ and $\mathcal{D}(X^*) \supseteq \mathcal{D}$. The set $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a partial *-algebra with respect to the following operations. The usual sum $X_1 + X_2$, the scalar multiplication λX, the involution $X \to X^\dagger = X^*$upharpoonright $\mathcal{D}$ restricted to $\mathcal{D}$ and the (weak) partial multiplication $X_1 \Box X_2 = X_1^{\dagger *} X_2$ defined whenever $X_2 \in R^w(X_1)$ and $X_1 \in L^w(X_2)$, that is, if and only if $X_2\mathcal{D} \subset \mathcal{D}(X_1^{\dagger *})$ and $X_1^*\mathcal{D} \subset \mathcal{D}(X_2^*)$. When we regard $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ as a partial *-algebra with these operations, we denote it by $\mathcal{L}_w^\dagger(\mathcal{D}, \mathcal{H})$. A *-subalgebra $\mathcal{M}$ of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is called a partial O* -algebra, that is, a subspace of $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ such that $X^\dagger \in \mathcal{M}$ whenever $X \in \mathcal{M}$ and $X_1 \Box X_2 \in \mathcal{M}$ for any $X_1, X_2 \in \mathcal{M}$ such that $X_2 \in R^w(X_1)$. In Antoine et al.[2] a theory for such algebras have been studied and a detailed literature provided.we now have the following definitions;

Definition 2: A *- Homomorphism of a partial *-algebra $\mathcal{U}_1$ into another one $\mathcal{U}_2$ is a linear map $\Phi\colon \mathcal{U}_1 \to \mathcal{U}_2$ such that

(1) $\Phi(x^*) = \Phi(x)^*$ for every $x \in \mathcal{U}_1$
(2) whenever $x \in L(y)$ in $\mathcal{U}_1$ then $\Phi(x) \in L(\Phi(y))$ in $\mathcal{U}_2$ and $\Phi(x)\Phi(y) = \Phi(xy)$.

The map Φ is a *-isomorphism if it is a bijection and Φ^{-1} is also a *-homomorphism. The image $\Phi(\mathcal{U}_1)$ need not be a patial *- algebra of $\mathcal{U}_2$ if Φ is not a *-isomorphism

Definition 3: We define a map, which we will call a $\hat{\gamma}$ *-linear map.

Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be partial *-algebras. A linear map of a partial *-algebra $\mathcal{U}_1$ into $\mathcal{U}_2$ satisfying the following properties

(1) $\hat{\gamma}(R(\mathcal{U}_1)) \subseteq R(\mathcal{U}_2)$

is called a $\hat{\gamma}$ -linear map on $\mathcal{U}_1$.

Definition 4: A representation of a partial *-algebra $\mathcal{U}$ is a *-homomorphism of $\mathcal{U}$ into $\mathcal{L}^{\dagger}_{w}(\mathcal{D}, \mathcal{H})$ for some $\mathcal{D} \subset \mathcal{H}$, that is a linear map $\pi \colon \mathcal{U} \to \mathcal{L}^{\dagger}_{w}(\mathcal{D}, \mathcal{H})$ such that

(1) $\pi(x^*) = \pi(x)^*$ for every $x \in \mathcal{U}$
(2) $x \in L(y)$ in $\mathcal{U}$ implies $\pi(x) \in L^w(\pi(Y))$ and $\pi(x)\Box\pi(y) = \pi(xy)$.

3. Biweights

In this Section we will state the definition of a biweight given in Antoine et al [3], but first we will give an outline of the GNS-construction for a biweight as may be found in [2]and [3].
Let φ be a positive sesquilinear form on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$ where $\mathcal{D}(\varphi)$ is a subspace of a partial *-algebra $\mathcal{U}$. Then we have $\varphi(x, y) = \overline{\varphi(y, x)}$ $\forall x, y \in \mathcal{D}(\varphi)$ and $|\varphi(x, y)|^2 \leq \varphi(x, x)\varphi(y, y)$
$\forall x, y \in \mathcal{D}(\varphi)$ .The kernel $\mathcal{N}_\varphi = \{x \in \mathcal{D}(\varphi) \colon \varphi(x, y) = 0, \forall y \in \mathcal{D}(\varphi)\}$ of φ is a vector subspace of $\mathcal{D}(\varphi)$. We define the map $\lambda_\varphi \colon \mathcal{D}(\varphi) \to \mathcal{D}(\varphi) \setminus \mathcal{N}_\varphi$ by $\lambda_\varphi(x) = x + \mathcal{N}_\varphi$. This give the coset containing x and thus $\mathcal{D}(\varphi) \setminus \mathcal{N}_\varphi$ is a pre-Hilbert space with respect to the inner product $\langle \lambda_\varphi(x), \lambda_\varphi(y) \rangle = \varphi(x, y)$. We denote the completion of $\mathcal{D}(\varphi) \setminus \mathcal{N}_\varphi$ with respect to the inner product by $\mathcal{H}_\varphi$.

