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CERTAIN NEW SUBCLASSES OF ANALYTIC FUNCTIONS INVOLVING GENERALIZED HYPERGEOMETRIC FUNCTION

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ABSTRACT. In this work the author considered new subclasses of analytic univalent functions using certain linear operator. With the operator, the author was able to study several new and existing subclasses of analytic univalent functions in the unit disk. The results presented in this paper include, coefficient estimates (which were later used to investigate certain subclasses of analytic functions with fixed finitely many coefficients) and distortion theorems for functions belonging to these subclasses. Furthermore, some relationships between these subclasses were also discussed.

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1. INTRODUCTION

A single-valued function $f(z)$ is said to be univalent in a chosen domain D if it never takes on the same value twice, meaning that if $f(z_1) = f(z_2)$ for $z_1, z_2 \in D$ then $z_1 = z_2$. Also, a set X is said to be starlike with respect to $\omega_0 \in X$ if the line segment joining ω_0 to every other point $\omega \in X$ lies entirely in X. If a function $f(z)$ maps D onto a domain that is starlike with respect to ω_0, then $f(z)$ is said to be starlike with respect to ω_0. In particular, if ω_0 is the origin, then we say that $f(z)$ is a starlike function in D. In like manner, a set X is said to be convex if the line segment joining any two points of X lies entirely in X. If a function $f(z)$ maps D onto a convex domain, then we say that $f(z)$ is a convex function in D(see [2] and [14]).

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Now, let A denotes the class of functions $f(z)$ of the form

$$f(z)=\sum_{k=0}^{\infty}a_{k+1}z^{k+1}\qquad(a_1=1)\tag{1}$$

which are analytic in the unit disk $D=\{z:z\in Cand\mid z\mid<1\}$. Let S denotes the class of all functions in A which are univalent in D. Then a function $f(z)$ belonging to S is said to be starlike of order γ denoted by $S^*(\gamma)$ if and only if

$$\Re\left\{\frac{zf'(z)}{f(z)}\right\}>\gamma\quad(0\leq\gamma<1)\quad(z\in D).$$

Also, a function $f(z)$ belonging to S is said to be convex of order γ denoted by $K(\gamma)$ if and only if

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\}>\gamma\quad(0\leq\gamma<1)\quad(z\in D).$$

Here, we shall note that
$(i.)$ $f(z)\in K(\gamma)$ if and only if $zf'(z)\in S^*(\gamma)$.
$(ii.)$ $S^*(\gamma)\subseteq S^*(0)\equiv S^*$.
$(iii.)$ $K(\gamma)\subseteq K(0)\equiv K$.
$(iv.)$ $K(\gamma)\subset S^*(\gamma)\subset S,\ \ 0\leq\gamma<1$.
The two classes above ($S^*(\gamma)$ *and* $K(\gamma)$) were first introduced by Roberton[11] and were later studied by Schild[12], MacGregor[7], Pinchuk[10], Jack[6] among others.
For the sake of our present investigation, we define the function $f(z)^\alpha$ as

$$f(z)^\alpha=\sum_{k=0}^{\infty}a_{k+1}(\alpha)z^{k+\alpha},\quad(a_1=1)\quad(z\in D).\tag{2}$$

for real α ($\alpha>0$) (see [3] and [5] among others).
Using Catas operator[1], we write for function $f(z)^\alpha$ that

$$I^m(\lambda,l)f(z)^\alpha=\sum_{k=0}^{\infty}\left(\frac{1+\lambda(\alpha+k-1)+l}{1+l}\right)^m a_{k+1}z^{\alpha+k}\tag{3}$$

$$l\geq0,\ \lambda\geq0,\ m\in N_0,\ \alpha>0\ and\ z\in D.$$

From (3), a function $f(z)^\alpha\in A$ is said to be in the class $V(l,m,\alpha,\lambda,\gamma)$ if it satisfies the inequality

$$\Re\left\{\frac{z(I^m(\lambda,l)f(z)^\alpha)'}{I^m(\lambda,l)f(z)^\alpha}\right\}>-\gamma\tag{4}$$

for $\alpha > 0,\ \gamma > -1$ and $z \in D$.

On the other hand, a function $f(z)^\alpha$ belonging to A is said to be in the class

$W(l, m, \alpha, \lambda, \gamma)$ if it satisfies the inequality

$$\Re\left\{1 + \frac{z\Big(I^m(\lambda, l) f(z)^\alpha\Big)''}{\Big(I^m(\lambda, l) f(z)^\alpha\Big)'}\right\} > -\gamma \tag{5}$$

for $\alpha > 0,\ \gamma > -1$ and $z \in D$.

We observe that,

$$f(z)^\alpha \in W(l, m, \alpha, \lambda, \gamma) \Leftrightarrow z\big(f(z)^\alpha\big)' \in V(l, m, \alpha, \lambda, \gamma)$$

and

$$W(l, m, \alpha, \lambda, \gamma) \subset V(l, m, \alpha, \lambda, \gamma).$$

Recall that the Pochhammer symbol (or the shifted factorial) $(\lambda)_k$ is defined, in term of the Gamma Function Γ, as

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \begin{cases} 1\ for\ (k = 0) \\ \lambda(\lambda+1)(\lambda+2)...(\lambda+k-1)\ for\ (k \in N) \end{cases} \tag{6}$$

or

$$(\lambda)_k = \frac{\Gamma(\lambda + k)}{\Gamma(\lambda)} = \frac{1}{\lambda + k}\left\{\Pi_{j=1}^{k+1}(\lambda + j - 1)\right\}, \quad k = N_0 = N \cup \{0\}. \tag{7}$$

Now let $\alpha_j (j = 1, 2, ..., p)$ and $\beta_j (j = 1, 2, ..., q)$ be complex numbers with $\beta_j \neq 0, -1, -2, ...\ \ and\ \ j = 1, 2, ..., q$. The genralized hypergeometric function ${}_pF_q$ is defined by

$${}_pF_q = {}_p F_q\big(\alpha_1, ..., \alpha_p\ ; \beta_1, ..., \beta_q\big) = \sum_{k=0}^{\infty} \frac{(\alpha_1)_k ... (\alpha_p)_k}{(\beta_1)_k ... (\beta_q)_k} \frac{z^k}{k!}\ (p \leq q + 1),$$

while $\alpha_j (j = 1, 2, ..., p)$ and $\beta_j (j = 1, 2, ..., q)$ are determined using the concept defined in (6) or (7).

We note that the series ${}_pF_q$ converges absolutely for $|z| < \infty$ if $p < q + 1$, and for $z \in D$ if $p = q + 1$. Supposing we set

$$\omega = \sum_{j=1}^{q} \beta_j - \sum_{j=1}^{p} \alpha_j, \tag{8}$$

it is well-known that the ${}_pF_q$ series, with $p = q + 1$, is absolutely convergent for $|z| = 1$ if $\Re(\omega) > 0$ and conditionally convergent for $|z| = 1 (z \neq 0)$ if $-1 < \Re(\omega) \leq 0$(see [14]).

