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FLEXURAL VIBRATION TO PARTIALLY DISTRIBUTED MASSES OF NON-UNIFORM RAYLEIGH BEAM RESTING ON VLASOV FOUNDATION WITH GENERAL BOUNDARY CONDITIONS

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ABSTRACT. In this study, the dynamic response to partially distributed moving masses of non-prismatic Rayleigh beam with classical boundary conditions and resting on Vlasov elastic foundation moving with variable velocities is investigated. A procedure involving generalized Galerkin's method, the use of the expression of the Heaviside function in series form, a modification of the Strubles asymptotic method and the use of the Fresnel sine and Fresnel cosine functions is developed to solve the dynamical problem and closed form solutions for both the moving distributed force and moving distributed mass models which is valid for all variants of classical boundary conditions are obtained. The closed form solutions obtained are evaluated numerically and results show that an increase in the values of foundation stiffness, shear modulus, axial force and rotatory inertia correction factor reduces the response amplitudes of both clamped-clamped non-uniform Rayleigh beam and the clamped-free non-uniform Rayleigh beam. Resonance conditions for the dynamical systems are obtained for the illustrative end conditions considered. Finally, useful conclusions are drawn from the investigation of the flexural vibration of non-uniform beams resting on Vlasov foundation and under partially distributed masses moving at varying velocities.

Keywords and phrases: Partially distributed masses, Rayleigh beam, Vlasov foundation, Resonance.

1. INTRODUCTION

This paper is concerned with the problem of flexural motion of non-uniform Rayleigh beam on Vlasov foundation and under partially distributed masses moving at varying velocities for all variants of classical end supports. It is sequel to an earlier paper [1] that considered the response to simply supported non-prismatic

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Rayleigh beam to travelling partially distributed loads. In particular, this paper is a generalization of the theory advanced in paper [1]. Beams resting on elastic foundations are very often encountered in the analysis of building, geotechnical highways and railway structures. However the vibration of Beam-like structures under the action of moving loads on elastic foundation has been a subject of technological importance and many papers have been published on it during the past years. Among the earlier researchers into this subject were Fryba [2], Oni [3], Andi and Oni [4], Oni and Awodola [5], Omolofe and Ogunbamike [6], Alaa [7], Isede and Gbadeyan [8] and Abdelghany [9]. The importance of this problem is manifested in numerous applications in the field of transportation. For instance, bridges, overhead cranes, cable ways, rails, roadways, tunnels and pipelines are examples of structural elements designed to support moving loads. However, in most of the studies available in literature, the scope has been limited to structural members having uniform cross-section whether the inertia of the moving load is considered or not. In most of the existing works in literature on the vibration of elastic beam on moving load, the loads are simplified by considering them as point loads or concentrated loads; a problem which has continued to motivate a considerable number of researchers [10-13]. Evidently, such single vector line segment acting at a particular point as it moves does not model the forces involved in the physical situation accurately. However, distributed moving load problems have been the subject of recent investigation [14-15] being more practically realistic than the concentrated problem. Esmailzadeh and Ghorashi [14] studied the problem of vibration of beam traversed by uniform partially distributed masses. The work was extended by the same authors by considering the vibration of Timoshenko beams subjected to a travelling mass [15]. Recently, Oni and Ogunyebi [16] carried out the dynamic analysis of prestressed Bernoulli-Euler beams with general boundary conditions under travelling distributed loads. Furthermore, the analysis of structures resting on elastic foundation is usually based on a relatively simple model of the foundations response to applied load. Generally, the analysis of vibration of beams on an elastic foundation is developed on the assumption that the reaction forces of the foundation are proportional at every point to the deflection of the beam at that point. The vertical deformation characteristics of the foundation are defined by means of continuous, closely spaced strings. The constant of proportionality of these springs is known as the modulus of subgrade reaction K. This simple representation of elastic foundation was introduced by Winkler in 1867. The Winkler model (one parameter model) which has been originally developed for the analysis of railroad tracks, is very simple but does not accurately represents the characteristics of many practical foundations. One of the most important deficiencies of the Winkler model is that a displacement discontinuity appears between the loaded and the unloaded parts of the foundation surface. In order to eliminate the deficiency of Winkler model, improved theories have been introduced on refinement of Winkler model, by visualizing various types of interconnections such as shear layers and beams along the winkler springs [17], Pasternak [18]. These theories have been attempted to find an applicable and simple model of representation of foundation medium. Also known as Vlasov foundation, two parameter foundation model are more accurate than the one-parameter foundation model. Thus, this paper presents the flexural vibration under partially distributed masses of non-uniform Rayleigh beam resting on Vlasov elastic foundation with general boundary conditions.

2. MATHEMATICAL MODEL

The problem of the flexural vibrations of finite non-uniform Rayleigh beam resting on elastic foundation and under partially distributed loads moving when it is under the action of a moving load of mass M which is moving with a variable velocity determined by the position $f(t)$ of the mass M at any time is governed by the fourth order partial differential equation Fryba [2]

$$\begin{aligned}&\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 V(x,t)}{\partial x^2}\right]-N\frac{\partial^2 V(x,t)}{\partial x^2}+\mu(x)\frac{\partial^2 V(x,t)}{\partial t^2}\\&-\mu(x)R^0\frac{\partial^4 V(x,t)}{\partial x^2\partial t^2}-G\frac{\partial^2 V(x,t)}{\partial x^2}+KV(x,t)\\&=MgH[x-f(t)]\left[1-\frac{1}{g}\frac{\mathrm{d}^2 V(x,t)}{\mathrm{d}t^2}\right]\end{aligned}\tag{1}$$

where $EI(x)$ is the variable flexural rigidity of the structure, N is the axial force, R^0 is the rotatory inertia factor, $\mu(x)$ is the variable mass per unit length of the beam, $V(x,t)$ is the transverse displacement, x is the spatial coordinate, the time t is assumed to be limited to that interval of time within which the mass on the beam, that is

$$0\le f(t)\le L\tag{2}$$

g is the acceleration due to gravity and $\frac{\mathrm{d}^2}{\mathrm{d}t^2}$ is a convective acceleration operator defined in [3]

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} = \frac{\partial^2}{\partial t^2} + 2\frac{\mathrm{d}}{\mathrm{d}t}f(t)\frac{\partial^2}{\partial x \partial t} + \left(\frac{\mathrm{d}f(t)}{\mathrm{d}t}\right)^2 \frac{\partial^2}{\partial x^2} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}f(t)\frac{\partial}{\partial x} \tag{3}$$

The distance covered by the load on the same structure at any given instance of time is given as

$$f(t) = x_0 + ct + \frac{1}{2}at^2 \tag{4}$$

where x_0 is the point of application of force $P(x,t)$ at the instance $t = 0$, c is the initial velocity and a is the constant acceleration of motion and $H[x - f(t)]$ is the Heaviside function defined as

$$H\Big[x - f(t)\Big] = \begin{cases} 0 & \text{if } x < f(t), \\ 1 & \text{if } x > f(t). \end{cases} \tag{5}$$

The boundary conditions of the structure under consideration is arbitrary, while the initial conditions are

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t} \tag{6}$$

Adopting the examples in [1],in the non-uniform Rayleigh beam structure, $I(x)$ and $\mu(x)$ are taken to be of the form

$$I(x) = I_0\left(1 + \sin\frac{\pi x}{L}\right)^3 \tag{7}$$

and

$$\mu(x) = \mu_0\left(1 + \sin\frac{\pi x}{L}\right) \tag{8}$$

Substituting equations (3), (4), (7) and (8) into equation (1) after some simplifications and rearrangements one obtains

$$\frac{\partial^2}{\partial x^2}\left[EI_0\left(1 + \sin\frac{\pi x}{L}\right)^3 \frac{\partial^2 V(x,t)}{\partial x^2}\right] - N\frac{\partial^2 V(x,t)}{\partial x^2} - G\frac{\partial^2 V(x,t)}{\partial x^2}$$

$$+\mu_0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^2 V(x,t)}{\partial t^2} - \frac{\partial}{\partial x}\left[\mu_0 R^0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^3 V(x,t)}{\partial x \partial t^2}\right]$$

$$+KV(x,t) + MH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]$$

$$\left[\frac{\partial^2 V(x,t)}{\partial t^2} + 2(c+at)\frac{\partial^2 V(x,t)}{\partial x \partial t} + \left(c+at^2\right)\frac{\partial^2 V(x,t)}{\partial x^2} + a\frac{\partial V(x,t)}{\partial x}\right] = MgH\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] \tag{9}$$

3. OPERATIONAL SIMPLIFICATION

This section seeks to obtain the analytical solution to the problem of the dynamic response of a non-uniform Rayleigh beam resting on Vlasov foundation for arbitrary boundary conditions. The Generalized Galerkin method discussed in [1] is employed to simplify and reduce equation (9) to a sequence of second order ordinary differential equations. This method requires that the solution of equation (9) be of the form

$$V_n(x,t) = \sum_{m=1}^{\infty} Z_m(t) U_m(x) \tag{10}$$

where $U_m(x)$ is chosen such that the pertinent boundary conditions are satisfied. Equation (10) is substituted into equation (9) and after some simplifications and rearrangements one obtains

$$\begin{aligned}\sum_{m=1}^{n}\Bigg\{\Bigg[\frac{EI_0}{\mu_0}\Bigg(&\frac{5}{2}U_m^{iv}(x) + \frac{15}{4}\sin\frac{\pi x}{L}U_m^{iv}(x) - \frac{1}{4}\sin\frac{3\pi x}{L}U_m^{iv}(x) \\ &-\frac{3}{2}\cos\frac{2\pi x}{L}U_m^{iv}(x)\Bigg) + \left(\frac{9\pi^2}{4L^2}\sin\frac{3\pi x}{L}U_m^{"}(x) - \frac{15\pi^2}{4L^2}\sin\frac{\pi x}{L}U_m^{"}(x) + \right. \\ &\left.+\frac{6\pi^2}{L^2}\cos\frac{2\pi x}{L}U_m^{"}(x)\right)\Bigg]Z_m(t) - \left(\frac{N+G}{\mu_0}\right)U_m^{"}(x)Z_m(t) + \\ &\left(1+\sin\frac{\pi x}{L}\right)U_m(x)\ddot{Z}_m(t) - R^0\left[\left(1+\sin\frac{\pi x}{L}\right)U_m^{"}(x)\ddot{Z}_m(t)\right. \\ &\left.+\frac{\pi}{L}\cos\frac{\pi x}{L}U_m^{"}(x)\ddot{Z}_m(t)\right] + \frac{K}{\mu_0}U_m(x)Z_m(t) \\ &+\frac{MH}{\mu_0}\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]\Big[U_m(x)\ddot{Z}_m(t) \\ &+(c+at)U'_m(x)\dot{Z}_m(t) + (c+at)^2 U"_m(x)Z_m(t) \\ &+aU'_m(x)Z_m(t)\Big]\Bigg\} - \frac{PH}{\mu_0}\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right] = 0\end{aligned} \tag{11}$$