Definition 5: Let φ be a sesquilinear form on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$. A subspace $\mathcal{B}(\varphi)$ of $\mathcal{D}(\varphi)$ is said to be a *precore* for φ if

(1) $\mathcal{B}(\varphi) \subseteq R\mathcal{U}$
(2) $\{ax \colon a \in \mathcal{U}, x \in \mathcal{B}(\varphi)\} \subseteq \mathcal{D}(\varphi)$
(3) $\varphi(ax, y) = \varphi(x, a^*y)$ $\forall a \in \mathcal{U}$, $x, y \in \mathcal{B}(\varphi)$
(4) $\varphi(a^*x, by) = \varphi(x, (ab)y)$ $a \in L(b)$, $\forall x, y \in \mathcal{B}(\varphi)$
The subspace $\mathcal{B}(\varphi)$ is called a *core* if in addition
(5) $\lambda_\varphi(\mathcal{B}(\varphi))$ is dense in $\mathcal{H}_\varphi$

We denote by $\mathcal{P}_\varphi$ the set of all *precores* for φ and $\mathcal{B}_\varphi$ the set of all *cores* $\mathcal{B}(\varphi)$ for φ

Definition 6: A positive sesquilinear form φ on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$ such that $\mathcal{B}_\varphi \neq \emptyset$ is called a biweight.

We put $\mathcal{D}(\pi_\varphi) = \{\lambda_\varphi(x) \colon x \in \mathcal{B}(\varphi)\}$
and $\pi_\varphi(a)\lambda_\varphi = \lambda_\varphi(ax)$ $\forall a \in \mathcal{U}$, $x \in \mathcal{B}(\varphi)$. This is a well-defined linear map and from conditions (2)and(5) is a *-representation [1]

and [3]. We denote by π_φ its closure. We have the following

Remark[1] ; The triple $(\pi_\varphi, \lambda_\varphi, \mathcal{H}_\varphi)$ is called the GNS construction for the biweight φ.

Definition 7: A positive sesquilinear form φ on $\mathcal{D}(\varphi) \times \mathcal{D}(\varphi)$ is called a *representable form* if there exist a precore $\mathcal{B}(\varphi)$ for φ and a *-representation π_φ defined on a dense subspace $\mathcal{D}(\pi_\varphi)$ of $\mathcal{H}_\varphi$ with $\lambda_\varphi(\mathcal{B}(\varphi)) \subseteq \mathcal{D}(\varphi)$ such that $\varphi(ax, by) = \langle \pi_\varphi(a)\lambda_\varphi(x), \pi_\varphi(b)\lambda_\varphi(y)\rangle$.

Definition 8: Let $\mathcal{U}_1$, $\mathcal{U}_2$ be partial *-algebras, and φ a biweight on $\mathcal{U}_2$ with domain $\mathcal{D}(\varphi)$ and core $\mathcal{B}(\varphi)$, a *-linear map $\theta\colon \mathcal{U}_1 \to \mathcal{U}_2$ is said to be biweight- preserving if φ_θ is a biweight on $\mathcal{U}_1$ with core $\mathcal{B}(\varphi_\theta)$, where φ_θ is given by $\varphi_\theta = \varphi \circ \theta$

4. Biweights-Preserving Maps

In this Section we introduce the notion of γ^*- linear map which is not a *-homomorphism and state condition for the composition $\varphi_\gamma = \varphi \circ \gamma$ to have a nonzero subspace that satisfies the condition of a precore for φ_γ and we also state the conditon for the map φ_γ to be representable and a biweight on $\mathcal{U}_1$. We start with the following,