Now, for functions

$$f_i(z)^\alpha = \sum_{k=0}^{\infty} a_{i,k+1} z^{\alpha + k}\ \ (a_{i,1} = 1;\ i = 1, 2) \tag{9}$$

we define the convolution of $f_1(z)^\alpha$ and $f_2(z)^\alpha$ as

$$f_1(z)^\alpha * f_2(z)^\alpha = \sum_{k=0}^{\infty} a_{1,k+1}a_{2,k+1}z^{\alpha+k}. \tag{10}$$

At this juncture, we can now define the following:

$$\begin{aligned} L(a,b,c;z)f(z)^\alpha &= I^m(\lambda,l)f(z)^\alpha * z^\alpha.{}_2F_1(a,b,c;z) \\ = \sum_{k=0}^{\infty} \left(\frac{1+\lambda(\alpha+k-1)+l}{1+l}\right)^m &\frac{(a)_k(b)_k}{(c)_k k!} a_{k+1}(\alpha)z^{\alpha+k} \end{aligned} \tag{11}$$

$l \geq 0,\ \lambda \geq 0,\ m \in N_0 = N \cup \{0\},\ \alpha > 0\ a_1 = 1,\ z \in D$ and $c \neq 0$, -1, $-2, ...$ Obviously,

$$V(l,m,\alpha,\lambda,\gamma) = L(2,1,1)W(l,m,\alpha,\lambda,\gamma), \quad \alpha > 0,\ \gamma > -1$$

or inversely

$$W(l,m,\alpha,\lambda,\gamma) = L(1,1,2)V(l,m,\alpha,\lambda,\gamma), \quad \alpha > 0,\ \gamma > -1.$$

A function $f(z)^\alpha$ belonging to A is said to in the class $V(a,b,c,l,$ $m,\alpha,\ \lambda;\gamma)$ if $L(a,b,c)f(z)^\alpha$ is an element of $V(l,m,\alpha,\lambda,\gamma)$ for $\gamma > -1$. Also, a function $f(z)^\alpha$ belonging to A is said to be in the class $W(a,b,c,l,m,\alpha,\lambda;\ \gamma)$ if $z\big(I^m(\lambda,l)f(z)^\alpha\big)'$ is an element of $V(l,m,\alpha,\lambda,\gamma)$ for $\gamma > -1$.

$$V(a,b,c,l,m,\alpha,\lambda,\gamma) = L(2,1,1)W(a,b,c,l,m,\alpha,\lambda,\gamma)$$

and

$$W(a,b,c,l,m,\alpha,\lambda,\gamma) = L(1,1,2)V(a,b,c,l,m,\alpha,\lambda,\gamma).$$

Furthermore, let $V_0(l,m,\alpha,\lambda,\gamma)$ and $W_0(l,m,\alpha,\lambda,\gamma)$ denote the classes obtained respectively by taking the intersection of $V(l,m,\alpha,$ $\lambda,\gamma)$ and $W(l,m,\alpha,\lambda,\gamma)$ with S. Then,

$$\begin{gathered} V_0(l,m,\alpha,\lambda,\gamma) = V(l,m,\alpha,\lambda,\gamma) \cap S,\ \alpha > 0,\ \gamma > -1 \\ W_0(l,m,\alpha,\lambda,\gamma) = W(l,m,\alpha,\lambda,\gamma) \cap S\ \alpha > 0,\ \gamma > -1 \\ V_0(a,b,c,l,m,\alpha,\lambda;\gamma) = V(a,b,c,l,m,\alpha,\lambda;\gamma) \cap S \\ W_0(a,b,c,l,m,\alpha,\lambda;\gamma) = W(a,b,c,l,m,\alpha,\lambda;\gamma) \cap S. \end{gathered}$$

Finally, in the present investigation, we give coefficient estimate and distortion theorems for functions belonging to the various subclasses of A earlier defined. In addition to these, the authors derived some relationship involving the said subclasses. Also, with the aids of the coefficient estimate presented, certain subclasses of analytic functions with fixed finitely many coefficients were considered.

1. Coefficient estimates

The first four theorems presented in this paper are highly foundational for the present investigation.

Theorem 2.1. Let the function $f(z)^\alpha$ be defined by (2). If

$$\sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \left|\frac{(a)_k(b)_k}{(c)_k k!}\right| \left|a_{k+1}(\alpha)\right| \leq 1 \tag{12}$$

$$T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$, $\gamma > -1$ and $z \in D$. Then $f(z)^\alpha \in V(a,b,c,l,m,\alpha,\lambda;\gamma)$. The result is sharp.

Proof. Since

$$\Re\left\{\frac{z\Big(L(a,b,c)f(z)^\alpha\Big)'}{L(a,b,c)f(z)^\alpha}\right\} > -\gamma \ \ (z \in D).$$

It suffices to see that

$$\left|\frac{z\big(L(a,b,c)f(z)^\alpha\big)'}{L(a,b,c)f(z)^\alpha} - \alpha\right| < \alpha+\gamma \ \ (z \in D) \tag{13}$$

for real α and $\gamma > -1$.

It implies that

$$\left|\frac{z\big(L(a,b,c)f(z)^\alpha\big)'}{L(a,b,c)f(z)^\alpha} - \alpha\right| \leq \frac{\sum_{k=1}^{\infty} kT_k^m \left|\frac{(a)_k(b)_k}{(c)_k k!}\right| \left|a_{k+1}(\alpha)\right|}{1-\sum_{k=1}^{\infty} T_k^m \left|\frac{(a)_k(b)_k}{(c)_k k!}\right| \left|a_{k+1}(\alpha)\right|} \leq \alpha+\gamma \tag{14}$$

$$(z \in D) \ \textit{and} \ T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

provided

$$\sum_{k=1}^{\infty} kT_k^m \left|\frac{(a)_k(b)_k}{(c)_k k!}\right| \left|a_{k+1}(\alpha)\right| \leq (\alpha+\gamma)\left[1-\sum_{k=1}^{\infty} T_k^m \left|\frac{(a)_k(b)_k}{(c)_k k!}\right| \left|a_{k+1}(\alpha)\right|\right], \tag{15}$$

which is equivalent to inequality (12).

Obviously, the result (12) is sharp for the functions given by

$$f(z)^\alpha = z^\alpha + \frac{(\alpha+\gamma)_k (c)_k k!}{(1+\alpha+\gamma)_k (a)_k (b)_k} T_k^{-m} z^{k+1} \ \ (k \in N) \tag{16}$$

where

$$T_k^{-m} = \left(\frac{1+\lambda(\alpha-1)}{1+\lambda(\alpha+k-1)+1}\right)^m$$

and this complete the proof of the theorem 2.1.
Corollary 2.2. Let the function $f(z)^\alpha$ be defined by (2). If

$$\sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \left| a_{k+1}(\alpha) \right| \leq 1 \quad z \in D \tag{17}$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$ and $\gamma > -1$. Then $f(z)^\alpha \in V(l, m, \alpha, \lambda; \gamma)$. The result is sharp for the function given by

$$f(z)^\alpha = z^\alpha + \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} T_k^{-m} z^{k+1} \quad (k \in N) \tag{18}$$

where T_k^{-m} is as defined above.
Remark A. In its special cases when $m = 0$ and $b = 1$, Corollary 2.2 yields the corresponding result for the class $V(\gamma)$ due to Srivastava and Owa[14, p.6, Theorem 2.1]. Also, when $m = 0$, $b = 1$ and $-1 < \gamma \leq 0$, Corollary 2.2 yields the corresponding result for the class $S^*(\gamma)$ due to Silverman[13, p.10, Theorem 1].
Theorem 2.3. Let the function $f(z)^\alpha$ be defined by (2). If

$$\sum_{k=1}^{\infty} \frac{(1+\alpha)_k(1+\alpha+\gamma)_k}{(\alpha)_k(\alpha+\gamma)_k} T_k^m \left| \frac{(a)_k(b)_k}{(c)_k k!} \right| \left| a_{k+1}(\alpha) \right| \leq 1 \tag{19}$$

$$T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l} \right)^m$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$, $\gamma > -1$ and $z \in D$. Then $f(z)^\alpha \in W(a, b, c, l, m, \alpha, \lambda; \gamma)$. The result is sharp.
Proof. Since

$$\Re \left\{ \frac{z\Big(L(a,b,c)f(z)^\alpha\Big)''}{\Big(L(a,b,c)f(z)^\alpha\Big)'} + 1 \right\} > -\gamma \quad (z \in D).$$

