

**EFFECT OF PERTURBATION IN THE CORIOLIS
FORCE ON THE STABILITY OF $L_{4,5}$ IN
THE RELATIVISTIC R3BP**

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ABSTRACT. An investigation of motion of a test particle near the triangular points $L_{4,5}$ in the restricted three-body problem (R3BP) when a small perturbation is given to the Coriolis force within the framework of the post-Newtonian approximation is carried out. It is seen that there is no explicit effect of this perturbation on the positions of triangular point, whereas the relativistic factor has. We also observe that the stability region is affected by both the relativistic factor and a small perturbation in the Coriolis force. It is also found that both the coordinates of triangular points are affected by the mass ratio μ , contrary to the classical case where only the abscissa is affected.

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1. INTRODUCTION

The planar circular restricted three-body problem describes the motion of the third body of infinitesimal mass moving in the gravitational field of two massive bodies called primaries, which revolve around their common center of mass in circular orbits under the influence of their mutual gravitational attraction. This problem possesses five points of equilibrium, three of them called collinear points L_1, L_2, L_3 and are unstable; the other two are called the triangular points L_4, L_5 , and are stable for the mass ratio $\mu < 0.03852...$ (Szebehely [1]). Wintner [2] showed that the stability of these triangular points is due to the existence of Coriolis force as the coordinate system is rotating. Several studies (Szebehely [3]; Bhatnagar and Hallan [4]; AbdulRaheem and Singh [5]; Singh [6] and Singh and Begha [7]) have described the effects of small perturbations in the Coriolis and centrifugal forces on the motion of the third body. Szebehely [3] investigated the stability of triangular points by keeping the centrifugal force constant, and found that the Coriolis force

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is a stabilizing force. Later, SubbaRao and Sharma [8] proved that this fact is not always true as they observed an increase in both the Coriolis and centrifugal force due to oblateness of the primary. This was confirmed by AbdulRaheem and Singh [5]. Bhatnagar and Hallan [4] extended the work of Szebehely [3] by considering the effect of perturbations in the Coriolis and centrifugal forces.

Of recent Singh [9] investigated the effects of small perturbations in the Coriolis and centrifugal forces, radiation pressures and triaxiality of the two stars (primaries) on the positions and stability of an infinitesimal mass (third body) in the frame work of the planar circular restricted three-body problem (R3BP). The author observed that the positions of the three collinear and two triangular equilibrium points are affected by radiation, triaxiality and a small perturbation in the centrifugal force, but are unaffected by that of the Coriolis force. The collinear points are found to remain unstable, while the triangular points are seen to be stable for $0 < \mu < \mu_c$ and unstable for $\mu_c \leq \mu \leq \frac{1}{2}$, where μ_c is the critical mass ratio influenced by small perturbations in the Coriolis and centrifugal forces, radiation and triaxiality. The author also noticed that the Coriolis force possesses a stabilizing behavior, while the centrifugal force has a destabilizing behavior. Therefore, the overall effect is that the size of the region of stability decreases with increase in the values of the parameters involved.

The theory of general relativity is currently the most successful gravitational theory describing the nature of space and time, and is well confirmed by observations. Especially, it has passed “classical test” such as the deflection of light, the perihelion shift of Mercury and the Shapiro time delay, and also a systematic test using the remarkable binary pulsar “PSR1913+11” (Will [10]). In addition, future space astrometric missions such as the Space Interferometry Mission (SIM) and Galactic Astrometric Instrument for Astrophysics (GAIA) require a general relativistic modeling of the solar system within the accuracy of a micro arc second (Klioner [11]). In this context, it is worth and interesting to examine the CR3BP in general relativity compared with Newtonian gravity.

Brumberg [12, 13] studied the relativistic problem of three bodies in detail and compiled important results on relativistic celestial mechanics. The author did not only obtained the equations of motion for the general problem of three bodies but also deduced the equations of motion of the restricted problem of three bodies.

Bhatnagar and Hallan [14] studied the existence and linear stability of the triangular points $L_{4,5}$ in the relativistic R3BP and found that they are always unstable in the whole range $0 \leq \mu \leq \frac{1}{2}$ in contrast to the classical R3BP where they are stable for $\mu < \mu_0$, where μ is the mass ratio and $\mu_0 \leq 0.03852\dots$ is the Routh's value.