Definition 9: Let $\mathcal{U}_1$, $\mathcal{U}_2$ be partial *-algebras, and let θ be a non-multiplicative *-linear map $\theta\colon \mathcal{U}_1 \to \mathcal{U}_2$,we denote the collection of all such maps by $\mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$ and $\forall a \in \mathcal{U}_1$ and $\theta \in \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2)$ we define the map $\gamma\colon \mathcal{U}_1 \times \mathcal{L}(\mathcal{U}_1, \mathcal{U}_2) \to \mathcal{U}_2$,with the following properties

(1) $\gamma(a, \theta)^* = \gamma(a^*, \theta)$
(2) $\gamma(ax, \theta) = \theta(a)\hat{\gamma}(x)$
(3) $\gamma(a^*x, \theta) = \theta(a^*)\hat{\gamma}(x)$, $\forall a \in \mathcal{U}_1, x \in R(\mathcal{U}_1)$

γ is well-defined but is not a *-homomorphism since, we have $\gamma(x_1x_2) \neq \gamma(x_1)\gamma(x_2)$ but a $\hat{\gamma}$ *-linear map. In the sequel for any $x \in R(\mathcal{U}_1)$, we denote $\hat{\gamma}(x)$ by $\gamma(x, \theta) \in R(\mathcal{U}_2)$.

For a partial *-algebra $\mathcal{U}_2$ with a biweight φ having domain $\mathcal{D}(\varphi)$ and $\mathcal{U}_1$ a partial *-algebra, we define the map φ_γ on $\mathcal{D}(\varphi_\gamma)$ as follows; put $\mathcal{D}(\varphi_\gamma) = \{a \in \mathcal{U}_1 \colon \gamma(a, \theta) \in \mathcal{D}(\varphi)\}$ for $\gamma(x, \theta), \gamma(y, \theta) \in \mathcal{D}(\varphi)$ and $x, y \in \mathcal{D}(\varphi_\gamma)$ we have $\varphi_\gamma(x, y) = \varphi(\gamma(x, \theta), \gamma(y, \theta))$, define a subspace $\mathcal{B}(\varphi_\gamma)$ of $\mathcal{D}(\varphi_\gamma)$ as the subspace $\mathcal{B}(\varphi_\gamma) = \{x \in R(\mathcal{U}_1) \colon \gamma(ax, \theta) \in \mathcal{D}(\varphi), \forall a \in \mathcal{U}_1\}$.
We note that $\varphi_\gamma(x, y) = \varphi(\gamma(x, \theta), \gamma(y, \theta))$ is a positive sesquilinear form on $\mathcal{D}(\varphi_\gamma) \times \mathcal{D}(\varphi_\gamma)$.
We have the following,

Proposition 1: The subspace $\mathcal{B}(\varphi_\gamma)$ is a precore for the positive sesquilinear form φ_γ given by $\varphi_\gamma(x,y) = \varphi(\gamma(x,\theta),\gamma(y,\theta))$ on $\mathcal{D}(\varphi_\gamma) \times \mathcal{D}(\varphi_\gamma)$.

Proof. We note that $\mathcal{B}(\varphi_\gamma)$ satisfies condition (i) that is, $\mathcal{B}(\varphi_\gamma) \subseteq R(\mathcal{U}_1)$ for condition (ii) let $a \in \mathcal{U}_1$ such that $\gamma(a,\theta) \in \mathcal{D}(\varphi)$ and $x \in \mathcal{B}(\varphi_\gamma)$ such that $\gamma(ax,\theta) \in \mathcal{D}(\varphi)$ implies from the definition of φ_γ that the set $\{ax\colon a \in \mathcal{U}_1, x \in \mathcal{B}(\varphi_\gamma)\} \subseteq \mathcal{D}(\varphi_\gamma)$. As for condition (iii) and (iv) we apply the properties of γ stated in definition 3.1 and the defintion of φ_γ
condition (iii) we have;

$$\begin{aligned}\varphi_\gamma(ax,y) =& \varphi(\gamma(ax,\theta),\gamma(y,\theta))\\ =&\varphi(\theta(a)\gamma(x,\theta),\gamma(y,\theta))\\ =&\varphi(\gamma(x,\theta),\theta(a)^*\gamma(y,\theta))\\ =&\varphi(\gamma(x,\theta),\gamma(a^*y,\theta))\\ =&\varphi_\gamma(x,a^*y)\end{aligned}$$

and for condition (iv) we have

$$\begin{aligned}\varphi_\gamma(a^*x,by) =& \varphi(\gamma(a^*x,\theta),\gamma(by,\theta))\\ =&\varphi(\theta(a^*)\gamma(x,\theta),\gamma(by,\theta))\\ =&\varphi(\theta(a)^*\gamma(x,\theta),\gamma(by,\theta))\\ =&\varphi(\gamma(x,\theta),\theta(a)\gamma(by,\theta))\\ =&\varphi(\gamma(x,\theta),\gamma((ab)y,\theta))\\ =&\varphi_\gamma(x,(ab)y).\end{aligned}$$