We can say that

$$\left| \frac{z\Big(L(a,b,c)f(z)^\alpha\Big)''}{\Big(L(a,b,c)f(z)^\alpha\Big)'} + 1 - \alpha \right| < \alpha + \gamma \quad (z \in D) \tag{20}$$

for real α and $\gamma > -1$.
It also implies that

$$\left|\frac{z\Big(L(a,b,c)f(z)^{\alpha}\Big)''}{\Big(L(a,b,c)f(z)^{\alpha}\Big)'}+1-\alpha\right|$$

$$\leq \frac{\sum_{k=1}^{\infty} k\Big(\frac{\alpha+k}{\alpha}\Big)T_k^m\Big|\frac{(a)_k(b)_k}{(c)_k k!}\Big|\big|a_{k+1}(\alpha)\big|}{1-\sum_{k=1}^{\infty}\Big(\frac{\alpha+k}{\alpha}\Big)T_k^m\Big|\frac{(a)_k(b)_k}{(c)_k k!}\Big|\big|a_{k+1}(\alpha)\big|}\leq \alpha+\gamma \tag{21}$$

$$(z\in D)\ and\ T_k^m=\left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

provided that

$$\sum_{k=1}^{\infty} k\Big(\frac{\alpha+k}{\alpha}\Big)T_k^m\Big|\frac{(a)_k(b)_k}{(c)_k k!}\Big|\big|a_{k+1}(\alpha)\big|$$
$$\leq(\alpha+\gamma)\Big[1-\sum_{k=1}^{\infty}\Big(\frac{\alpha+k}{\alpha}\Big)T_k^m\Big|\frac{(a)_k(b)_k}{(c)_k k!}\Big|\big|a_{k+1}(\alpha)\big|\Big], \tag{22}$$

which is equivalent to inequality (19).
Clearly, the result (19) is sharp for the functions given by

$$f(z)^{\alpha}=z^{\alpha}+\frac{(\alpha)_k(\alpha+\gamma)_k(c)_k k!}{(\alpha+1)_k(1+\alpha+\gamma)_k(a)_k(b)_k}T_k^{-m}z^{k+1}\quad (k\in N) \tag{23}$$

where

$$T_k^{-m}=\left(\frac{1+\lambda(\alpha-1)}{1+\lambda(\alpha+k-1)+1}\right)^m$$

and this complete the proof of the Theorem 2.3.
Corollary 2.4. Let the function $f(z)^{\alpha}$ be defined by (2). If

$$\sum_{k=1}^{\infty}\frac{(1+\alpha)_k(1+\alpha+\gamma)_k}{(\alpha)_k(\alpha+\gamma)_k}T_k^m\big|a_{k+1}(\alpha)\big|\leq 1\quad z\in D \tag{24}$$

for $\alpha>0$, $l\geq 0$, $\lambda\geq 0$, $m\in N_0=N\cup\{0\}$ and $\gamma>-1$. Then $f(z)^{\alpha}\in W(l,m,\alpha,\lambda;\gamma)$. The result is sharp for the function given by

$$f(z)^{\alpha}=z^{\alpha}+\frac{(\alpha)_k(\alpha+\gamma)_k}{(\alpha+1)_k(1+\alpha+\gamma)_k}T_k^{-m}z^{k+1}\quad (k\in N) \tag{25}$$

where T_k^{-m} is as defined above.
Remark B. In its special cases when $m=0$ and $b=1$, Corollary 2.4 yields the corresponding result for the class $W(\gamma)$ due to Srivastava and Owa[14, p.7, Theorem 2.2]. Also, when $m=0$, $b=1$

and $-1 < \gamma \leq 0$, Corollary 2.2 yields the corresponding result for the class $K(\gamma)$ due to Silverman[13].

Theorem 2.5. Let the function $f(z)^\alpha$ be defined by (2). If

$$\sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \frac{(a)_k (b)_k}{(c)_k k!} \left| a_{k+1}(\alpha) \right| \leq 1 \tag{26}$$

$$T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l} \right)^m$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$, $\gamma > -1$ and $z \in D$. Then $f(z)^\alpha \in V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$. The result is sharp.

Proof. Assuming that (26) holds true. Then it follows immediately from Theorem 2.1 that $f(z)^\alpha \in V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$. Conversely, assume that $f(z)^\alpha \in V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$. Then, by definition,

$$\Re \left\{ \frac{z\Big(L(a,b,c)f(z)^\alpha \Big)'}{L(a,b,c)f(z)^\alpha} \right\} = \Re \left\{ \frac{\sum_{k=0}^{\infty} \frac{(\alpha+1)_k}{(\alpha)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!} a_{k+1} z^{\alpha+k}}{\sum_{k=0}^{\infty} T_k^k \frac{(a)_k(b)_k}{(c)_k k!} a_{k+1} z^{\alpha+k}} \right\}$$

$$> -\gamma \tag{27}$$

for real α, $\gamma > -1$ and $m \in N_0$. Now, choose values of z on the real axis so that

$$\frac{z\big(L(a,b,c)f(z)^\alpha\big)'}{L(a,b,c)f(z)^\alpha}$$

is real. Letting $z \to 1^-$, (27) immediately yields

$$1 - \sum_{k=0}^{\infty} \frac{(\alpha+1)_k}{(\alpha)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!} \left| a_{k+1}(\alpha) \right| \geq \gamma \left[\sum_{k=0}^{\infty} T_k^m \frac{(a)_k(b)_k}{(c)_k k!} \left| a_{k+1}(\alpha) \right| - 1 \right], \tag{28}$$

which implies our result in (26). By taking the functions defined by

$$f(z)^\alpha = z^\alpha + \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} T_k^{-m} \frac{(c)_k k!}{(a)_k(b)_k} z^{\alpha+k} \quad k \in N \tag{29}$$

where T_k^{-m} is as defined earlier.

Corollary 2.6. Let the functions $f(z)^\alpha$ defined by (2) be in the class $V_0(a,b,c,\,l,m,\alpha,\lambda;\gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then

$$\left| a_{k+1}(\alpha) \right| \leq \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} T_k^{-m} \frac{(c)_k k!}{(a)_k(b)_k}$$

where

$$T_k^{-m} = \left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda(\alpha+k-1)+l}\right)^m.$$

The result is sharp for the function defined by (29).