In order to determine $Z_m(t)$, it is required that the expression on the left hand side of the equation be orthogonal to the function $U_k()$. Hence

$$\int_0^L \Bigg\{ \sum_{m=1}^{n} \Bigg\{ \Bigg[\frac{EI_0}{\mu_0}\left(\frac{5}{2}+\frac{15}{4}\sin\frac{\pi x}{L}-\frac{1}{4}\sin\frac{3\pi x}{L}-\frac{3}{2}\cos\frac{2\pi x}{L}\right)U_m^{iv}(x)$$
$$+\left(\frac{9\pi^2}{4L^2}\sin\frac{3\pi x}{L}-\frac{15\pi^2}{4L^2}\sin\frac{\pi x}{L}+\frac{6\pi^2}{L^2}\cos\frac{2\pi x}{L}\right)U_m^{"}(x)\Bigg] Z_m(t)$$
$$-\left(\frac{N+G}{\mu_0}\right)U_m^{"}(x)Z_m(t)+\left(1+\sin\frac{\pi x}{L}\right)U_m(x)\ddot{Z}_m(t)$$
$$-R^0\Bigg[\left(1+\sin\frac{\pi x}{L}\right)U_m^{"}(x)\ddot{Z}_m(t)+\frac{\pi}{L}\cos\frac{\pi x}{L}U_m^{"}(x)\ddot{Z}_m(t)\Bigg]$$
$$+\frac{K}{\mu_0}U_m(x)Z_m(t)+\frac{MH}{\mu_0}\Bigg[x-\left(x_0+ct+\frac{1}{2}at^2\right)\Bigg]\Bigg[U_m(x)\ddot{Z}_m(t)$$
$$+(c+at)U_m'(x)\dot{Z}_m(t)+(c+at)^2U"_m(x)Z_m(t)$$
$$+aU_m'(x)Z_m(t)\Bigg]\Bigg\}-\frac{PH}{\mu_0}\Bigg[x-\left(x_0+ct+\frac{1}{2}at^2\right)\Bigg]\Bigg\}=0 \tag{12}$$

Further rearrangement of equation (12) yield

$$\sum_{m}^{n}\Bigg\{\Bigg[D_1(m,k)-R^0(D_2(m,k)+D_3(m,k))\Bigg]\ddot{Z}_m(t)$$
$$+\Bigg[\alpha_1(T_0+T_1)+\alpha_2D_4(m,k)+\alpha_3D_5(m,k)\Bigg]Z_m+$$
$$\frac{M}{\mu_0}\Bigg[D_1(t)\ddot{Z}_m(t)+D_2(t)\dot{Z}_m(t)+D_3(t)Z_m$$
$$+D_4(t)Zm(t)\Bigg]\Bigg\}=\frac{Mg}{\mu_0}D_5(t) \tag{13}$$

where

$$\alpha_1=\frac{EI_0}{\mu_0}; \qquad \alpha_2=\frac{G+N}{\mu_0}; \qquad \alpha_3=\frac{K}{\mu_0} \tag{14}$$

$$T_0=D_6+D_7-(D_8+D_9); \qquad T_1=D_{10}-D_{11}+D_{12} \tag{15}$$

$$D_1(t)=\int_0^L H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U_m(x)U_k(m)\mathrm{d}x;$$
$$D_2(t)=\int_0^L 2(c+at)H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U'_m(x)U_k(x)\mathrm{d}x$$
$$D_3(t)=\int_0^L (c+at)^2H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U^n_m(x)U_k(x)\mathrm{d}x;$$
$$D_4(t)=\int_0^L aH\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U'_m(x)U_k(x)\mathrm{d}x$$
$$D_5(t)=\int_0^L H\left[x-\left(x_0+ct+\frac{1}{2}at^2\right)\right]U_k(x)\mathrm{d}x;$$
$$D_1(m,k)=\int_0^L \left(1+\sin\frac{\pi x}{L}\right)U_m(x)U_k(x)\mathrm{d}x$$
$$D_2(m,k)=\int_0^L \left(1+\sin\frac{\pi x}{L}\right)U^n_m(x)U_k(x)\mathrm{d}x;$$
$$D_3(m,k)=\int_0^L \cos\frac{\pi x}{L}U'_m(x)U_k(x)\mathrm{d}x;$$
$$D_4(m,k)=\int_0^L U^n_m(x)U_k(x)\mathrm{d}x;$$
$$D_5(m,k)=\int_0^L U_m(x)U_k(x)\mathrm{d}x;\ \ D_6(m,k)=\frac{5}{2}\int_0^L U^{iv}_m(x)U_k(x)\mathrm{d}x;$$
$$D_7(m,k)=\frac{15}{4}\int_0^L \sin\frac{\pi x}{L}U^{iv}_m(x)U_k(x)\mathrm{d}x$$
$$D_8(m,k)=\frac{1}{4}\int_0^L \sin\frac{3\pi x}{L}U^{iv}_m(x)U_k(x)\mathrm{d}x;$$
$$D_9(m,k)=\frac{3}{2}\int_0^L \cos\frac{2\pi x}{L}U^{iv}_m(x)U_k(x)\mathrm{d}x$$
$$D_{10}(m,k)=\frac{9\pi^2}{4L^2}\int_0^L \sin\frac{3\pi x}{L}U^n_m(x)U_k(x)\mathrm{d}x;$$
$$D_{11}(m,k)=\frac{15\pi^2}{4L^2}\int_0^L \sin\frac{\pi x}{L}U^n_m(x)U_k(x)\mathrm{d}x$$
$$D_{12}(m,k)=\frac{6\pi^2}{L^2}\int_0^L \cos\frac{2\pi x}{L}U^n_m(x)U_k(x)\mathrm{d}x; \tag{16}$$

In order to evaluate the integrals (16), use is made of the Fourier sine series representation for the Heaviside unit step function namely;

$$H = \frac{1}{4} + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi\left[x - \left(x_0 + ct + \frac{1}{2}at^2\right)\right]}{2n+1}, \quad 0 < x < L, \tag{17}$$

Thus, in view of (9), using (17) in equation (13) after some simplifications and rearrangements one obtains

$$\begin{aligned}
&\Big[D_1(k,m) - R\Big[D_2(k,m) + D_3(m,k)\Big]\Big]\ddot{Z}_m(t) \\
&+\Big[\alpha_1(T_0+T_1) + \alpha_2 D_4(m,k) + \alpha_3 D_5(m,k)\Big]Z_m + \\
&\frac{M}{\mu_0}\Big\{\Big[D_{1A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{1B}(k,m,n) \\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{1C}(n,m,k)\Big]\ddot{Z}_m(t) \\
&+2(c+at)\Big(D_{2A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{2B}(k,m,n) \\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{2C}(k,m,n)\Big)\dot{Z}_m(t) \\
&+(c+at)^2\Big(D_{3A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{3B}(k,m,n) \\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{3C}(k,m,n)\Big)Z_m(t) \\
&+a\Big(D_{4A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{4B}(k,m,n) \\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{4C}(k,m,n)\Big)\dot{Z}_m(t)\Big\} \\
&= \frac{PL}{\mu_o\lambda_k\Omega_1(k,m)}\Big[-cos\lambda_k + A_k sin\lambda_k + B_k cosh\lambda_k + C + ksinh\lambda_k + \\
&cos\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2) - A_k sin\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2) \\
&-B_k cosh\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2) - C_k sinh\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2)\Big]
\end{aligned} \tag{18}$$

where

$$
\begin{aligned}
D_1(k,m) &= D_{11} + D_{12} \\
D_{11} = \int_0^L U_m(x)U_k(x)dx \qquad D_{12} &= \int_0^L \sin\frac{\pi x}{L} U_m(x)U_k(x)dx \\
D_{1A}(k,m) &= \frac{1}{4}\int_0^L U_m(x)U_k(x)dx \\
D_{1B}(k,m) &= \int_0^L \sin(2n+1)\pi x U_m(x)U_k(x)dx \\
D_{1C}(k,m) &= \int_0^L \cos(2n+1)\pi x U_m(x)U_k(x)dx \\
D_{2A}(k,m) &= \frac{1}{4}\int_0^L U'_m(x)U'_k(x)dx \\
D_{2B}(k,m) &= \int_0^L \sin(2n+1)\pi x U'_m(x)U_k(x)dx \\
D_{2C}(k,m) &= \int_0^L \cos(2n+1)\pi x U'_m(x)U_k(x)dx \\
D_{3A}(k,m) &= \frac{1}{4}\int_0^L U"_m(x)U_k(x)dx \\
D_{3B}(k,m) &= \int_0^L \sin(2n+1)\pi x U"_m(x)U_k(x)dx \\
D_{3C}(k,m) &= \int_0^L \cos(2n+1)\pi x U"_m(x)U_k(x)dx \\
D_{4A}(k,m) &= \frac{1}{4}\int_0^L U'_m(x)U_k(x)dx \\
D_{4B}(k,m) &= \int_0^L \sin(2n+1)\pi x U'_m(x)U_k(x)dx \\
D_{4C}(k,m) &= \int_0^L \cos(2n+1)\pi x U'_m(x)U_k(x)dx
\end{aligned} \tag{19}
$$

Equation (18) when further simplified and rearranged gives

$$
\begin{aligned}
\sum_{n=0}^{\infty}\Bigg\{\ddot{Z}_m(t) + \frac{\Omega_2(k,m)}{\Omega_1(k,m)}Z_m(t) + \frac{\Gamma_0 L}{\Omega_1(k,m)}\Bigg[\Bigg(D_{1A}(k,m) + \\
\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{1B}(k,m,n) \\
-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{1C}(n,m,k)\Bigg)\ddot{Z}_m(t) \\
+2(c+at)\Bigg(D_{2A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{2B}(k,m,n) \\
-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{2C}(k,m,n)\Bigg)\dot{Z}_m(t) \\
+(c+at)^2\Bigg(D_{3A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{3B}(k,m,n) \\
-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{3C}(k,m,n)\Bigg)Z_m(t) \\
+a\Bigg(D_{4A}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{4B}(k,m,n) \\
-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{4C}(k,m,n)\Bigg)\dot{Z}_m(t)\Bigg]\Bigg\} \\
= \frac{PL}{\mu_o\lambda_k\Omega_1(k,m)}\Bigg[-cos\lambda_k + A_k sin\lambda_k + B_k cosh\lambda_k + C + ksinh\lambda_k \\
+cos\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2) - A_k sin\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2) \\
-B_k cosh\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2) - C_k sinh\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2)\Bigg]
\end{aligned} \tag{20}
$$

where

$$
\begin{aligned}
\Omega_1(k,m) &= D_1(k,m) - R^0\Big[D_2(k,m) + D_3(k,m)\Big]; \\
\Omega_2(k,m) &= \alpha_1(T_0+T_1) + \alpha_2 D_4(k,m) + \alpha_3 D_5(k,m) \\
&\quad and \;\; \epsilon_o = \frac{M}{\mu_o L}
\end{aligned} \tag{21}
$$

$$
and \;\; \epsilon_o = \frac{M}{\mu_o L} \tag{22}
$$

Equation (20) is the transformed equation governing the problem of a non-uniform Rayleigh beam on a bi-Parametric Vlasov foundation and traversed by uniform partially distributed masses. In what follows, two cases of equation (20) are considered.