Hence $\mathcal{B}(\varphi_\gamma)$ is a precore for φ_γ □

We now state the condition for φ_γ to be representable on $\mathcal{U}_1$. Let φ_γ be a positive sesquilinear form on $\mathcal{U}_1$ from [2] this gives rise to a triplet $(\mathcal{H}_{\varphi_\gamma},\mathcal{D}_{\varphi_\gamma},\pi_{\varphi_\gamma})$ which we outline as follows; define the set $\mathcal{N}_{\varphi_\gamma} = \{x \in \mathcal{U}_1\colon \varphi_\gamma(x,y) = 0, \forall y \in \mathcal{D}(\varphi_\gamma)\}$ and we have the map $\lambda_{\varphi_\gamma}\colon \mathcal{D}(\varphi_\gamma) \to \mathcal{D}(\varphi_\gamma) \setminus \mathcal{N}_{\varphi_\gamma}$ defined by $\lambda_{\varphi_\gamma}(x) = x + \mathcal{N}_{\varphi_\gamma}$, $x \in \mathcal{D}(\varphi_\gamma)$. The set $\mathcal{D}(\varphi_\gamma) \setminus \mathcal{N}_{\varphi_\gamma}$ is a pre-Hilbert space.The map λ_{φ_γ} induces the inner product $\langle\lambda_{\varphi_\gamma}(x),\lambda_{\varphi_\gamma}(y)\rangle = \varphi_\gamma(x,y)$.Then $\mathcal{H}_{\varphi_\gamma}$ is the Hilbert space obtained by completing $\lambda_{\varphi_\gamma}(\mathcal{U}_1)$ in the norm topology determine by $\langle\cdot,\cdot\rangle$. Let $\mathcal{U}_\varphi$ be a nonzero subspace of $R(\mathcal{U}_1)$ such that $\lambda_{\varphi_\gamma}(\mathcal{U}_\varphi)$ is dense in $\mathcal{H}_{\varphi_\gamma}$. Then we denote $\lambda_{\varphi_\gamma}(\mathcal{U}_\varphi)$ by $\mathcal{D}_{\varphi_\gamma}$. For $a \in \mathcal{U}_1$ we have the map $\pi_{\varphi_\gamma}\colon \mathcal{U}_1 \to \mathcal{L}^\dagger(\mathcal{H}_{\varphi_\gamma},\mathcal{D}_{\varphi_\gamma})$ defined by $\pi_{\varphi_\gamma}(a)\lambda_{\varphi_\gamma}(x) = \lambda_{\varphi_\gamma}(a.x)$, for $a \in \mathcal{U}_1$ and $x \in \mathcal{U}_\varphi$.
From [4] this map is well-defined but may not be a *-representation

unless φ_γ is an invariant positive sesquilinear form given in the following;