Corollary 2.7. Let the functions $f(z)^\alpha$ defined by (2) be in the class $V_0(a, 1, a, l, m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$.

$$\sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \left|a_{k+1}(\alpha)\right| \leq 1 \tag{30}$$

$$T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$, $\gamma > -1$ and $z \in D$. The result is sharp for the functions given by

$$f(z)^\alpha = z^\alpha - \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} T_k^{-m} z^{\alpha+k} \quad k \in N \tag{31}$$

where T_k^{-m} is as defined earlier.

REMARK C.

(i.) For $m = 0$, $\alpha = 1$ and $b = 1$. Then Theorem 2.5 correspond to a result for the class $V_0(\gamma)$ proved by Srivastava and Owa[14].

(ii.) If $m = 0$, $\alpha = 1$, $-1 < \gamma \leq 0$ and $b = 1$, then Theorem 2.5 correspond to the result for the class $S^*(\gamma)$ proved by Silverman[13]

Theorem 2.8. Let the function $f(z)^\alpha$ be defined by (2). Then, $f(z)^\alpha \in W_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ if and only if

$$\sum_{k=1}^{\infty} \frac{(1+\alpha)_k(1+\alpha+\gamma)_k}{(\alpha)_k(\alpha+\gamma)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!} \left|a_{k+1}(\alpha)\right| \leq 1 \tag{32}$$

$$T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$, $\gamma > -1$ and $z \in D$. The result is sharp.

Proof. The proof is very much similar to that of Theorem 2.5 and so we shall ignore the details involved. Equality in (32) is attained by the functions given by

$$f(z)^\alpha = z^\alpha - \frac{(\alpha)_k}{(\alpha+1)_k} \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} T_k^{-m} \frac{(c)_k k!}{(a)_k(b)_k} z^{\alpha+k}, \quad k \in N \tag{33}$$

where T_k^{-m} is as earlier defined.

The following are immediate results from Theorem 2.8.

Corollary 2.9 Let the functions $f(z)^\alpha$ defined by (2) be in the class $W_0(a, b, c,\ l, m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then

$$\left|a_{k+1}(\alpha)\right| \leq \frac{(\alpha)_k}{(\alpha+1)_k}\frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}T_k^{-m}\frac{(c)_k k!}{(a)_k(b)_k} \tag{34}$$

where T_k^{-m} is as defined earlier. The result (34) is sharp for the function $f(z)^\alpha$ given by (33).

Corollary 2.10. Let the functions $f(z)^\alpha$ defined by (2) be in the class $W_0(a, 1, a, l, m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then

$$\sum_{k=1}^{\infty}\frac{(\alpha+1)_k}{(\alpha)_k}\frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k}T_k^m\left|a_{k+1}(\alpha)\right| \leq 1 \tag{35}$$

for $\alpha > 0$, $l \geq 0$, $\lambda \geq 0$, $m \in N_0$, $\gamma > -1$ and $z \in D$. The result (35) is sharp for the functions given by

$$f(z)^\alpha = z^\alpha - \frac{(\alpha)_k}{(\alpha+1)_k}\frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}T_k^{-m}z^{\alpha+k} \quad k \in N \tag{36}$$

where T_k^{-m} is as earlier defined.

REMARK D.

(i.) For $m = 0$, $\alpha = 1$ and $b = 1$. Then Corollary 2.10 corresponds to a result for the class $W_0(\gamma)$ proved by Srivastava and Owa[14].

(ii.) In particular, supposing we pose a constraint on γ such that $-1 < \gamma \leq 0$ for $m = 0$, $\alpha = 1$ and $b = 1$, then Corollary 2.10 corresponds to the result for the class $K^*(\gamma)$ proved by Silverman[13]. For recent works on the coefficient estimates interested reader can refer $[2-5]$ *and* $[8, 9]$.

2. Distortion Theorems

In order to determine the extreme points of the two classes $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ and $W_0(a, b, c, l, m, \alpha, \lambda; \gamma)$, the following Lemmas shall be proved.

Lemma 3.1. Let the class $V_0(a, b, c, l, m, \alpha,\ \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$ be convex. Then $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ is considered as a linear space over the field of real numbers.

Proof. Let the functions $f_i(z)^\alpha$ $(i = 1, 2)$ defined by

$$f_i(z)^\alpha = \sum_{k=0}^{\infty} a_{i,k+1}(\alpha)z^{\alpha+k} \ \ (a_{i,1}; a_{i,k+1}, \forall k \in N),\ [cf.eqn8]$$

be in the class $V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$. Then, we shall show that the function

$$\psi f_1(z)^\alpha + (1-\psi) f_2(z)^\alpha \quad (0 \le \psi \le 1) \tag{37}$$

is also in the class $V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$. It implies that

$$\psi f_1(z)^\alpha + (1-\psi) f_2(z)^\alpha = z^\alpha + \sum_{k=1}^{\infty}\Big[\psi a_{1,k+1}(\alpha) + (1-\psi) a_{2,k+1}(z)^\alpha\Big] z^{\alpha+k}, \tag{38}$$

which by virtue of Theorem 2.5, yields

$$\sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!}\Big[\psi\big|a_{1,k+1}(\alpha)\big| + (1-\psi)\big|a_{2,k+1}(z)^\alpha\big|\Big]$$

$$= \psi \sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!}\big|a_{1,k+1}(\alpha)\big| \tag{39}$$

$$+(1-\psi) \sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!}\big|a_{2,k+1}(\alpha)\big|$$

where

$$T_k^m = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l}\right)^m$$

and this complete the proof of Lemma 3.1.
Further, since the class $V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$ is convex as we have seen in Lemma 3.1, it has some extreme points given by Lemma 3.2 and Theorem 3.3 below.
Lemma 3.2. Let

$$f_0(z)^\alpha = z^\alpha \tag{40}$$

and

$$f_k(z)^\alpha = z^\alpha - \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}\frac{(c)_k k!}{(a)_k(b)_k} T_k^{-m} z^{\alpha+k} \quad k \in N \tag{41}$$

where

$$T_k^{-m} = \left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda(\alpha+k-1)+l}\right)^m$$

with $\frac{(a)_k(b)_k}{(c)_k k!} > 0$. Then, $f(z)^\alpha \in V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$ if and only if it can be expressed in the form

$$f(z)^\alpha = \sum_{k=0}^{\infty} \delta_k f_k(z)^\alpha, \tag{42}$$

where $\delta \geq 0$, $\forall k \in N_0$ and

$$\sum_{k=0}^{\infty} \delta_k = 1. \tag{43}$$

REMARK E. Lemma 3.2 simply states that the family $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ is an infinite-dimensional simplex with definite vertices defined by certain extremal conditions.