(a) Moving Force

If we neglect the inertia term in equation (20), we have the classical case of a moving force problem. Under this assumption and equation (20) after some simplifications and rearrangement becomes

$$\begin{aligned}\ddot{Z}_m(t) + \frac{\Omega_2(k,m)}{\Omega_1(k,m)}Z_m(t) = \frac{PL}{\mu_o\lambda_k\Omega_1(k,m)}\Big[- cos\lambda_k + A_k sin\lambda_k + \\ B_k cosh\lambda_k + C_k sinh\lambda_k + cos\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2) \\ -A_k sin\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2) - B_k cosh\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2) \\ -C_k sinh\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2)\Big]\end{aligned} \tag{23}$$

where

$$\begin{aligned}\ddot{Z}_m(t) + \sigma^2_{MF}Z_m(t) = Q_p\Big[- cos\lambda_k + A_k sin\lambda_k + \\ B_k cosh\lambda_k + C_k sinh\lambda_k + cos\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2) \\ -A_k sin\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2) - B_k cosh\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2) \\ -C_k sinh\frac{\lambda_k}{L}(x_0 + ct + \frac{1}{2}at^2)\Big]\end{aligned} \tag{24}$$

where

$$\sigma^2_{MF} = \frac{\Omega_2(k,m)}{\Omega_1(k,m)}; \qquad Q_p = \frac{PL}{\mu_o\lambda_k\Omega_1(k,m)} \tag{25}$$

Solving equation (24) in conjunction with the initial conditions, the solution is given by

$$
\begin{aligned}
Z_m(t) = \frac{P_{MF}}{\sigma_{MF}} \Bigg\{ & Sin\sigma_{MF}t \Bigg[P_{11}S\Big(d_{11}+d_{10}t\Big) + P_{12}C\Big(d_{11}+d_{10}t\Big) \\
& +P_{13}S\Big(d_{12}+d_{10}t\Big) + P_{14}C\Big(d_{12}+d_{10}t\Big) + P_{21}C\Big(d_{11}+d_{10}t\Big) \\
& +P_{22}S\Big(d_{11}+d_{10}t\Big) + P_{23}C\Big(d_{12}+d_{10}t\Big) - P_{24}S\Big(d_{12}+d_{10}t\Big) \\
& -Q_{11}erfi\Big(d_{21}+d_{20}t\Big) - Q_{12}erf\Big(d_{21}+d_{20}t\Big) - Q_{13}erfi\Big(d_{22}+d_{20}t\Big) - \\
& Q_{14}erfi\Big(d_{22}+d_{20}t\Big) + Q_{21}erf\Big(d_{21}+d_{20}t\Big) - Q_{22}erfi\Big(d_{21}+d_{20}\Big)t \\
& -Q_{23}erf\Big(d_{22}+d_{20}\Big)t + Q_{24}erfi\Big(d_{22}+d_{20}\Big)t - \frac{1}{2\sigma_{MF}}\Big[sin(\lambda_k+\sigma_{MF}t) \\
& -sin(\lambda_k-\sigma_{MF}t) + A_k\Big(cos(\lambda_k-\sigma_{MF}t) - cos(\lambda_k+i\sigma_{MF}t)\Big) \\
& -iB_k\Big(sinh(\lambda_k+i\sigma_{MF}t) - sinh(\lambda_k+\sigma_{MF}t)\Big) \\
& -iC_k\Big(cosh(\lambda_k+i\sigma_{MF}t) - cosh(\lambda_k-i\sigma_{MF}t)\Big)\Big] - F_2^*\Bigg] \\
& -Cos\sigma_{MF}t\Bigg[P_{12}S(d_{11}+d_{10}t) - P_{11}C\Big(d_{11}+d_{10}t\Big) + P_{13}C\Big(d_{12}+d_{10}t\Big) \\
& -P_{14}C\Big(d_{12}+d_{10}t\Big) + P_{21}S\Big(d_{11}+d_{10}t\Big) + P_{22}C\Big(d_{11}+d_{10}t\Big) - P_{23}S\Big(d_{12}+d_{10}t\Big) \\
& -P_{24}C\Big(d_{12}+d_{10}t\Big) + iQ_{11}erfi\Big(d_{21}+d_{20}t\Big) - iQ_{12}erf\Big(d_{21}+d_{20}t\Big) \\
& -iQ_{13}erf\Big(d_{22}+d_{20}t\Big) - iQ_{14}erf\Big(d_{22}+d_{20}t\Big) \\
& +\frac{1}{2\sigma_{MF}}\Big[cos(\lambda_k-\sigma_{MF}t) + cos(\lambda_k+\sigma_{MF}t) \\
& -A_k\Big(sin(\lambda_k-\sigma_{MF}t) + sin(\lambda_k+\sigma_{MF}t)\Big) \\
& -B_k\Big(cosh(\lambda_k+i\sigma_{MF}t) + cosh(\lambda_k-i\sigma_{MF}t)\Big) \\
& -C_k\Big(sinh(\lambda_k+i\sigma_{MF}t) - sinh(\lambda_k-i\sigma_{MF}t)\Big)\Big] + F_1^*\Bigg]\Bigg\}
\end{aligned} \tag{26}
$$

Thus using (26) in (10), one obtains

$$
\begin{aligned}
V(x,t) = \frac{1}{\rho_m(x)} \sum_{m=1}^{\infty} \frac{P_{MF}}{\sigma_{MF}} \Bigg\{ & Sin\sigma_{MF}t \Big[P_{11}S\Big(d_{11}+d_{10}t\Big) + P_{12}C\Big(d_{11}+d_{10}t\Big) \\
& + P_{13}S\Big(d_{12}+d_{10}t\Big) + P_{14}C\Big(d_{12}+d_{10}t\Big) + P_{21}C\Big(d_{11}+d_{10}t\Big) \\
& + P_{22}S\Big(d_{11}+d_{10}t\Big) + P_{23}C\Big(d_{12}+d_{10}t\Big) - P_{24}S\Big(d_{12}+d_{10}t\Big) \\
& - Q_{11}erfi\Big(d_{21}+d_{20}t\Big) - Q_{12}erf\Big(d_{21}+d_{20}t\Big) - Q_{13}erfi\Big(d_{22}+d_{20}t\Big) - \\
& Q_{14}erfi\Big(d_{22}+d_{20}t\Big) + Q_{21}erf\Big(d_{21}+d_{20}t\Big) - Q_{22}erfi\Big(d_{21}+d_{20}\Big)t \\
& - Q_{23}erf\Big(d_{22}+d_{20}\Big)t + Q_{24}erfi\Big(d_{22}+d_{20}\Big)t - \frac{1}{2\sigma_{MF}}\Big[sin(\lambda_k+\sigma_{MF}t) \\
& - sin(\lambda_k-\sigma_{MF}t) + A_k\Big(cos(\lambda_k-\sigma_{MF}t) - cos(\lambda_k+i\sigma_{MF}t)\Big) \\
& - iB_k\Big(sinh(\lambda_k+i\sigma_{MF}t) - sinh(\lambda_k+\sigma_{MF}t)\Big) \\
& - iC_k\Big(cosh(\lambda_k+i\sigma_{MF}t) - cosh(\lambda_k-i\sigma_{MF}t)\Big)\Big] - F_2^*\Big] \\
& - Cos\sigma_{MF}t\Big[P_{12}S(d_{11}+d_{10}t) - P_{11}C\Big(d_{11}+d_{10}t\Big) + P_{13}C\Big(d_{12}+d_{10}t\Big) \\
& - P_{14}C\Big(d_{12}+d_{10}t\Big) + P_{21}S\Big(d_{11}+d_{10}t\Big) + P_{22}C\Big(d_{11}+d_{10}t\Big) - P_{23}S\Big(d_{12}+d_{10}t\Big) \\
& - P_{24}C\Big(d_{12}+d_{10}t\Big) + iQ_{11}erfi\Big(d_{21}+d_{20}t\Big) - iQ_{12}erf\Big(d_{21}+d_{20}t\Big) \\
& - iQ_{13}erf\Big(d_{22}+d_{20}t\Big) - iQ_{14}erf\Big(d_{22}+d_{20}t\Big) \\
& + \frac{1}{2\sigma_{MF}}\Big[cos(\lambda_k-\sigma_{MF}t) + cos(\lambda_k+\sigma_{MF}t) \\
& - A_k\Big(sin(\lambda_k-\sigma_{MF}t) + sin(\lambda_k+\sigma_{MF}t)\Big) \\
& - B_k\Big(cosh(\lambda_k+i\sigma_{MF}t) + cosh(\lambda_k-i\sigma_{MF}t)\Big) \\
& - C_k\Big(sinh(\lambda_k+i\sigma_{MF}t) - sinh(\lambda_k-i\sigma_{MF}t)\Big)\Big] + F_1^*\Big]\Bigg\}\Big(sin\frac{\lambda_m x}{L} + A_m cos\frac{\lambda_m x}{L} \\
& + B_m sinh\frac{\lambda_m x}{L} + C_m cosh\frac{\lambda_m x}{L}\Big)
\end{aligned} \tag{27}
$$