Ragos *et al.* [15] investigated numerically the linear stability of the collinear libration points $L_{1,2,3}$ in the relativistic R3BP for several solar system cases, and found that they are unstable.

Douskos and Perdios [16] investigated the stability of the triangular points in the relativistic R3BP and contrary to the result of Bhatnagar and Hallan [14], they obtained a region of linear stability in the parameter space $0 \leq \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2}$. They also determined the positions of the collinear points and showed that they are always unstable.

Ahmed *et al.* [17] studied also the stability of triangular points in the relativistic R3BP. In contrast to the previous result of Bhatnagar and Hallan [14] they obtained a region of linear stability as $0 \leq \mu < \mu_0 + \frac{11387}{119232c^2}$.

Abd El-Salam and Abd El-Bar [18] studied the photogravitational restricted three-body problem within the framework of the post-Newtonian approximation. They obtained the locations of the triangular points in the series forms which are new analytical results. From the literature above, we are motivated to study the effect of a small perturbation in the Coriolis force on the same problem.

This paper is organized as follows: in section 2, the equations governing the motion are presented; section 3 describes the positions of equilibrium points, while their linear stability is analyzed in section 4; the discussion and numerical results are given in section 5 and 6 respectively. Finally, section 7 conveys the main findings of this paper.

2. EQUATIONS OF MOTION

The motion of an infinitesimal mass in the relativistic R3BP in a synodic coordinate system and dimensionless variables is controlled by the equations (Brumberg [12], Bhatnagar and Hallan [14]):

$$\begin{aligned}\ddot{\xi} - 2n\dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}} \right), \\ \ddot{\eta} + 2n\dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right),\end{aligned}\tag{1}$$

with

$$\begin{aligned}
 W = & \frac{1}{2}(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1}{c^2} \left[-\frac{3}{2} \left(1 - \frac{1}{3}\mu(1-\mu) \right) (\xi^2 + \eta^2) \right. \\
 & + \frac{1}{8} \left\{ \dot{\xi}^2 + \dot{\eta}^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right\}^2 \\
 & + \frac{3}{2} \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \left(\dot{\xi}^2 + \dot{\eta}^2 + 2(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right) \\
 & - \left(\frac{(1-\mu)^2}{\rho_1^2} + \frac{\mu^2}{\rho_2^2} \right) + \mu(1-\mu) \left\{ (4\dot{\eta} + \frac{7}{2}\xi) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \right. \\
 & \left. \left. - \frac{\eta^2}{2} \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) + \left(\frac{-1}{\rho_1\rho_2} + \frac{3\mu-2}{2\rho_1} + \frac{1-3\mu}{2\rho_2} \right) \right\} \right] ,
 \end{aligned} \tag{2}$$

$$n = 1 - \frac{3}{2c^2} \left(1 - \frac{1}{3}\mu(1-\mu) \right) , \tag{3}$$

$$\begin{aligned}
 \rho_1^2 &= (\xi + \mu)^2 + \eta^2, \\
 \rho_2^2 &= (\xi + \mu - 1)^2 + \eta^2,
 \end{aligned} \tag{4}$$

where $0 < \mu \leq \frac{1}{2}$ is the ratio of the mass of the smaller primary to the total mass of the primaries; ρ_1 and ρ_2 are distances of the infinitesimal mass from the bigger and smaller primary, respectively; n is the mean motion of the primaries; c is the velocity of light.

We now introduce a small perturbation in the Coriolis force by means of a parameter $\varphi = 1 + \varepsilon$, $|\varepsilon| \ll 1$, with unperturbed value unity. Consequently, the equations (1) may be written as in the form:

$$\begin{aligned}
 \ddot{\xi} - 2\varphi n \dot{\eta} &= \frac{\partial W}{\partial \xi} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\xi}} \right) , \\
 \ddot{\eta} + 2\varphi n \dot{\xi} &= \frac{\partial W}{\partial \eta} - \frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right) ,
 \end{aligned} \tag{5}$$