We now state the condition for φ_γ to be representable on $\mathcal{U}_1$. Let φ_γ be a positive sesquilinear form on $\mathcal{U}_1$ from [2] this gives rise to a triplet $(\mathcal{H}_{\varphi_\gamma}, \mathcal{D}_{\varphi_\gamma}, \pi_{\varphi_\gamma})$ which we outline as follows; define the set $\mathcal{N}_{\varphi_\gamma} = \{x \in \mathcal{U}_1 \colon \varphi_\gamma(x, y) = 0, \forall y \in \mathcal{D}(\varphi_\gamma)\}$ and we have the map $\lambda_{\varphi_\gamma} \colon \mathcal{D}(\varphi_\gamma) \to \mathcal{D}(\varphi_\gamma) \setminus \mathcal{N}_{\varphi_\gamma}$ defined by $\lambda_{\varphi_\gamma}(x) = x + \mathcal{N}_{\varphi_\gamma}$, $x \in \mathcal{D}(\varphi_\gamma)$. The set $\mathcal{D}(\varphi_\gamma) \setminus \mathcal{N}_{\varphi_\gamma}$ is a pre-Hilbert space.The map λ_{φ_γ} induces the inner product $\langle \lambda_{\varphi_\gamma}(x), \lambda_{\varphi_\gamma}(y)\rangle = \varphi_\gamma(x, y)$.Then $\mathcal{H}_{\varphi_\gamma}$ is the Hilbert space obtained by completing $\lambda_{\varphi_\gamma}(\mathcal{U}_1)$ in the norm topology determine by $\langle \cdot, \cdot \rangle$. Let $\mathcal{U}_\varphi$ be a nonzero subspace of $R(\mathcal{U}_1)$ such that $\lambda_{\varphi_\gamma}(\mathcal{U}_\varphi)$ is dense in $\mathcal{H}_{\varphi_\gamma}$. Then we denote $\lambda_{\varphi_\gamma}(\mathcal{U}_\varphi)$ by $\mathcal{D}_{\varphi_\gamma}$. For $a \in \mathcal{U}_1$ we have the map $\pi_{\varphi_\gamma} \colon \mathcal{U}_1 \to \mathcal{L}^\dagger(\mathcal{H}_{\varphi_\gamma}, \mathcal{D}_{\varphi_\gamma})$ defined by $\pi_{\varphi_\gamma}(a)\lambda_{\varphi_\gamma}(x) = \lambda_{\varphi_\gamma}(a.x)$, for $a \in \mathcal{U}_1$ and $x \in \mathcal{U}_\varphi$.
From [4] this map is well-defined but may not be a *-representation unless φ_γ is an invariant positive sesquilinear form given in the following;

Definition 10: We call φ_γ invariant if there is nonzero subspace $\mathcal{U}_\varphi$ be a nonzero subspace of $R(\mathcal{U}_1)$ such that

(1) $\lambda_{\varphi_\gamma}(\mathcal{U}_\varphi)$ is dense in $\mathcal{H}_\varphi$
(2) $\langle \pi_{\varphi_\gamma}(a)^\dagger \lambda_{\varphi_\gamma(x)}, \lambda_{\varphi_\gamma}(b.y)\rangle = \langle \lambda_{\varphi_\gamma(x)}, \lambda_{\varphi_\gamma}((a.b).y)\rangle$

Thus we will have $\pi_{\varphi_\gamma}(a) \Box \pi_{\varphi_\gamma}(b)\lambda_{\varphi_\gamma}(y) = \pi_{\varphi_\gamma}(a.b)\lambda_{\varphi_\gamma}(y)$

We now give a definition for a representable form for φ_γ;

Definition 11: A positive sesquilinear form φ_γ on $\mathcal{D}(\varphi_\gamma) \times \mathcal{D}(\varphi_\gamma)$ is called a *representable form* if there exist a precore $\mathcal{B}(\varphi_\gamma)$ for φ_γ and a *-representation π_{φ_γ} defined on a dense subspace $\mathcal{D}(\pi_{\varphi_\gamma})$ of $\mathcal{H}_{\varphi_\gamma}$ with $\lambda_{\varphi_\gamma}(\mathcal{B}(\varphi_\gamma)) \subseteq \mathcal{D}(\varphi_\gamma)$ such that
$\varphi_\gamma(ax, by) = \varphi(\gamma(ax, \theta), \gamma(by, \theta)) = \langle \pi_{\varphi_\gamma}(a)\lambda_{\varphi_\gamma}(x), \pi_{\varphi_\gamma}(b)\lambda_{\varphi_\gamma}(y)\rangle$
$\forall x, y \in \mathcal{B}(\varphi_\gamma)$ and $a, b \in \mathcal{U}_1$.
We now proof the condition for φ_γ to be representable

Proposition 2: Let $\mathcal{U}_1$ and $\mathcal{U}_2$ partial *-algebras. Let φ be a representable biweight with precore $\mathcal{B}(\varphi)$ and domain $\mathcal{D}(\varphi)$ in $\mathcal{U}_2$. Let $\hat{\theta}$ be a *-isomorphism from $\mathcal{U}_1$ into $\mathcal{U}_2$ then φ_γ is a representable form.