where

$$P_{11} = \tfrac{1}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Cos\left(\tfrac{b_1^2}{4a} - c_0\right); \qquad P_{12} = \tfrac{1}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Sin\left(\tfrac{b_1^2}{4a} - c_0\right)$$

$$P_{13} = \tfrac{1}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Sin\left(\tfrac{b_2^2}{4a} - c_0\right); \qquad P_{14} = \tfrac{1}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Cos\left(\tfrac{b_2^2}{4a} - c_0\right)$$

$$P_{21} = \tfrac{A_m}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Cos\left(\tfrac{b_1^2}{4a} - c_0\right) \qquad P_{22} = \tfrac{A_m}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Sin\left(\tfrac{b_1^2}{4a} - c_0\right)$$

$$P_{23} = \tfrac{A_m}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Cos\left(\tfrac{b_2^2}{4a} - c_0\right); \qquad P_{24} = \tfrac{A_m}{2\sqrt{a}}\sqrt{\tfrac{\pi}{2}}Sin\left(\tfrac{b_2^2}{4a} - c_0\right)$$

$$Q_{11} = \tfrac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a} - c_0}e^{2c_0}; \qquad Q_{12} = \tfrac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_2^3}{4a} - c_0}e^{\frac{b_3^2}{2a}}$$

$$Q_{13} = \tfrac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a} - c_0}e^{2c_0}; \qquad Q_{14} = \tfrac{B_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_2^4}{4a} - c_0}e^{\frac{b_4^2}{2a}}$$

$$Q_{21} = \tfrac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_3^2}{4a} - c_0}e^{\frac{b_3^2}{2a}} \qquad Q_{22} = \tfrac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_3^2}{4a} - c_0}e^{2c_0}$$

$$Q_{23} = \tfrac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a} - c_0}e^{\frac{b_4^2}{2a}}; \qquad Q_{24} = \tfrac{C_m\sqrt{\pi}}{8\sqrt{a}}e^{-\frac{b_4^2}{4a} - c_0}e^{2c_0}$$

$$\begin{aligned} d_{10} = \frac{2a}{\sqrt{2\pi a}}, \quad d_{11} = \frac{b_1}{\sqrt{2\pi a}}, \quad d_{12} = \frac{b_2}{\sqrt{2\pi a}}, \quad d_{20} = \frac{2a}{2\sqrt{a}}, \\ d_{21} = \frac{b_3}{2\sqrt{a}}, \quad d_{22} = \frac{b_4}{2}\sqrt{a} \end{aligned} \tag{28}$$

$$\begin{aligned} F_1^* = {} & -P_{11}C(d_{11}) + P_{12}S(d_{11}) + P_{13}C(d_{12}) - P_{14}S(d_{12}) + P_{21}S(d_{11}) + \\ & P_{22}C(d_{11}) - P_{23}S(d_{12}) - P_{24}C(d_{12}) + iQ_{11}erfi(d_{21}) - iQ_{12}erf(d_{21}) \\ & - iQ_{13}erfi(d_{22}) + iQ_{14}erfi(d_{22}) + iQ_{21}erf(d_{21}) + iQ_{22}erfi(d_{21}) \\ & - iQ_{23}erf(d_{22}) - iQ_{24}erfi(d_{22}) \\ & + \frac{1}{\gamma_{bj}}\left[Cos\lambda_m - A_mSin\lambda_m - B_mCosh\lambda_m - C_mSinh\lambda_m\right] \end{aligned} \tag{29}$$

$$F_2^* = P_{13}S(d_{12}) + P_{14}C(d_{12}) + P_{11}S(d_{11}) + P_{12}C(d_{11}) + P_{21}C(d_{11}) -$$
$$P_{22}S(d_{11}) - P_{23}C(d_{12}) - P_{24}S(d_{12}) + Q_{12}erf(d_{21}) - Q_{12}erfI(d_{21}) -$$
$$Q_{13}erfi(d_{22}) - Q_{14}erfi(d_{22}) + Q_{21}erf(d_{21})$$
$$Q_{23}erf(d_{22}) - iQ_{24}erfi(d_{22}) \tag{30}$$

Equation (27) represents the transverse displacement response to forces moving at variable velocities of a prestressed non-uniform Rayleigh beam resting on Vlasov elastic foundation and having arbitrary end support conditions.

(b) Moving Mass

If the moving load has mass commensurable with that of the structure, the inertia effect of the heavy load is not negligible. Thus,$\Gamma_0 \neq 0$, and we are required to solve the entire equation (20). This we term the moving mass problem. Unlike in (a), it is obvious that an exact analytical solution to this equation is not possible. Thus, we resort to an approximate analytical method discussed in [1]. It is a modification of the asymptotic method due to Struble. To this end, we rearrange equation (20) to take the form

$$\ddot{Z_m}(t) + \frac{2(c+at)\Gamma_0 R_2(k,m,t)}{1+\Gamma_0 R_1(k,m,t)} Z_m(t) + \frac{\gamma_{MF}^2 (c+at)^2 \Gamma_0 R_3(k,m,t)}{1+\Gamma_0 R_1(k,m,t)} Z_m(t) =$$

$$\frac{\Gamma_0 L^2 g T_{PM}}{\lambda_k \Omega_1(k,m)\Big[1+\Gamma_0 R_1(k,m,t)\Big]} \tag{31}$$

where

$$R_1(k,m,t) = D_{1a}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty} T_A(n,t)D_{1b}(k,m,n) - \frac{1}{\pi}\sum_{n=0}^{\infty} T_B(n,t)D_{1c}(k,m,n)$$
$$R_2(k,m,t) = D_{2a}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty} T_A(n,t)D_{2b}(k,m,n) - \frac{1}{\pi}\sum_{n=0}^{\infty} T_B(n,t)D_{2c}(k,m,n)$$
$$R_3(k,m,t) = D_{3a}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty} T_A(n,t)D_{3b}(k,m,n) - \frac{1}{\pi}\sum_{n=0}^{\infty} T_B(n,t)D_{3c}(k,m,n)$$
$$R_4(k,m,t) = D_{4a}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty} T_A(n,t)D_{4b}(k,m,n) - \frac{1}{\pi}\sum_{n=0}^{\infty} T_B(n,t)D_{4c}(k,m,n)$$
$$D_{1a}(k,m) = \frac{LD_{1A}(k,m)}{\Omega_1(k,m)}, \quad D_{1B}(k,m,n) = \frac{LD_{1B}(k,m,n)}{\Omega_1(k,m)},$$
$$D_{1C}(k,m,n) = \frac{LD_{1C}(k,m,n)}{\Omega_1(k,m)}, \quad D_{2a}(k,m) = \frac{LD_{2A}(k,m)}{\Omega_1(k,m)},$$
$$D_{2B}(k,m,n) = \frac{LD_{2B}(k,m,n)}{\Omega_1(k,m)}, \quad D_{2C}(k,m,n) = \frac{LD_{2C}(k,m,n)}{\Omega_1(k,m)}$$
$$D_{3a}(k,m) = \frac{LD_{3A}(k,m)}{\Omega_1(k,m)}, \quad D_{3B}(k,m,n) = \frac{LD_{3B}(k,m,n)}{\Omega_1(k,m)},$$
$$D_{3C}(k,m,n) = \frac{LD_{3C}(k,m,n)}{\Omega_1(k,m)}, \quad D_{4a}(k,m) = \frac{LD_{4A}(k,m)}{\Omega_1(k,m)},$$
$$D_{4B}(k,m,n) = \frac{LD_{4B}(k,m,n)}{\Omega_1(k,m)}, \quad D_{4C}(k,m,n) = \frac{LD_{4C}(k,m,n)}{\Omega_1(k,m)} \tag{32}$$

By means of this technique, one seeks the modified frequency corresponding to the frequency of the free system due to the presence of the distributed moving mass. An equivalent free system operator defined by the modified frequency then replaces equation (31). Thus, we set the right-hand side of equation (31) to zero and consider a parameter $\Gamma_1 < 1$ for any arbitrary ratio , defined as

$$\Gamma_1 = \frac{\Gamma_0}{1+\Gamma_0} \tag{33}$$

Evidently,

$$\Gamma_0 = \Gamma_1 + O(\Gamma_1^2) \tag{34}$$

which implies

$$\frac{1}{1+\Gamma_1 R_1(k,m,t)} = 1 - \Gamma_1 R_1(k,m,t) + O\Big(\Gamma_1^2\Big) \tag{35}$$

where

$$\left|\Gamma_1\Big(D_{1a}(k,m) + \frac{1}{\pi}\sum_{n=0}^{\infty} T_A(n,t)D_{1b}(k,m,n)\frac{1}{\pi}\sum_{n=0}^{\infty} T_A(n,t)D_{1b}(k,m,n)\Big) < 1\right| \tag{36}$$

when we set $\Gamma_1 = 0$ is a case corresponding to the case when the inertia effect of the mass of the system is negligible is obtained and the solution of (31) is of the form

$$Z_m(t) = \Lambda \cos\Big[\sigma_{MF}t - \phi(m,t)\Big] \tag{37}$$

where Λ, $\sigma_{MF}t$ are constants and $\phi(m,t)$ is as previously defined. However, since $\Gamma_1 < 1$, Struble's technique requires that the asytmptotic solution of the homogeneous part of equation (31) be of the form

$$Z_m(t) = \psi(m,t)Cos[\sigma_{MF}t - \phi(m,t)] + \Gamma_1 Z_1(t) + O(\Gamma_1^2) \tag{38}$$

where $\psi(m,t)$ and $\phi(m,t)$ are slowly varying functions of time. Substituting (38) and its derivatives into the homogeneous part of equation (31) one obtains

$$\begin{aligned}
&-2\psi(\dot{m},t)\sigma_{MF}sin\Big[\sigma_{MF}t-\phi(m,t)\Big]+2\psi(m,t)\sigma_{MF}\phi(\dot{m},t)cos\Big[\sigma_{MF}t-\phi(m,t)\Big]\\
&-2\Gamma(c+at)\Big[D_{2a}(m,k)\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{2b}(k,m,n)\\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{2c}(k,m,n)\Big]\Big[\psi(m,t)\sigma_{MF}sin\Big[\sigma_{MF}t-\phi(m,t)\Big]\Big]\\
&+\Big[(c+at)^2\Gamma_1\Big\{\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{3b}(k,m,n)\\
&+D_{3a}(k,m)-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{3c}(k,m,n)\Big\}+a\Gamma_1\Big\{D_{4a}(k,m)\\
&+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{4b}(k,m,n)\\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{4c}(k,m,n)\Big\}\\
&-\sigma_{MF}^2\Gamma_1\Big\{D_{1a}(k,m)+\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{1b}(k,m,n)\\
&-\frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1}D_{1c}(k,m,n)\Big\}\Big]\Big(\psi(m,t)cos\Big[\sigma_{MF}t\Big]-\phi(m,t)\Big)=0
\end{aligned} \tag{39}$$