where,

$$\begin{aligned}
 W = & \frac{1}{2}(\xi^2 + \eta^2) + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} + \frac{1}{c^2} \left[-\frac{3}{2} \left(1 - \frac{1}{3}\mu(1-\mu) \right) (\xi^2 + \eta^2) \right. \\
 & + \frac{1}{8} \left\{ \varphi \dot{\xi}^2 + \varphi \dot{\eta}^2 + 2\varphi(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right\}^2 \\
 & + \frac{3}{2} \left(\frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2} \right) \left\{ \varphi \dot{\xi}^2 + \varphi \dot{\eta}^2 + 2\varphi(\xi\dot{\eta} - \eta\dot{\xi}) + (\xi^2 + \eta^2) \right\} \\
 & - \frac{1}{2} \left(\frac{(1-\mu)^2}{\rho_1^2} + \frac{\mu^2}{\rho_2^2} \right) + \mu(1-\mu) \left\{ (4\varphi\dot{\eta} + \frac{7}{2}\xi) \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \right. \\
 & \left. \left. - \frac{\eta^2}{2} \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) + \left(\frac{-1}{\rho_1\rho_2} + \frac{3\mu-2}{2\rho_1} + \frac{1-3\mu}{2\rho_2} \right) \right\} \right] ,
 \end{aligned} \tag{6}$$

3. LOCATIONS OF TRIANGULAR POINTS

The equilibrium points are obtained from equations (5) after putting $\dot{\xi} = \dot{\eta} = \ddot{\xi} = \ddot{\eta} = 0$. These points are the solutions of the equations

$$\frac{\partial W}{\partial \xi} = 0 = \frac{\partial W}{\partial \eta} \text{ with } \dot{\xi} = \dot{\eta} = 0.$$

This is equivalent to

$$\begin{aligned} & \xi - \frac{(1-\mu)(\xi+\mu)}{\rho_1^3} - \frac{\mu(\xi-1+\mu)}{\rho_2^3} + \frac{1}{c^2} \left[-3\xi \left(1 - \frac{1}{3}\mu(1-\mu) \right) + \frac{1}{2}\xi(\xi^2 + \eta^2) \right. \\ & - \frac{3}{2}(\xi^2 + \eta^2) \left(\frac{(1-\mu)(\xi+\mu)}{\rho_1^3} + \frac{\mu(\xi-1+\mu)}{\rho_2^3} \right) + 3 \left(\frac{1-\mu}{\rho_1} - \frac{\mu}{\rho_2} \right) \xi \\ & + \frac{(1-\mu)^2(\xi+\mu)}{\rho_1^4} + \frac{\mu^2(\xi-1+\mu)}{\rho_2^4} + \mu(1-\mu) \left\{ \frac{7}{2} \left(\frac{1}{\rho_1} - \frac{1}{\rho_2} \right) \right. \\ & + \frac{7}{2}\xi \left(-\frac{(\xi+\mu)}{\rho_1^3} + \frac{(\xi-1+\mu)}{\rho_1^3} \right) + \frac{3}{2}\eta^2 \left(\frac{\mu(\xi+\mu)}{\rho_1^5} + \frac{(1-\mu)(\xi-1+\mu)}{\rho_2^5} \right) \\ & \left. \left. + \left(\frac{\xi+\mu}{\rho_1^3\rho_2} + \frac{\xi-1+\mu}{\rho_1\rho_2^3} - \frac{(3\mu-2)(\xi+\mu)}{2\rho_1^3} - \frac{(1-3\mu)(\xi-1+\mu)}{2\rho_2^3} \right) \right\} \right] = 0, \end{aligned}$$

and

$$\eta F = 0, \quad (7)$$

where,

$$\begin{aligned} F = & \left(1 - \frac{(1-\mu)}{\rho_1^3} - \frac{\mu}{\rho_2^3} \right) + \frac{1}{c^2} \left[-3 \left(1 - \frac{1}{3}\mu(1-\mu) \right) + \frac{1}{2}(\xi^2 + \eta^2) \right. \\ & + 3 \left(\frac{(1-\mu)}{\rho_1} + \frac{\mu}{\rho_2} \right) - \frac{3}{2}(\xi^2 + \eta^2) \left(\frac{(1-\mu)}{\rho_1^3} + \frac{\mu}{\rho_2^3} \right) + \left(\frac{(1-\mu)^2}{\rho_1^4} + \frac{\mu^2}{\rho_2^4} \right) \\ & + \mu(1-\mu) \left\{ \frac{7}{2}\xi \left(-\frac{1}{\rho_1^3} + \frac{1}{\rho_2^3} \right) - \left(\frac{\mu}{\rho_1^3} + \frac{1-\mu}{\rho_2^3} \right) \right. \\ & \left. \left. + \frac{3}{2}\eta^2 \left(\frac{\mu}{\rho_1^5} + \frac{(1-\mu)}{\rho_2^5} \right) + \frac{1}{\rho_1^3\rho_2} + \frac{1}{\rho_1\rho_2^3} - \frac{(3\mu-2)}{2\rho_1^3} - \frac{(1-3\mu)}{2\rho_2^3} \right\} \right], \end{aligned}$$