Proof. Let φ_γ be a positive sesquilinear form on $\mathcal{D}(\varphi_\gamma) \times \mathcal{D}(\varphi_\gamma)$ for the set $\mathcal{N}_{\varphi_\gamma} = \{x \in \mathcal{U}_1 \colon \varphi_\gamma(x, y) = 0, \forall y \in \mathcal{D}(\varphi_\gamma)\}$ we have the map $\lambda_{\varphi_\gamma} \colon \mathcal{D}(\varphi_\gamma) \to \mathcal{D}(\varphi_\gamma) \setminus \mathcal{N}_{\varphi_\gamma}$ defined by $\lambda_{\varphi_\gamma}(x) = x + \mathcal{N}_{\varphi_\gamma}$, $x \in \mathcal{D}(\varphi_\gamma)$. The set $\mathcal{D}(\varphi_\gamma) \setminus \mathcal{N}_{\varphi_\gamma}$ is a pre-Hilbert space. The map λ_{φ_γ} induces the inner product $\langle \lambda_{\varphi_\gamma}(x), \lambda_{\varphi_\gamma}(y)\rangle = \varphi_\gamma(x, y)$.Thus we define

a map $\hat{\theta}\colon \lambda_{\varphi_\gamma}(\mathcal{D}(\varphi_\gamma)) \to \lambda_\varphi(\mathcal{D}(\varphi))$ by $\hat{\theta}(\lambda_{\varphi_\gamma}(x)) = \lambda_\varphi(\theta(x))$,where $\theta(x) = \gamma(x, \theta)$. This map is well-defined , since if $\lambda_{\varphi_\gamma}(x) = 0$,then $\varphi(x, x) = 0$ which implies that $\varphi_\gamma(x, y) = 0$, and if $\hat{\theta}$ is injective and $\lambda_\varphi(\theta(x)) = 0$ then this implies that
$\varphi(\gamma(x, \theta), \gamma(y, \theta)) = \varphi_\gamma(x, y) = 0$ and thus $\lambda_{\varphi_\gamma}(x) = 0$.We have that $\hat{\theta}$ is an isometric map, that is, $\|\theta(\lambda_{\varphi_\gamma}(x))\|^2 = \|\lambda_\varphi(\theta(x))\|^2 = \varphi(\gamma(x, \theta), \gamma(x, \theta)) = \varphi_\gamma(x, y) = \|\lambda_{\varphi_\gamma}(x)\|^2$. Now since $\hat{\theta}$ is injective it extends to a unitary operator, denote by the same symbol, from $\mathcal{H}_{\varphi_\gamma}$ onto $\mathcal{H}_\varphi$. Since φ is a representable form on $\mathcal{U}_2$ there exists a dense domain $\mathcal{D}(\pi_\varphi) \subseteq \mathcal{H}_\varphi$ and a *-representation $\pi_\varphi\colon \mathcal{U}_2 \to \mathcal{L}^\dagger(\mathcal{D}(\pi_\varphi), \mathcal{H}_\varphi)$ such that
$\varphi(\gamma(ax, \theta), \gamma(by, \theta)) = \langle \pi_\varphi(\theta(a))\lambda_\varphi(\theta(x)), \pi_\varphi(\theta(b))\lambda_\varphi\theta(y)\rangle, \forall \gamma(ax, \theta), \gamma(by, \theta) \in \mathcal{D}(\varphi)$.
We put $\mathcal{D}_{\varphi_\gamma} = \hat{\theta}^{-1}\mathcal{D}(\pi_\varphi)$ then $\mathcal{D}_{\varphi_\gamma}$ is a dense subspace of $\mathcal{H}_{\varphi_\gamma}$. Let $\mathcal{B}_{\varphi_\gamma} \subseteq \mathcal{D}(\varphi_\gamma)$ and the map $\hat{\theta}(\lambda_{\varphi_\gamma}(\mathcal{B}_{\varphi_\gamma})) \subseteq \lambda_\varphi(\theta(\mathcal{B}(\varphi_\gamma))) \subseteq \mathcal{D}(\pi_\varphi)$, where $\theta(\mathcal{B}(\varphi_\gamma)) = \gamma(\mathcal{B}(\varphi_\gamma), \theta)$, this implies that $\hat{\theta}(\lambda_{\varphi_\gamma}(\mathcal{B}_{\varphi_\gamma})) \subseteq \mathcal{D}(\pi_\varphi)$. Now put $\pi_{\varphi_\gamma}(a) = \hat{\theta}^{-1}\pi_\varphi(\theta(a))\hat{\theta}$, then we have the following $\varphi_\gamma(ax, by) = \varphi(\gamma(ax, \theta), \gamma(by, \theta)) = \varphi(\theta(a)\gamma(x, \theta), \theta(b)\gamma(y, \theta))$
$= \langle \pi_\varphi(\theta(a))\lambda_\varphi(\gamma(x, \theta)), \pi_\varphi(\theta(b))\lambda_\varphi(\gamma(y, \theta))\rangle$
$= \langle \pi_\varphi(\theta(a))\lambda_\varphi(\theta(x))), \pi_\varphi(\theta(b))\lambda_\varphi(\theta(y))\rangle$
$= \langle \pi_\varphi(\theta(a))\hat{\theta}(\lambda_{\varphi_\gamma}(x)), \pi_\varphi(\theta(b))\hat{\theta}(\lambda_{\varphi_\gamma}(y))\rangle$
$= \langle \hat{\theta}^{-1}(\pi_\varphi\theta(a))\hat{\theta}(\lambda_{\varphi_\gamma}(x)), \hat{\theta}^{-1}(\pi_\varphi\theta(b))\hat{\theta}(\lambda_{\varphi_\gamma}(y))\rangle$
$= \langle \pi_{\varphi_\gamma}(a))\lambda_{\varphi_\gamma}(x)), \pi_{\varphi_\gamma}(b))\lambda_{\varphi_\gamma}(y))\rangle$
Thus we have shown that φ_γ is a representable form. □