retaining terms to $O(\Gamma_1)$ only.
In order to obtain the modified frequency, we extract the variational part of the equation describing the behavior of $\psi(m,t)$ and $\phi(m,t)$ during the motion of the mass. Thus, noting the following

trigonometric identities

$$
\begin{aligned}
&\frac{cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} cos\Big[\sigma_{MF}t-\phi(m,t)\Big] \\
&= \frac{1}{2(2n+1)}\Big\{cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2) \\
&+\ \sigma_{MF}t-\phi(m,t)) + cos((2n+1)\pi(x_0+ct+\frac{1}{2}at^2)-\sigma_{MF}t+\phi(m,t))\Big\} \\
&\frac{sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} cos\Big[\sigma_{MF}t-\phi(m,t)\Big] \\
&= \frac{1}{2(2n+1)}\Big\{sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2) \\
&+\ \sigma_{MF}t-\phi(m,t)) + sin((2n+1)\pi(x_0+ct+\frac{1}{2}at^2)-\sigma_{MF}t+\phi(m,t))\Big\} \\
&\frac{cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} sin\Big[\sigma_{MF}t-\phi(m,t)\Big] \\
&= \frac{1}{2(2n+1)}\Big\{sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2) \\
&+\ \sigma_{MF}t-\phi(m,t)) + sin((2n+1)\pi(x_0+ct+\frac{1}{2}at^2)-\sigma_{MF}t+\phi(m,t))\Big\} \\
&\frac{sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{2n+1} sin\Big[\sigma_{MF}t-\phi(m,t)\Big] \\
&= \frac{1}{2(2n+1)}\Big\{cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2) \\
&+\ \sigma_{MF}t-\phi(m,t)) + cos((2n+1)\pi(x_0+ct+\frac{1}{2}at^2)-\sigma_{MF}t+\phi(m,t))\Big\}
\end{aligned} \tag{40}
$$

and neglecting terms that do not contribute to the variational equations, equation (39) reduces to

$$
\begin{aligned}
&-2\dot{\psi}(m,t)\sigma_{MF}\sin[\sigma_{pp}t-\phi(m,t)] + 2\psi(m,t)\sigma_{MF}\phi(m,t)\cos[\sigma_{MF}t-\phi(m,t)] \\
&-2c\Gamma_1\sigma_{MF}t\psi(m,t)D_{2a}(k,m)\sin[\sigma_{MF}t-\phi(m,t)] + \big[c^2\Gamma_1 D_{3a}(m,t) \\
&+a\Gamma_1 D_{4a}(k,m) - \sigma_{MF}^2\Gamma_1 D_{1a}(k,m)]\psi(m,t)\sigma_{MF}\cos[\sigma_{MF}t-\phi(m,t)] = 0
\end{aligned} \tag{41}
$$

However,the variational equations of the problem are obtained by setting coefficients of $\sin[\sigma_{MF}t-\phi(m,t)]$ and $\cos[\sigma_{MF}t-\phi(m,t)]$ in equation (41) to zero.

Thus, we obtain

$$\Big[2\dot{\psi}(m,t)+2c\Gamma_1\psi(m,t)D_{2a}(k,m)\Big]sin\Big[\sigma_{MF}t-\phi(m,t)\Big] \tag{42}$$

and

$$\Big[2\sigma_{MF}\dot{\phi}(m,t)+\Gamma_1(c^2D_{3a}(k,m)+aD_{4a}(k,m)-\sigma_{MF}^2D_{1a}(k,m))\Big]\psi(m,t) \tag{43}$$

Solving equations (42) and (43), we have

$$\psi(m,t)=A_0\mathbf{e}^{-\Gamma_1cD_2a(k,m)t} \tag{44}$$

and

$$\phi(m,t)=\left[\frac{\sigma_{MF}^2D_{1a}(k,m)-c^2D_{3a}(k,m)-aD_{4a}(k,m)}{2\sigma_{MF}}\right]\Gamma_1t+\theta_m \tag{45}$$

where A_0 and θ_m are constants.
Solving equations (42) and (43) respectively, we have

$$Z_m(t)=A_0\mathbf{e}^{-\Gamma_1cD_2a(k,m)t}\cos[\sigma_{MF}t-\theta_m] \tag{46}$$

where

$$\sigma_{MM}=\sigma_{MF}\left\{1-\frac{\Gamma_1}{2}\left[D_{1a}(k,m)-\frac{c^2D_{3a}(k,m)+aD_{4a}(k,m)}{\sigma_{MF}^2}\right]\right\} \tag{47}$$

is called the modified frequency representing the frequency of the free system due to the presence of moving mass. Thus, the homogeneous part of equation (31) can be written as

$$\frac{\mathrm{d}^2Z_m(t)}{\mathrm{d}t^2}+\sigma_{MM}^2Z_m(t)=0 \tag{48}$$

Consequently, the entire equation (31) reduces to

$$\begin{aligned}\frac{\mathrm{d}^2Z_m(t)}{\mathrm{d}t^2}+\sigma_{MM}^2Z_m(t)&=\frac{\Gamma_1L^2g}{k\pi\Omega_1(k,m)}[-\cos\lambda_k+A_k\sin\lambda_k+B_k\cosh\lambda_k\\&+C_ksinh\lambda_k+\cos\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2)-A_k\sin\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2)\\&-B_k\cosh\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2)-C_k\sinh\frac{\lambda_k}{L}(x_0+ct+\frac{1}{2}at^2)]\end{aligned} \tag{49}$$

Solving equation (49) in conjunction with the initial conditions, one obtains the expression for$Z_m(t)$. Thus, in view of (10)

$$V_n(x,t)=\sum_{m=1}^{\infty}\frac{\Gamma_1 Lg\sqrt{\pi}}{\lambda_k\varphi_{11}\sqrt{2a}\tau(x)}\Bigg\{\frac{\sin\Omega_{mm}t}{\Omega_{mm}}\Bigg[\cos\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)$$

$$+\sin\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)$$

$$+\sin\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right)$$

$$-\sin\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2}{\sqrt{2\pi a}}\right)$$

$$-\sin\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2}{\sqrt{2\pi a}}\right)-\frac{1}{2\Omega_{mm}}\Big[\sin(\lambda_k+\Omega_{mm}t)-\sin(\lambda_k-\Omega_{mm}t)\Big]\Bigg]$$

$$-\frac{\cos\Omega_{mm}t}{\Omega_{mm}}\Bigg[\cos\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)$$

$$-\sin\left[\frac{b_1^2}{4a}-C_0\right]C\left(\frac{b_1+2at}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)$$

$$+\sin\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2+2at}{\sqrt{2\pi a}}\right)-\cos\left(\frac{b_1^2}{4a}-C_0\right)S\left(\frac{b_1}{\sqrt{2\pi a}}\right)$$

$$+\sin\left(\frac{b_1^2}{4a}-C_0\right)C\left(\frac{b_1}{\sqrt{2\pi a}}\right)+\cos\left(\frac{b_2^2}{4a}-C_0\right)S\left(\frac{b_2}{\sqrt{2\pi a}}\right)$$

$$-\sin\left(\frac{b_2^2}{4a}-C_0\right)C\left(\frac{b_2}{\sqrt{2\pi a}}\right)-\frac{\cos\lambda_k}{\Omega_{mm}}+\frac{1}{2\Omega_{mm}}$$

$$\Big[\cos(\lambda_k-\Omega_{mm}t)+\cos(\lambda_k\Omega_{mm}t)\Big]\Bigg]\Bigg\}\left(\sin\frac{m\pi x}{L}\right) \tag{50}$$

which represents the transverse displacement response to distributed masses, moving with non-uniform velocity of a prestressed Rayleigh beam resting on elastic foundation having arbitrary end support conditions. By way of illustrating the foregoing analysis, we consider in succession (i) Clamped-Clamped end conditions and (ii) Cantilever end conditions noting that simply supported end conditions have been treated in an earlier paper [1] whose results are in agreement with results obtained using the methods in this paper.

(i) Non-uniform Rayleigh beam with Clamped-Clamped end conditions
At the clamped end, both deflection and slope vanish at the boundary. Thus,

$$V(0,t) = 0 = V(L,t), \qquad \frac{\partial V(0,t)}{\partial x} = 0 = \frac{\partial V(L,t)}{\partial x} \tag{51}$$

and for normal modes

$$U_m(0) = 0 = U_m(L), \qquad \frac{\partial U_m(0)}{\partial x} = 0 = \frac{\partial U_m(L)}{\partial x} \tag{52}$$

which implies that

$$U_k(0) = 0 = U_k(L), \qquad \frac{\partial U_k(0)}{\partial x} = 0 = \frac{\partial U_k(L)}{\partial x} \tag{53}$$

Thus, it can be shown that

$$A_m = \frac{\sinh \lambda_m - \sin \lambda_m}{\cos \lambda_m - \cosh \lambda_m} = \frac{\cos \lambda_m - \cosh \lambda_m}{\sin \lambda_m + \sinh \lambda_m} = -C_m \quad \textit{and} \quad B_m = -1 \tag{54}$$

In view of (54), the frequency equation is given as

$$\cos \lambda_m \cosh \lambda_m = 1 \tag{55}$$

It follows from equation (55), that

$$\lambda_1 = 4.73004, \qquad \lambda_2 = 7.85320, \qquad \lambda_3 = 10.99561 \tag{56}$$

The expression for A_k, B_k, C_k and the corresponding frequency equation are obtained by a simple replacement of m and k in equation (54) and (55). Substituting (54) and (55) into equations (29) and (50) respectively to obtain the displacement response to a moving force and moving mass respectively of a clamped-clamped uniform Rayleigh beam resting on a Pasternak elastic foundation.