Proof. In the first place, we shall assume that the function $f(z)^\alpha$ has the expression (42), that is,

$$f(z)^\alpha = z^\alpha - \sum_{k=1}^{\infty} \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} \frac{(c)_k k!}{(a)_k (b)_k} T_k^{-m} \delta_k z^{\alpha+k} \quad k \in N \tag{44}$$

where T_k^{-m} is as defined above.
Using (26) and (44), we obtain

$$\sum_{k=1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} \frac{(a)_k (b)_k}{(c)_k k!} T_k^m \cdot \frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k} \frac{(c)_k k!}{(a)_k (b)_k} T_k^{-m} \delta_k \tag{45}$$

$$= \sum_{k=1}^{\infty} \delta_k = \sum_{k=1}^{\infty} \delta_k - \delta_0$$

$$1 - \delta_0 \leq 1 (since \sum_{k=0}^{\infty} \delta_k)$$

which (in view of Theorem 2.5) implies that $f(z)^\alpha \in V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$.
Conversely, we assume that $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k (b)_k}{(c)_k} > 0$. Using corollary 2.6, we may say write that

$$\delta_k = \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} \frac{(a)_k (b)_k}{(c)_k k!} T_k^m \left| a_{k+1}(\alpha) \right| \quad (k \in N) \tag{46}$$

and

$$\delta_0 = 1 - \sum_{k=1}^{\infty} \delta_k. \tag{47}$$

This ends the proof.
The following result on the extreme points of the class $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ follows immediately from Lemma 3.2.

Theorem 3.3. The extreme points of the class $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k (b)_k}{(c)_k} > 0$ are the functions $f_k(z)^\alpha$ $(k \geq)$ given by (40) and (41). In like manner, we can establish the following theorem.

Theorem 3.4. The extreme points of the class $W_0(a, b, c, l, m, \alpha,$

$\lambda;\gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$ are the functions $f_k(z)^\alpha$ $(k \geq)$ given by (39) and

$$f_k(z)^\alpha = z^\alpha - \frac{(\alpha)_k}{(1+\alpha)_k}\frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}\frac{(c)_k k!}{(a)_k(b)_k}T_k^{-m}z^{\alpha+k} \quad k \in N \quad (48)$$

REMARK F.
As already observed in Remark E, each Theorem 3.3 and Theorem 3.4 can be re-expressed in terms of infinite-dimensional simplex. Corresponding to Theorem 3.3 and Theorem 3.4, we can now prove the distortion theorems for functions belonging to the classes $V_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ and $W_0(a, b, c, l, m, \alpha, \lambda; \gamma)$ as follow.
Theorem 3.5. Let the function $f(z)^\alpha$ defined by (2) be in the class $V_0(a, b, c, l,\ m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then

$$\left|z\right|^\alpha - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m\left|z\right|^{\alpha+1} \leq \left|f(z)^\alpha\right|$$

$$\leq \left|z\right|^\alpha + \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m\left|z\right|^{\alpha+1} \quad (49)$$

and

$$\alpha\left|z\right|^{\alpha-1} - \frac{(\alpha+\gamma)(\alpha+1)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m\left|z\right|^\alpha \leq \left|\left(f(z)^\alpha\right)'\right|$$

$$\alpha\left|z\right|^{\alpha-1} + \frac{(\alpha+\gamma)(\alpha+1)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m\left|z\right|^\alpha \quad (50)$$

for $z \in D$. The results (49) and (50) are sharp.
Proof. Since

$$\frac{(\alpha+\gamma)_k(c)_k k!}{(1+\alpha+\gamma)_k(a)_k(b)_k}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m\left|z\right|^{\alpha+k}$$

is a decreasing function of k, we have for $f_k(z)^\alpha$ $(k \geq 0)$ defined by (40) and (41) that

$$f_1\big(|z|^\alpha\big) \leq \left|f_k(z)^\alpha\right| \leq -f_1\big(|z|^\alpha\big) \quad (z \in D), \quad (51)$$

which readily yields the assertion (49) of the Theorem 3.5.
To prove the statement (50) of Theorem 3.5, we shall note that

$$|(f(z)^\alpha)'| \geq \alpha|z|^{\alpha-1} - \underbrace{max}_{k\in N}\left\{\frac{(\alpha+1)_k}{(\alpha)_k}\frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}\frac{(c)_k k!}{(a)_k(b)_k}T_k^{-m}|z|^k\right\} \quad (52)$$

and

$$|(f(z)^\alpha)'| \leq \alpha|z|^{\alpha-1} + \underbrace{max}_{k\in N}\left\{\frac{(\alpha+1)_k}{(\alpha)_k}\frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}\frac{(c)_k k!}{(a)_k(b)_k}T_k^{-m}|z|^k\right\} \quad (53)$$

for $z \in D$. Thus, we need to show that function

$$G\big(a,b,c,l,m,\alpha,\lambda,\gamma,k,|z|^{\alpha}\big) = \frac{(\alpha+1)_k}{(\alpha)_k}\frac{(\alpha+\gamma)_k}{(1+\alpha+\gamma)_k}\frac{(c)_k k!}{(a)_k(b)_k}T_k^{-m}|z|^k \quad (54)$$

is decreasing in k $(k \in N)$. For $|z|^{\alpha} \neq 0$,

$$G\big(a,b,c,l,m,\alpha,\lambda,\gamma,k,|z|^{\alpha}\big) \geq G\big(a,b,c,l,m,\alpha,\lambda,\gamma,k+1,|z|^{\alpha}\big) \quad (55)$$

if and only if

$$G\big(a,b,c,l,m,\alpha,\lambda,\gamma,k,|z|^{\alpha}\big) = (\alpha+k)(1+\alpha+\gamma+k)(a+k)(b+k)$$
$$-(\alpha+\gamma+k)(\alpha+k+1)(k+1)(c+k)T_k^{m*}|z| \geq 0 \quad (56)$$

where

$$T_k^{m*} = \left(\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha+k)+l}\right)^m.$$

Now, since G_1 is a decreasing function of $|z|$ for fixed $a,b,c,l,m,\alpha,$ $\lambda,\gamma,k,|z|$, we have

$$G_1\big(a,b,c,l,m,\alpha,\lambda,\gamma,k,|z|\big) \geq G_1\big(a,b,c,l,m,\alpha,\lambda,\gamma,k,1\big)$$
$$= (\alpha+k)(\alpha+\gamma+k+1)\Big[(a+k)(b+k)-(k+1)(c+k)T_k^{m*}\Big] \quad (57)$$
$$-\gamma(k+1)(c+k)T_k^{m*} \geq 0,$$

provided that the parameters involved are constrained to satisfy, either set of the inequalities preceding (50). Appropriate substitution in (52) and (53) give rise to the assertion (50) of the Theorem 3.5.

Finally, the bounds of (49) and (50) are attained by the function

$$f(z)^{\alpha} = z^{\alpha} - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m z^{\alpha+1}. \quad (58)$$

Corollary 3.6. Let the function $f(z)^{\alpha}$ defined by (2) be in the class $V_0(a,b,c,l,\ m,\alpha,\lambda;\gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$ and $a \geq c$. Then, the unit disk D is mapped onto a domain that contains the disk $|\omega| < r_0$, where r_0 is given by

$$r_0 = 1 - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m. \quad (59)$$

Corollary 3.7. Let the function $f(z)^{\alpha}$ defined by (2) be in the class $V_0(a,b,c,0,\ m,\alpha,1;\gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$ and $a \geq c$. Then,

$$\big|z\big|^{\alpha} - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{\alpha}{1+\alpha}\right)^m \big|z\big|^{\alpha+1} \leq \Big|f(z)^{\alpha}\Big|$$
$$\leq \big|z\big|^{\alpha} + \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{\alpha}{1+\alpha}\right)^m \big|z\big|^{\alpha+1} \quad (60)$$

for $z \in D$ and

$$\alpha\left|z\right|^{\alpha-1} - \frac{(\alpha+\gamma)(\alpha+1)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{\alpha}{1+\alpha}\right)^m \left|z\right|^{\alpha} \leq \left|\left(f(z)^{\alpha}\right)'\right|$$

$$\alpha\left|z\right|^{\alpha-1} + \frac{(\alpha+\gamma)(\alpha+1)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{\alpha}{1+\alpha}\right)^m \left|z\right|^{\alpha} \tag{61}$$

for $z \in D$. The results (60) and (61) are sharp for the function

$$f(z)^{\alpha} = z^{\alpha} - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{\alpha}{1+\alpha}\right)^m z^{\alpha+1}. \tag{62}$$

REMARK G.