Proposition 3: $\lambda_{\varphi_\gamma}(\mathcal{B}(\varphi_\gamma))$ is dense in the Hilbert space $\mathcal{H}_{\varphi_\gamma}$.

Proof. Let φ be a biweight on $\mathcal{U}_2$ and let $x \in \mathcal{D}(\varphi)$, for φ there exists a sequence (z_n) ,$z_n \in \mathcal{B}(\varphi)$, such that $\varphi(x - z_n, x - z_n) \to 0$. Thus we can define a map $\hat{i}\colon \mathcal{B}(\varphi) \to \mathcal{B}(\varphi_\gamma)$ by $\hat{i}(z_n) = \hat{z_n}$ and put $\lambda_{\varphi_\gamma}(\hat{x}) = \hat{\theta}^{-1}(\lambda_\varphi(\theta(x)))$ where $\hat{\theta}$ is a *-isomorphism on $\lambda_{\varphi_\gamma}(\mathcal{D}(\varphi_\gamma))$ into $\lambda_\varphi(\mathcal{D}(\varphi))$.Thus we have $\lambda_{\varphi_\gamma}(\hat{i}(x)) = \hat{\theta}^{-1}(\lambda_\varphi(\gamma(\hat{x}, \theta)))$ and thus $\gamma(\hat{x}, \theta) \in \mathcal{D}(\varphi)$. Hence for $\hat{x} \in \mathcal{D}(\varphi_\gamma)$ we let $\gamma(\hat{x} - \hat{z}_n) = (x - z_n)$, hence we have
$\|\lambda_{\varphi_\gamma}(\hat{x} - \hat{z_n})\|^2 = \langle \lambda_{\varphi_\gamma}(\hat{x} - \hat{z_n}), \lambda_{\varphi_\gamma}(\hat{x} - \hat{z_n})\rangle$
$= \varphi_\gamma((\hat{x} - \hat{z_n}), (\hat{x} - \hat{z_n}))$
$= \varphi((\gamma(\hat{x} - \hat{z}_n), (\gamma(\hat{x} - \hat{z}_n)) = \varphi(x - z_n, x - z_n) \to 0.$ □

This shows that $\lambda_{\varphi_\gamma}(\mathcal{B}(\varphi_\gamma))$ is dense in the Hilbert space $\mathcal{H}_{\varphi_\gamma}$ and we can summarized the above disscusion with the following;

Theorem 1: Let $\mathcal{U}_1$, $\mathcal{U}_2$ be partial *-algebras, and φ is a biweight on $\mathcal{U}_2$ then a γ *-linear map from $\mathcal{U}_1$ into $\mathcal{U}_2$ is a biweight-preserving map if it satisfies the properties [1-3] in definition(9) and $\varphi \circ \gamma$ is a biweight on $\mathcal{U}_1$.

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