(ii) Non-uniform Rayleigh beam with Cantilever end end conditions
In this case the beam type structure is clamped at one end and free at the other end. Accordingly, the boundary conditions are

$$V(0,t) = 0 = \frac{\partial V(0,t)}{\partial x}, \qquad \frac{\partial^2 V(0,t)}{\partial x^2} = 0 = \frac{\partial^3 V(L,t)}{\partial x^3} \tag{57}$$

and for normal modes

$$U_m(0) = 0 = \frac{\partial U_m(0)}{\partial x}, \qquad \frac{\partial^2 U_m(L)}{\partial x^2} = 0 = \frac{\partial^3 U_m(L)}{\partial x^3} \tag{58}$$

which implies that

$$U_m(0) = 0 = \frac{\partial U_k(0)}{\partial x}, \qquad \frac{\partial^2 U_k(L)}{\partial x^2} = 0 = \frac{\partial^3 U_k(L)}{\partial x^3} \tag{59}$$

It can be shown that

$$A_m = \frac{-\sin\lambda_m - \sinh\lambda_m}{\cos\lambda_m + \cosh\lambda_m} = \frac{-\cos\lambda_m - \cosh\lambda_m}{\sinh\lambda_m - \sin\lambda_m} = -C_m \quad and \quad B_m = -1 \tag{60}$$

and the frequency equation for both end conditions is

$$\cos\lambda_m \cosh\lambda_m = 1 \tag{61}$$

and we have that

$$\lambda_1 = 1.875, \qquad \lambda_2 = 4.694, \qquad \lambda_3 = 7.855 \tag{62}$$

using (60), (61) and (62) in equations (54) and (55), we obtain the transverse displacement response respectively to a moving force and moving mass of a cantilever Rayleigh beam with a non-uniform cross section resting on a Pasternak elastic foundation.

4. DISCUSSION OF CLOSE FORM SOLUTION

The response amplitude of a dynamical system such as this may grow without bound. Conditions under which this happens are termed resonance conditions. For both illustrative examples, we observe that the non-uniform Rayleigh beam traversed by a moving partially distributed force at variable velocity reaches a state of resonance whenever

$$\Omega_{MF} = \frac{m\pi c_c}{L} \tag{63}$$

while the same non-uniform beam under the action of a moving partially distributed mass experiences resonance effect when

$$\Omega_{MM} = \frac{m\pi c_c}{L} \tag{64}$$

Evidently,

$$\Omega_{MM} = \Omega_{MF}\left\{1 - \frac{\Gamma_1}{2}\left[D_1 a(k,m) - \frac{\left[c^2 D_3 a(k,m) + a D_4 a(k,m)\right]}{\Omega_{MF}^2}\right]\right\} \tag{65}$$

Equations (64) and (65) show that for the same natural frequency, the critical speed for the system consisting of a non-uniform Rayleigh beam resting on an elastic foundation and traversed by partially distributed force is greater than that traversed by partially distributed mass. Thus, resonance is reached earlier in the moving distributed mass system than in moving distributed force system.

NUMERICAL CALCULATIONS AND DISCUSSIONS

In this section, calculations of practical interests in the field of structural dynamics are presented for both illustrative examples considered. A non-prismatic Rayleigh beam with length $12.192m$ has been considered. The mass is assumed to travel at initial velocity $8.128m/s$. Furthermore E, I and μ are chosen to be $3.1 \times 10^10N/m^2$, $2.87698 \times 10^-3m^4$ and $2758.291kg/m$ respectively. The values of the foundation stiffness K varied between $0N/m^3$ and $4000000N/m^3$, axial force$N$ is varied between $0N$ and 2×10^6N and shear modulus$G$ is varied between $0N/m$ and $3 \times 10^5N/m$. The transverse deflections of Rayleigh beam are calculated and plotted against time for various values of foundation stiffness K, axial force N, shear modulus G and rotatory inertia R^0. In figure 5.1, the deflection profile of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed forces moving at variable velocity for various values of foundation stiffness$K$ and fixed values of axial force N shear modulus G and rotatory inertia correction factor R^0 is displayed. The figure shows that as K increases, the transverse displacement of the non-uniform Rayleigh beam decreases. For various travelling time t, the transverse displacement of the beam for various values of axial force N and for fixed values of foundation stiffness K, shear modulus G and rotatory inertia correction factor R^0 are shown in figure 5.2. It is observed that higher values of axial force N reduce the deflection of the beam. Also figure 5.3 displays the response amplitudes of the clamped-clamped non-uniform Rayleigh beam to partially distributed forces travelling at variable velocity for various values of shear modulus G and for fixed values of foundation stiffness K, axial force N and rotatory inertia correction factor R^0. It is seen from the figure that as the values of shear modulus increases, the response amplitude of the simply supported non-uniform Rayleigh beam under the action of partially distributed forces travelling at variable velocity decreases. In figure 5.4 the deflection profile of clamped-clamped uniform Rayleigh beam under the action of partially distributed forces is displayed. It is clearly shown that as we increase the values of rotatory inertia correction factor R^0, for fixed values of foundation stiffness K, axial force N and shear modulus G the deflection of the non-uniform beam decreases. In figure 5.5, the transverse displacement of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses for various values of foundation stiffness$K$ and fixed values of axial force N shear modulus G and rotatory inertia correction factor R^0 is displayed. Results and analyses are the same with the cases in figure 5.1. The deflection profile of the clamped-clamped beam under the action of partially distributed masses moving at variable velocity for various values of axial force N is shown in figure 5.6. Results and analyses obtained are similar to the ones in figure 5.2. Furthermore, figure 5.7 shows that higher values of shear modulus G increase the displacement amplitudes of the clamped-clamped Rayleigh beam for fixed values of foundation stiffness K, axial force N and rotatory inertia correction factor R^0. For the same clamped-clamped beam traversed by non-uniform partially distributed masses, figure 5.8 depicts that as the values of rotatory inertia R^0 increases, the deflection of the beam reduces for fixed values of foundation stiffness K, axial force and shear modulus G.

Figure 5.9 displays the comparison of the transverse displacement response of moving force and moving mass cases of the clamped-clamped non-uniform Rayleigh beam traversed by a moving load travelling at variable velocity for fixed values of $K = 400000$, $N = 200000$, $G = 100000$ and $R^0 = 0.5$. It is shown that the moving distributed force deflection is higher than that of the moving distributed mass.In figure 5.10, the deflection profile of a cantilever non-uniform Rayleigh beam under the action of partially distributed forces moving at variable velocity for various values of foundation stiffness K and for fixed values of axial force N, shear modulus G and rotatory inertia correction factor R^0 is displayed. The figure shows that as K increases, the deflection of the beam decreases. Also for various travelling time t, the deflection profile of the beam for various values of axial force N and for fixed values of foundation stiffness K, shear modulus G and rotatory inertia correction factor R^0 are shown in figure 5.11. It is observed that higher values of axial force reduce the deflection profile of the beam. Figure 5.12 displays the displacement response of the cantilever non-uniform Rayleigh beam to partially distributed forces travelling at variable velocity for various values of shear modulus G and for fixed values of foundation stiffness K, axial force N and rotatory inertia correction factor R^0. It was observed that increase in the shear modulus reduces the transverse displacement of the beam. The transverse displacement response of a clamped-free non-uniform Rayleigh beam under the action of partially distributed forces moving at variable velocity for various values of rotatory inertia correction factor R^0 and fixed values of foundation stiffness K, axial force N and shear modulus G is displayed in figure 5.13.

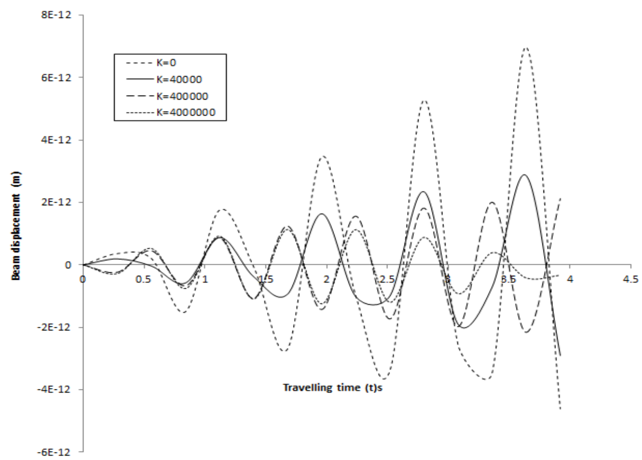

Fig. 5.1. Transverse displacement of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of foundation stiffness K

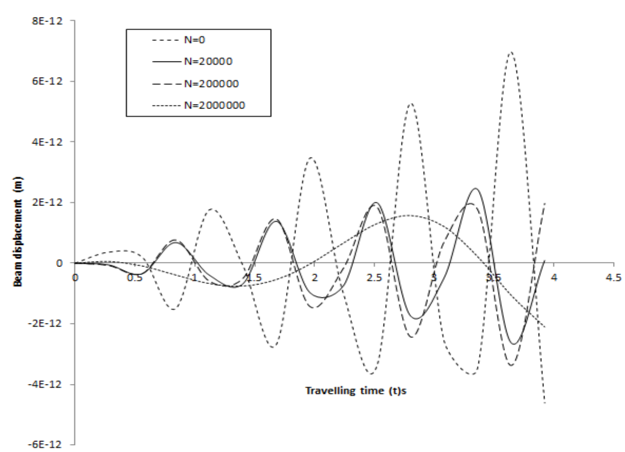

Fig. 5.2. Deflection profile of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of Axial force N

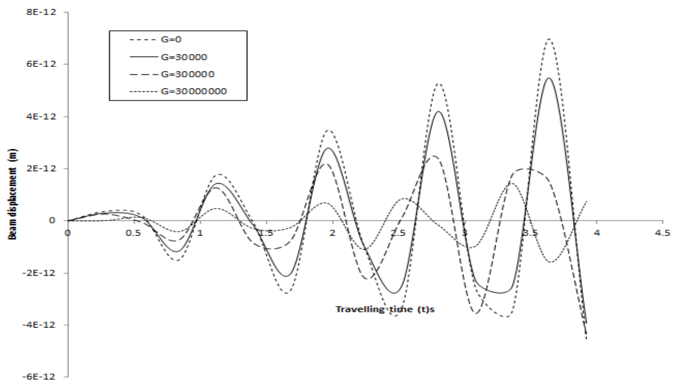

Fig. 5.3. Transverse displacement of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of shear modulus G

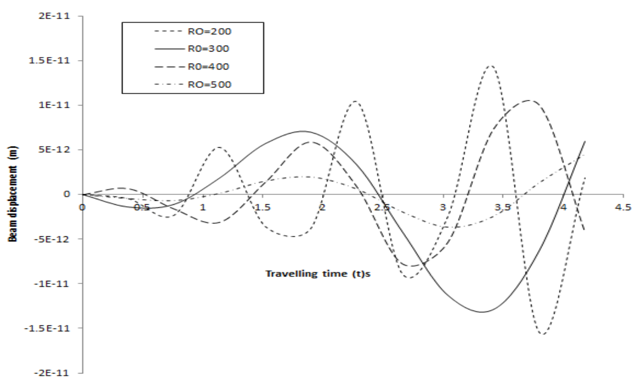

Fig. 5.4. Response amplitude of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of rotatory inertia correction factor R^0