(i.) Supposing $m = 0$, $\alpha = 1$ and $b = 1$. Then Corollary 3.7 immediately yields the corresponding distortion Theorem for the class $V_0(a, c; \gamma)$ proved by Srivastava and Owa[14].

(ii.) In its special case when we pose a constraint on γ such that $-1 < \gamma \leq 0$ for $m = 0$, $\alpha = 1$ and $b = 1$, then Corollary 3.7 yields the corresponding distortion Theorem for the class $S^*(\gamma)$ proved by Silverman[13].

Also, applying the above technique mutatis mutandi, we can prove the theorem below.

Theorem 3.8. Let the function $f(z)^{\alpha}$ defined by (2) be in the class $W_0(a, b, c, l,\ m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then

$$\left|z\right|^{\alpha} - \frac{(\alpha)}{(1+\alpha)}\frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m \left|z\right|^{\alpha+1}$$

$$\leq \left|f(z)^{\alpha}\right| \tag{63}$$

$$\leq \left|z\right|^{\alpha} + \frac{(\alpha)}{(1+\alpha)}\frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m \left|z\right|^{\alpha+1}$$

for $z \in D$ and

$$\alpha\left|z\right|^{\alpha-1} - \frac{\alpha(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m \left|z\right|^{\alpha} \leq \left|\left(f(z)^{\alpha}\right)'\right|$$

$$\leq \alpha\left|z\right|^{\alpha-1} + \frac{\alpha(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m \left|z\right|^{\alpha} \tag{64}$$

for $z \in D$. The results (62) and (63) are sharp for the function

$$f(z)^{\alpha} = z^{\alpha} - \frac{(\alpha)}{(1+\alpha)}\frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)}\left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l}\right)^m z^{\alpha+1}. \tag{65}$$

Corollary 3.9. Let the function $f(z)^{\alpha}$ defined by (2) be in the class $W_0(a, b, c, l,\ m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$ and $a \geq c$. Then, the unit disk D is mapped onto a domain that contains the disk $|\omega| < r_1$, where r_1 is given by

$$r_1 = 1 - \frac{(\alpha)}{(1+\alpha)} \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)} \left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda\alpha+l} \right)^m . \quad (66)$$

Corollary 3.10. Let the function $f(z)^\alpha$ defined by (2) be in the class $W_0(a, b, c,\, 0, m, \alpha, 1; \gamma)$ with $\frac{(a)_k (b)_k}{(c)_k} > 0$ and $a \geq c$. Then,

$$\left|z\right|^\alpha - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)} \left(\frac{\alpha}{1+\alpha} \right)^{m+1} \left|z\right|^{\alpha+1} \leq \left| f(z)^\alpha \right|$$

$$\leq \left|z\right|^\alpha + \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)} \left(\frac{\alpha}{1+\alpha} \right)^{m+1} \left|z\right|^{\alpha+1} \quad (67)$$

for $z \in D$ and

$$\alpha\left|z\right|^{\alpha-1} - \frac{(\alpha+1)(\alpha+\gamma)(\alpha+1)(c)}{(1+\alpha+\gamma)(a)(b)} \left(\frac{\alpha}{1+\alpha} \right)^{m+1} \left|z\right|^\alpha \leq \left| \left(f(z)^\alpha \right)' \right|$$

$$\alpha\left|z\right|^{\alpha-1} + \frac{(\alpha+1)(\alpha+\gamma)(\alpha+1)(c)}{(1+\alpha+\gamma)(a)(b)} \left(\frac{\alpha}{1+\alpha} \right)^{m+1} \left|z\right|^\alpha \quad (68)$$

for $z \in D$. The results (66) and (67) are sharp for the function

$$f(z)^\alpha = z^\alpha - \frac{(\alpha+\gamma)(c)}{(1+\alpha+\gamma)(a)(b)} \left(\frac{\alpha}{1+\alpha} \right)^{m+1} z^{\alpha+1}. \quad (69)$$

For the recent investigation on distortion theorem, see $[2-5]$ and [8].

3. Starlikeness and Convexity

At this juncture, our interest is in the starlikeness for functions belonging to the class $V_0(a, b, c, 0, m, \alpha, 1; \gamma)$ and the convexity for functions belonging to the class $W_0(a, b, c, 0, m, \alpha, 1; \gamma)$. Obviously, for $-1 < \gamma \leq 0$, the classes $V(\alpha, \gamma)$ and $W(\alpha, \gamma)$ are respectively the class of starlike functions of order γ and the class of convex functions of order γ.

Theorem 4.1. Let the function $f(z)^\alpha$ defined by (2) be in the class $V_0(a, b, c, l,\, m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k (b)_k}{(c)_k} > 0$. Then, $f(z)^\alpha$ is starlike of order σ $(0 \leq \sigma < 1)$ in the unit disk $|z| < r_2$, where

$$r_2 = \underbrace{inf}_{k \in N} \left[\frac{(\alpha-\sigma)(1+\alpha+\gamma)_k (a)_k (b)_k}{(\alpha+k-\sigma)(\alpha+\gamma)_k (c)_k k!} T_k^m \right]^{1/k} \quad (70)$$

and

$$T_k^m = (\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l})^m.$$

Proof. It suffices to show that

$$\left|\frac{z\big(f(z)^\alpha\big)'}{f(z)^\alpha} - \alpha\right| < \alpha - \sigma \tag{71}$$

for $|z| < r_2$. It implies that

$$\left|\frac{z\big(f(z)^\alpha\big)'}{f(z)^\alpha} - \alpha\right| \le \frac{\sum_{k=1}^{\infty} k\big|a_{k+1}(\alpha)\big||z|^k}{1-\sum_{k=1}^{\infty}\big|a_{k+1}(\alpha)\big||z|^k} \le \alpha - \sigma \tag{72}$$

that is,

$$\frac{\sum_{k=1}^{\infty}(\alpha+k-\sigma)\big|a_{k+1}(\alpha)\big||z|^k}{\alpha-\sigma} \le 1. \tag{73}$$

With the aid of Theorem 2.5, (72) is true only if

$$\frac{\sum_{k=1}^{\infty}(\alpha+k-\sigma)\big|a_{k+1}(\alpha)\big||z|^k}{\alpha-\sigma} \le \sum_{k=1}^{\infty}\frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k}T_k^m\frac{(a)_k(b)_k}{(c)_k k!}\big|a_{k+1}(\alpha)\big|. \tag{74}$$

Hence, we have that

$$\big|z\big|^k \le \frac{(\alpha-\sigma)(1+\alpha+\gamma)_k(a)_k(b)_k}{(\alpha+k-\gamma)(\alpha+\gamma)_k(c)_k k!}T_k^m \tag{75}$$

where T_k^m is as earlier defined and this complete the proof of Theorem 4.1.

Corollary 4.2. Let the function $f(z)^\alpha$ defined by (2) be in the class $V_0(a, 1, a, l,\ m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then, $f(z)^\alpha$ is starlike of order σ $(0 \le \sigma < 1)$ in the unit disk $|z| < r_3$, where

$$r_3 = \underbrace{inf}_{k\in N}\left[\frac{(\alpha-\sigma)(1+\alpha+\gamma)_k}{(\alpha+k-\sigma)(\alpha+\gamma)_k}T_k^m\right]^{1/k} \tag{76}$$

REMARK H. In its special case when $m = 0$, $\alpha = 1$ and $b = 1$, Corollary 4.2 immediately yields the result for starlikeness due to Srivastava and Owa[14].

Theorem 4.3. Let the function $f(z)^\alpha$ defined by (2) be in the class $W_0(a, b, c, l,\ m, \alpha, \lambda; \gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then, $f(z)^\alpha$ is convex of order σ $(0 \le \sigma < 1)$ in the unit disk $|z| < r_4$, where

$$r_4 = \underbrace{inf}_{k\in N}\left[\frac{\alpha(\alpha-\sigma)(1+\alpha+\gamma)_k(a)_k(b)_k}{(\alpha+k-\sigma)(\alpha+\gamma)_k(c)_k k!}T_k^m\right]^{1/k} \tag{77}$$

and
$$T_k^m = (\frac{1+\lambda(\alpha+k-1)+l}{1+\lambda(\alpha-1)+l})^m.$$

Proof. It suffices to show that
$$\left|\frac{z\big(f(z)^\alpha\big)''}{(f(z)^\alpha)'}+(1-\alpha)\right| < \alpha-\sigma \tag{78}$$
for $|z| < r_4$. It implies that
$$\left|\frac{z\big(f(z)^\alpha\big)''}{(f(z)^\alpha)'}+(1-\alpha)\right| \le \frac{\sum_{k=1}^{\infty} k(\alpha+k)a_{k+1}(\alpha)z^{\alpha+k-1}}{\sum_{k=1}^{\infty}(\alpha+k)a_{k+1}(\alpha)z^{\alpha+k-1}} \le \alpha-\sigma \tag{79}$$
that is,
$$\frac{\sum_{k=1}^{\infty}\frac{\alpha+k}{(\alpha)}(\alpha+k-\sigma)\big|a_{k+1}(\alpha)\big||z|^k}{\alpha-\sigma} \le 1. \tag{80}$$
By means of Theorem 2.8, we have that
$$\big|z\big|^k \le \frac{\alpha(\alpha-\sigma)(1+\alpha+\gamma)_k(a)_k(b)_k}{(\alpha+k-\sigma)(\alpha+\gamma)_k(c)_k k!}T_k^m \quad k\in N, \tag{81}$$
and this complete the proof.

Corollary 4.4. Let the function $f(z)^\alpha$ defined by (2) be in the class $W_0(a,1,a,\ l,m,\alpha,\lambda;\gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then, $f(z)^\alpha$ is starlike of order σ $(0 \le \sigma < 1)$ in the unit disk $|z| < r_3$, where
$$r_3 = \underbrace{inf}_{k\in N}\left[\frac{\alpha(\alpha-\sigma)(1+\alpha+\gamma)_k}{(\alpha+k-\sigma)(\alpha+\gamma)_k}T_k^m\right]^{1/k} \tag{82}$$

4. Further application of Theorem 2.5 and Theorem 2.8

In this section, we shall establish some relationships between the various subclasses of A, which are further consequences of Theorem 2.5 and Theorem 2.8.

Theorem 5.1. Let $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then,
$$W_0(a,b,c,l,m,\alpha,\lambda;\gamma) \subset V_0\left(a,b,c,l,m,\alpha,\lambda;-\frac{1+\alpha}{2+\alpha+\gamma}\right). \tag{83}$$
The result is sharp.

Proof. Let the function $f(z)^\alpha$ defined by (2) be in the class

$W_0(a,b,c,l,m,\alpha,\lambda;\,\gamma)$ with $\frac{(a)_k(b)_k}{(c)_k} > 0$. we shall note that

$$\frac{(1+\alpha)_k(1+\alpha+\gamma)_k}{(\alpha)_k(\alpha+\gamma)_k} \geq \frac{\left(1+\alpha-\frac{1+\alpha}{2+\alpha+\gamma}\right)_k}{\left(\alpha-\frac{1+\alpha}{2+\alpha+\gamma}\right)_k} \tag{84}$$

for $\alpha \geq 1$, $\gamma > -1$ and $k \in N$, we then find that

$$\sum_{k=1}^{\infty} \frac{\left(1+\alpha-\frac{1+\alpha}{2+\alpha+\gamma}\right)_k (a)_k(b)_k}{\left(1+\alpha-\frac{1+\alpha}{2+\alpha+\gamma}\right)_k (c)_k k!} T_k^m$$

$$\leq \sum_{k=1}^{\infty} \frac{(\alpha+1)_k}{(\alpha)_k}\frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m \frac{(a)_k(b)_k}{(c)_k k!}\big|a_{k+1}(\alpha)\big| \leq 1, \tag{85}$$

and by Theorem 2.8, it implies that

$$f(z)^\alpha \in V_0\left(a,b,c,l,m,\alpha,\lambda;\frac{-(1+\alpha)}{2+\alpha+\gamma}\right), \tag{86}$$

since the result is the best possible for the function $f(z)^\alpha$ given by (65), this obviously ends the proof of Theorem 5.1.

Corollary 5.2.

$$W_0(a,1,a,l,0,\alpha,\lambda;\gamma) \subset V_0\left(a,1,a,l,0,\alpha,\lambda;\frac{-(1+\alpha)}{2+\alpha+\gamma}\right) \tag{87}$$

The result is sharp for the function $f(z)^\alpha$ given by (69).

Theorem 5.3. Let $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then, if $a \geq c$ and $b = 1$

$$V_0(a,1,c,l,m,\alpha,\lambda;\gamma) \subset V_0(l,m,\alpha,\lambda;\gamma), \tag{88}$$

and if $a < c$, for $b = 1$

$$V_0(a,1,c,l,m,\alpha,\lambda;\gamma) \supset V_0(l,m,\alpha,\lambda;\gamma). \tag{89}$$

Proof. We note that $\frac{(a)_k(b)_k}{(c)_k}$ is an increasing function of k if $a \leq c$ and is decreasing in k if $a < c$. Using Theorem 2.8 and Corollary 2.9, then the proof is completed.

In the same manner, we have the next Theorem.