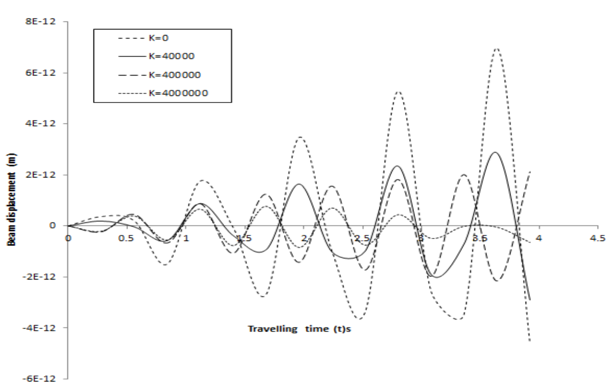

Fig. 5.5. Displacement response of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses

traveling at a variable velocity for various values of foundation stiffness K

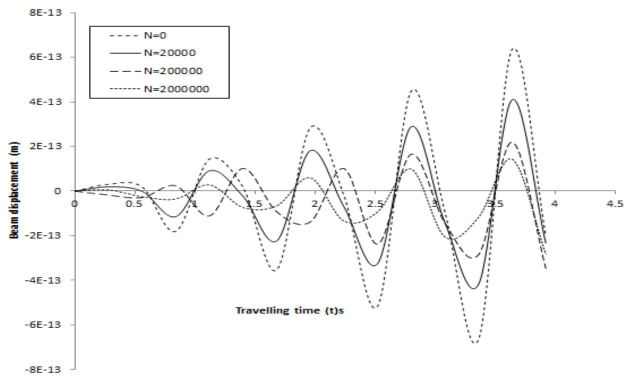

Fig. 5.6. Deflection profile of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of Axial force N

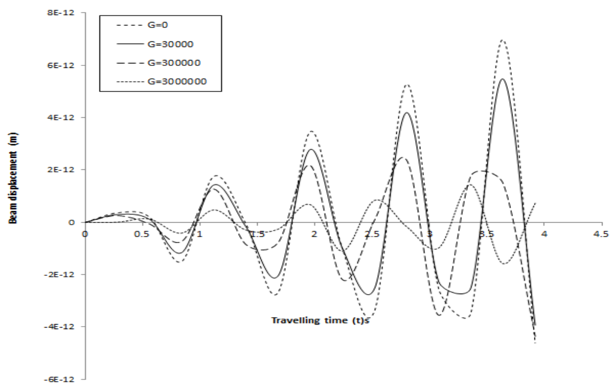

Fig. 5.7. Transverse displacement of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of shear modulus G

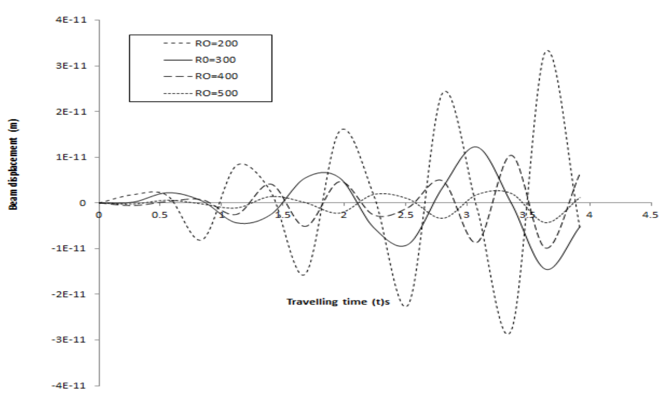

Fig. 5.8. Response amplitude of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of rotatory inertia correction factor R^0

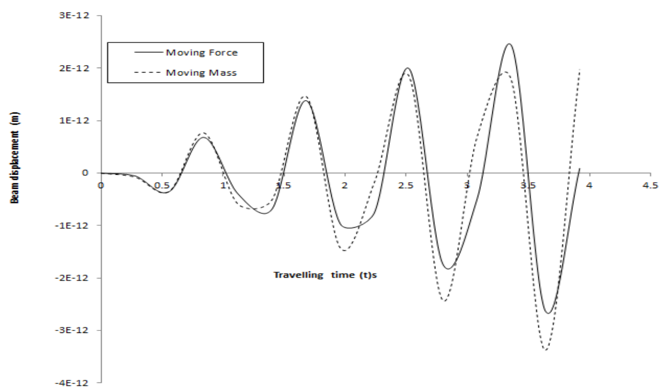

Fig. 5.9. Comparison of the displacement response of moving force and moving mass cases for a uniform clamped-clamped Rayleigh beam for fixed value of K=400000, N=200000, G=100000, R^0=0.5

The figure shows that as R^0 increases, the dynamic deflection of the beam decreases. Figure 5.14 shows the transverse displacement of cantilever non-uniform Rayleigh beam under the action of partially distributed masses moving at variable velocity for various values of foundation stiffness K and for fixed values of axial force N, shear modulus G and rotatory inertia correction factor R^0. Results and analyses similar to figure 5.10 are obtained. Also the response amplitude of the same non-uniform Rayleigh beam under the action of partially distributed masses for various values of axial force N and fixed values of foundation stiffness K, shear modulus G and rotatory inertia correction factor R^0 is displayed in figure 5.15. Results and analyses similar those of moving force in figure 5.11 are obtained. The response amplitude of the cantilever beam under the action of partially distributed masses moving at variable velocity for various values of shear modulus G is shown in figure 5.16. Results obtained are similar to those in figure 5.12. Figure 5.17 shows that higher values of rotatory inertia correction factor R0 increase the deflection profile of the cantilever beam under the action of partially distributed masses moving at variable velocity for fixed values of foundation stiffness K, axial force N and shear modulus G.

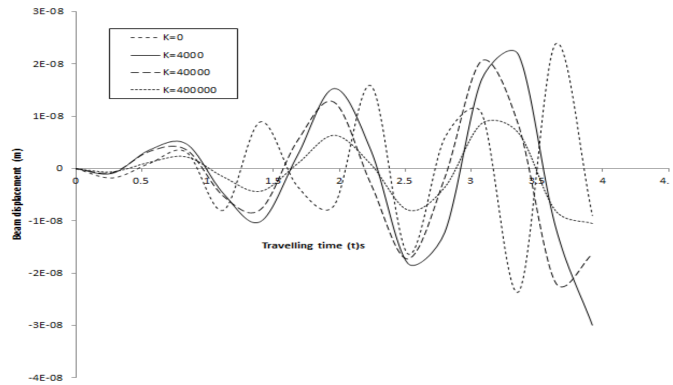

Fig. 5.10. Transverse displacement of a cantilever non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of foundation stiffness K

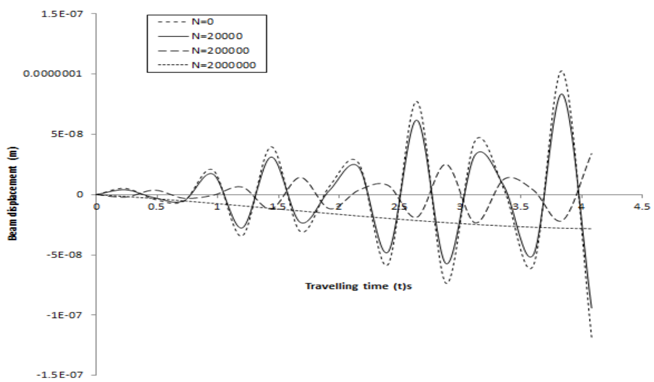

Fig. 5.11. Deflection profile of a cantilever non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of Axial force N

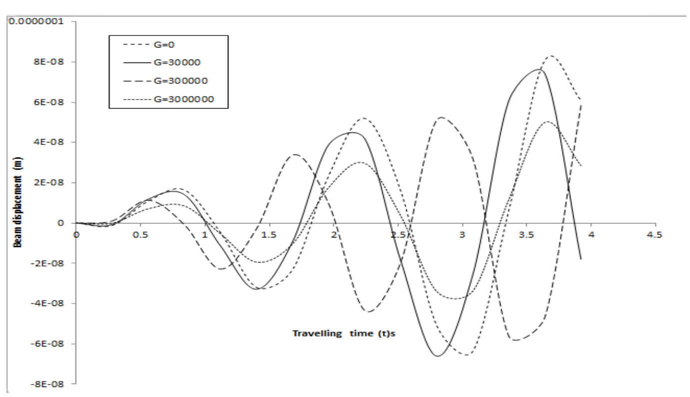

Fig. 5.12. Transverse displacement of a cantilever non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of shear modulus G

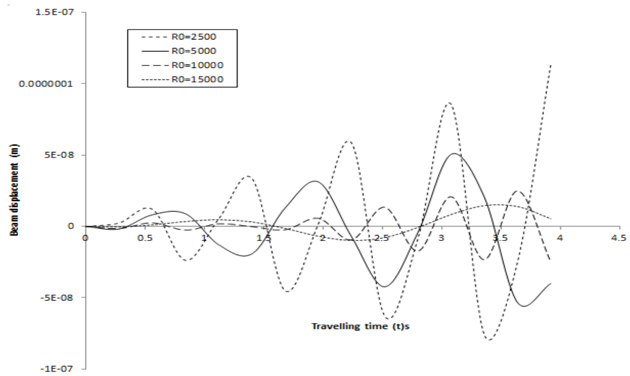

Fig. 5.13. Response amplitude of a cantilever non-uniform Rayleigh beam under the action of partially distributed forces traveling at a variable velocity for various values of rotatory inertia correction factor R^0

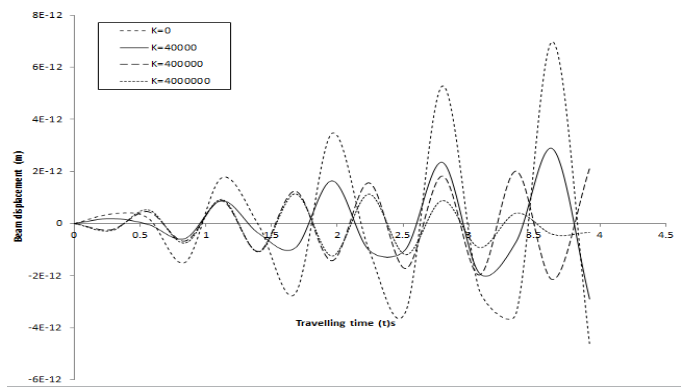

Fig. 5.14. Displacement response of a cantilever non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of foundation stiffness K