Theorem 5.4. Let $\frac{(a)_k(b)_k}{(c)_k} > 0$. Then, if $a \geq c$ and $b = 1$

$$W_0(a,1,c,l,m,\alpha,\lambda;\gamma) \subset W_0(l,m,\alpha,\lambda;\gamma), \tag{90}$$

and if $a < c$, for $b = 1$

$$W_0(a,1,c,l,m,\alpha,\lambda;\gamma) \supset W_0(l,m,\alpha,\lambda;\gamma). \tag{91}$$

5. Subclasses of $V_0(a,b,c,l,m,\alpha,\lambda;\gamma)$ and $W_0(a,b,c,l,m,\alpha,\lambda;\gamma)$ with fixed finitely many coefficients

Finally, we shall introduce the subclasses $V_n(a,b,c,l,m,\alpha,\lambda;\gamma)$ and $W_n(a,b,\ c,l,m,\alpha,\lambda;\gamma)$ of analytic functions with fixed finitely many coefficients.
In view of Corollary 2.6, we denote by $V_n(a,b,c,l,m,\alpha,\lambda;\gamma)$ $V_0(a,b,$ $c,l,m,\alpha,\ \lambda;\gamma)$ consisting of functions of the form

$$f(z)^\alpha = z^\alpha - \sum_{i=1}^{n} A_i p_i z^{i+1} - \sum_{k=n+1}^{\infty} a_{k+1}(\alpha) z^{k+1} \quad (a_{k+1}(\alpha) \geq 0), \tag{92}$$

where,

$$A_i = \frac{(\alpha+\gamma)_i (c)_i i!}{(1+\alpha+\gamma)_i (a)_i (b)_i} T_i^{-m} \quad 0 \leq p_i \leq 1 \quad (i = 1, 2, ..., n), \tag{93}$$

where

$$T_i^{-m} = \left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda(\alpha+i-1)+l}\right), \ \frac{(a)_k (b)_k}{(c)_k} > 0 \ \ and \ \ 0 \leq \sum_{i=1}^{n} p_i \leq 1.$$

Also, in view of Corollary 2.9, we denote by $W_n(a,b,c,l,m,\alpha,\lambda;\gamma)$ $W_0(a,b,c,l,\ m,\alpha,\lambda;\gamma)$ consisting of functions of the form

$$f(z)^\alpha = z^\alpha - \sum_{i=1}^{n} B_i p_i z^{i+1} - \sum_{k=n+1}^{\infty} a_{k+1}(\alpha) z^{k+1} \quad (a_{k+1}(\alpha) \geq 0), \tag{94}$$

where,

$$B_i = \frac{(\alpha)_i (\alpha+\gamma)_i (c)_i i!}{(1+\alpha)_i (1+\alpha+\gamma)_i (a)_i (b)_i} T_i^{-m} \quad 0 \leq p_i \leq 1 \quad (i = 1, 2, ..., n), \tag{95}$$

where

$$T_i^{-m} = \left(\frac{1+\lambda(\alpha-1)+l}{1+\lambda(\alpha+i-1)+l}\right), \ \frac{(a)_k (b)_k}{(c)_k} > 0 \ \ and \ \ 0 \leq \sum_{i=1}^{n} p_i \leq 1.$$

Furthermore, we define the class $V_n(a,b,c,l,m,\alpha,\lambda;\gamma)$ and $W_n(a,b,$ $c,l,m,\alpha,\lambda;\ \gamma)$ by

$$V_n(l,m,\alpha,\lambda;\gamma) = V_n(a,1,a,l,m,\alpha,\lambda;\gamma) \tag{96}$$

and

$$W_n(l,m,\alpha,\lambda;\gamma) = W_n(a,1,a,l,m,\alpha,\lambda;\gamma). \tag{97}$$

Theorem 6.1. Let the function $f(z)^\alpha$ be defined by (92). Then, $f(z)^\alpha \in V_n(a,b,c,l,m,\alpha,\lambda;\gamma)$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} \frac{(a)_k(b)_k}{(c)_k k!} a_{k+1}(\alpha) T_k^m \geq 1 - \sum_{i=1}^{n} p_i. \tag{98}$$

The result is sharp.

Proof. Putting

$$a_i = A_i p_i \quad (i = 1, 2, ..., n)$$

in (26), then we have (98). The result is the best possible for the function of the form

$$f(z)^\alpha = z^\alpha - \sum_{i=1}^{n} A_i p_i z^{i+1} - \frac{(\alpha+\gamma)_k (c)_k k! \big(1 - \sum_{i=1}^{n} p_i\big)}{(1+\alpha+\gamma)_k T_k^{-m} (a)_k (b)_k} z^{k+1} \tag{99}$$

$$(k \geq n+1)$$

where T_k^{-m} is as defined earlier.

Corollary 6.2. Let the function $f(z)^\alpha$ be defined by (92) $a = c$ and $b = 1$. Then $f(z)^\alpha \in V_n(a,1,a,l,m,\alpha,\lambda;\gamma)$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m a_{k+1}(\alpha) \geq 1 - \sum_{i=1}^{n} p_i. \tag{100}$$

The result is sharp for the function

$$f(z)^\alpha = z^\alpha - \sum_{i=1}^{n} \frac{(\alpha+\gamma)_i p_i}{(1+\alpha+\gamma)_i} T_k^{-m} z^{i+1}$$

$$- \frac{(\alpha+\gamma)_k \big(1 - \sum_{i=1}^{n} p_i\big)}{(1+\alpha+\gamma)_k} T_k^{-m} z^{k+1} \ (k \geq n+1) \tag{101}$$

Theorem 6.3. Let the function $f(z)^\alpha$ be defined by (94). Then, $f(z)^\alpha \in W_n(a,b,c,l,m,\alpha,\lambda;\gamma)$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{(1+\alpha)_k (1+\alpha+\gamma)_k}{(\alpha)_k (\alpha+\gamma)_k} \frac{(a)_k(b)_k}{(c)_k k!} T_k^m a_{k+1}(\alpha) \geq 1 - \sum_{i=1}^{n} p_i. \tag{102}$$

The result is sharp for the function

$$f(z)^\alpha = z^\alpha - \sum_{i=1}^{n} B_i p_i z^{i+1}$$

$$- \frac{(\alpha)_k (\alpha+\gamma)_k (c)_k k! \big(1 - \sum_{i=1}^{n} p_i\big)}{(1+\alpha)_k (1+\alpha+\gamma)_k (a)_k (b)_k} T_k^{-m} z^{k+1} \quad (k \geq n+1) \tag{103}$$

The proof is much similar to that of Theorem 6.1, thus we ignore the details involved.

Corollary 6.4. Let the function $f(z)^\alpha$ be defined by (94) $a = c$ and $b = 1$. Then $f(z)^\alpha \in W_n(a, 1, a, l, m, \alpha, \lambda; \gamma)$ if and only if

$$\sum_{k=n+1}^{\infty} \frac{(1+\alpha)_k}{(\alpha)_k} \frac{(1+\alpha+\gamma)_k}{(\alpha+\gamma)_k} T_k^m a_{k+1}(\alpha) \geq 1 - \sum_{i=1}^{n} p_i. \tag{104}$$

The result is the best possible for the function

$$f(z)^\alpha = z^\alpha - \sum_{i=1}^{n} \frac{(\alpha)_i}{(1+\alpha)_i} \frac{(\alpha+\gamma)_i p_i}{(1+\alpha+\gamma)_i} T_k^{-m} z^{i+1}$$
$$- \frac{(\alpha)_k}{(1+\alpha)_k} \frac{(\alpha+\gamma)_k \left(1 - \sum_{i=1}^{n} p_i\right)}{(1+\alpha+\gamma)_k} T_k^{-m} z^{k+1} \ (k \geq n+1). \tag{105}$$

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