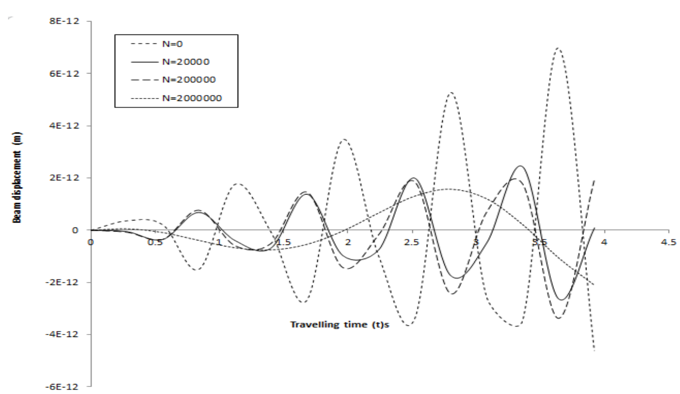

Fig. 5.15. Deflection profile of a cantilever non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of Axial force N

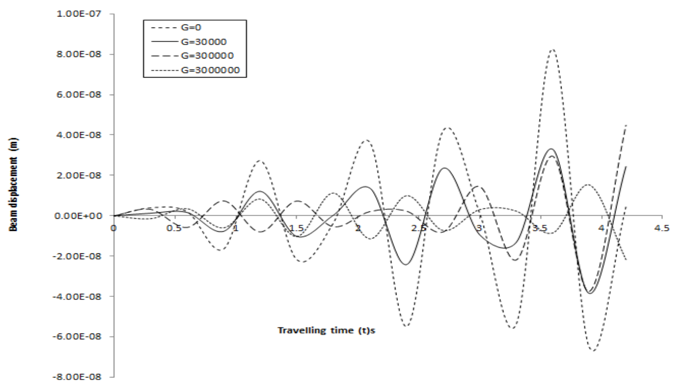

Fig. 5.16. Transverse displacement of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of shear modulus G

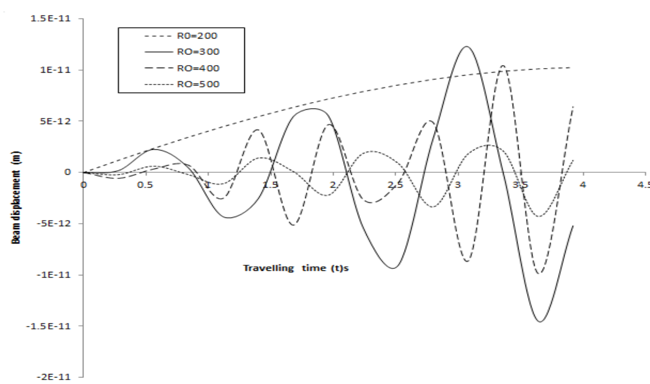

Fig. 5.17. Response amplitude of a clamped-clamped non-uniform Rayleigh beam under the action of partially distributed masses traveling at a variable velocity for various values of rotatory inertia correction factor R^0

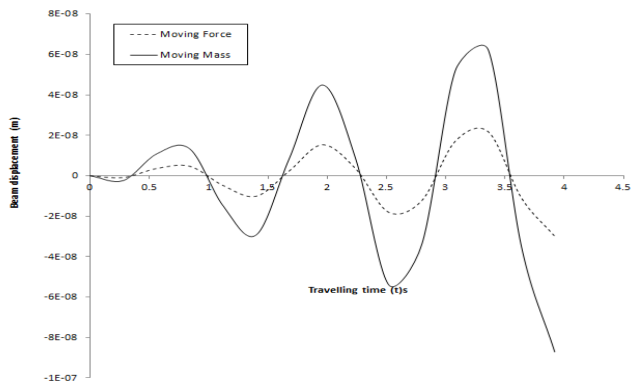

Fig. 5.18. Comparison of the displacement response of moving force and moving mass cases for a uniform clamped-clamped Rayleigh beam for fixed value of K=400000, N=200000, G=100000, R^0=0.5

Finally, figure 5.18 depicts the comparison of the transverse displacement response of moving force and moving mass of a cantilever-free Rayleigh beam traversed by a moving load travelling at variable velocity for fixed values of $K = 400000$, $N = 200000$, $G = 100000$ and $R^0 = 0.5$. It is clearly shown that moving mass deflection is higher than that of the moving force showing that moving force deflection is not always the upper bound of the deflection of the dynamical system.

CONCLUDING REMARKS

This paper presents an analytical solution for the transverse displacement of a non-uniform Rayleigh beam on a bi-parametric subgrade and under partially distributed masses moving at varying velocities. The versatile method of Galerkin has been used to reduce the governing fourth order singular partial differential equation with variable coefficients to a sequence of second order ordinary differential equations with variable coefficients. These equations are treated using a modification of Strubles asymptotic techniques. The resulting second order ordinary differential equation is solved using the method of integral transformations. Numerical analyses were carried out and the results show the following interesting features:

(i) for the moving distributed force and moving distributed mass problems the response amplitudes of the beam traversed by distributed load moving with variable velocity decrease with an increase in the values of foundation stiffness K for fixed values of N, G and R^0.

(ii) higher values of axial force N reduce the response amplitudes for both the moving force and moving mass problems.

(iii) greater values of the subgrade's shear modulus G and rotatory inertia r^0 for fixed values of foundation stiffness K, axial force N and shear modulus G are required for a noticeable effect on the response amplitudes due to moving force and moving mass in the vibrating system.

(iv) the response amplitudes of the Rayleigh beam decrease with an increase in the values of shear modulus G for fixed values of K, N and R^0.

(v) as K increases, the response amplitude of the non-uniform Rayleigh beams decreases. However the effect of K is more noticeable than that of G.

(vi) for the non-uniform beam problem under the actions of a partially distributed load moving with variable velocity, the transverse displacement of the moving force is not always greater than that of moving mass. This has previously been reported in literature [4] and therefore, inertia of the moving load must always be taken into consideration for accurate and safe assessment of the response to moving distributed load of elastic structural members.

Finally, for this dynamical system, for the same natural frequency, the critical speed for moving mass problem is greater/smaller than that of the moving force problem. Hence, resonance is reached earlier in moving mass problem.

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REFERENCES

[1] S.T. Oni and O.K. Ogunbamike, *Dynamic behaviour of non-prismatic beam on Pasternak foundation and under partially distributed masses moving at varying velocities*, Journal of the Nigerian Mathematical Society **33** 285-310, 2014.

[2] L, Fryba, *Vibration of solids and structures under moving loads.*, Groningen: Noordhoff, 1972.

[3] S.T. Oni and O.K. Ogunbamike, *Flexural motions of a uniform beam under the actions of a concentrated mass travelling with variable velocity*, Abacus, Journal of Mathematical Association of Nigeria **31** (2a) 79-93, 2004.

[4] E.A. Andi and S.T. Oni, *Dynamic analysis under uniformly distributed moving masses of rectangular plate with general boundary conditions*, Abacus, Journal of Computational Engineering. http//dx.doi.org/10.1155/2014/140407

[5] S.T. Oni and T.O. Awodola, *Dynamic behaviour under moving concentrated masses of elastically supported finite Bernoulli-Euler beam on Winkler elastic foundation*, Latin American Journal of Solids and Structures **7** 2-20, 2010.

[6] B. Omolofe and O.K. Ogunbamike, *Influence of some vital structural parameters on the dynamic characteristics of axially prestressed beam under moving masses*, International Journal of Engineering Research and Technology **3** (1) 816-828, 2014.

[7] T.A. Alaa, *Simulation of transient dynamic loads on beams with variable velocities*, European Journal of Scientific Research **62** (4) 481-490, 2011.

[8] H.A. Isede and J.A. Gbadeyan, *On the dynamic analysis of a tapered Timoshenko beam under a uniform partially distributed moving load*, Journal of the Nigerian Mathematical Society **32** 109-141, 2013.

[9] S.M. Abdelghany, K.M. Ewis, A.A. Mamoud and M.M. Nassar, *Dynamic response of non-uniform beam subjected to moving load and resting on non-linear viscoelastic foundation*, Beni-Suef University Journal of Basic and Applied Sciences **4** (3) 192-199, 2015.

[10] H.H. Jeffcott, *On the vibration of beams under the action of moving loads*, Philosophy mag **48** (7, 8) 66-67.

[11] S. Sadiku and H.H.E. Leipholz, *On the dynamics of elastic systems with moving concentrated masses*, Ingenieur Achieves **57** 223-242, 1989.

[12] M. Milomir, M.M. Stanisic and J.C. Hardin, *On the response of beams to an arbitrary number of concentrated moving masses*, Journal of the Franklin Institute **287** (2) 115-123, 1969.

[13] B. Omolofe and S.T. Oni, *On the response of beams to an arbitrary number of concentrated moving massesTransverse motions of rectangular plate resting on elastic foundation and under concentrated masses at varying velocities*, Latin American journal of Solids and structures **12** (7), July, 2015. http://dx.doi.org/10.1590/1679-78251429.

[14] E. Esmailzadeh and M. Ghorashi, *Vibration analysis of beams traverse by uniform partially distributed moving masses*, Journal of Sound and Vibration **184** (1) 9-17, 1995.

[15] E. Esmailzadeh and M. Ghorashi, *Vibration analysis of Timoshenko beams subjected to traveling mass*, Journal of Sound and Vibration **199** (4) 615-628, 1997.

[16] S.T. Oni and S.N. Ogunyebi, *Dynamical Analysis of a prestressed elastic beam with general boundary conditions under the action of uniform distributed masses*, In proceeding of conference of Nigerian Mathematical Society, Akure, Nigeria. May 15th -19th, 2007.

[17] A.D. Kerr, *Elastic and Viscoelastic foundation models*, Journal of Applied Mechanics transactions of the ASME **33** 491-498, 1964.

[18] P.L. Pasternak, *On a new method analysis of an elastic foundation by means of two foundation constant (in Russian)*, Journal of the Franklin Institute **287** (2) 115-123, 1969.Gosudastrennve Izdatellstve Literaturi postroetelstrvvi Arkhiteektre, Moscow.1954.

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