The triangular points are the solutions of equations (7) with $\eta \neq 0$. Since $\frac{1}{c^2} \ll 1$ and in the case $\frac{1}{c^2} \rightarrow 0$, one can obtain $\rho_1 = \rho_2 = 1$; we assume in the relativistic R3BP that $\rho_1 = 1+x$ and $\rho_2 = 1+y$ where, $x, y \ll 1$ may depend upon the relativistic and Coriolis effects. Substituting these values in the equations (4), solving them for ξ, η and ignoring terms of second and higher powers of x and y , we get

$$\begin{aligned} \xi &= x - y + \frac{1-2\mu}{2}, \\ \eta &= \pm \left(\frac{\sqrt{3}}{2} + \frac{x+y}{\sqrt{3}} \right), \end{aligned}$$

Now substituting the values of $\rho_1, \rho_2, \xi, \eta$ from above in equations (7) with $\eta \neq 0$, and neglecting second and higher order terms in $x, y, \frac{1}{c^2}$, we have

$$\begin{aligned} (1-\mu)x - \mu y - \frac{3\mu(1-2\mu)(1-\mu)}{8c^2} &= 0, \\ (1-\mu)x + \mu y + \frac{7(\mu-\mu^2)}{8c^2} &= 0. \end{aligned} \quad (8)$$

Solving these equations for x and y , we get

$$x = -\frac{\mu(2+3\mu)}{8c^2},$$

$$y = -\frac{(1-\mu)(5-3\mu)}{8c^2}.$$

Thus, the coordinates of the triangular points $(\xi, \pm\eta)$ denoted by L_4 and L_5 respectively are,

$$\xi = \frac{1-2\mu}{2} \left(1 + \frac{5}{4c^2}\right),$$

$$\eta = \pm \frac{\sqrt{3}}{2} \left\{1 + \frac{1}{12c^2} (-5 + 6\mu - 6\mu^2)\right\}. \quad (9)$$

4. LINEAR STABILITY

Let (a, b) be the coordinates of the triangular point L_4 . We set $\xi = a + \alpha$, $\eta = b + \beta$, $(\alpha, \beta \ll 1)$ in the equations (5) and for simplicity we substitute the value of $\varphi = 1 + \varepsilon$. First, we compute the terms on their right hand side, neglecting second and higher order terms of small quantities to obtain

$$\left(\frac{\partial W}{\partial \xi}\right)_{\xi=a+\alpha, \eta=b+\beta} = A\alpha + B\beta + C\dot{\alpha} + D\dot{\beta}$$

where,

$$A = \frac{3}{4} \left\{1 + \frac{1}{2c^2} (2 - 19\mu + 19\mu^2)\right\},$$

$$B = \frac{3\sqrt{3}}{4} (1 - 2\mu) \left(1 - \frac{2}{3c^2}\right),$$

$$C = \frac{\sqrt{3} (1 - 2\mu)}{2c^2} (1 + \varepsilon)$$

$$D = \frac{6 - 5\mu + 5\mu^2}{2c^2} (1 + \varepsilon)$$

Similarly, we get

$$\left(\frac{\partial W}{\partial \eta}\right)_{\xi=a+\alpha, \eta=b+\beta} = A_1\alpha + B_1\beta + C_1\dot{\alpha} + D_1\dot{\beta}$$

where,

$$A_1 = \frac{3\sqrt{3}}{4} (1 - 2\mu) \left(1 - \frac{2}{3c^2}\right),$$

$$B_1 = \frac{9}{4} \left\{1 + \frac{7(-2 + 3\mu - 3\mu^2)}{6c^2}\right\},$$

$$C_1 = \frac{1}{2c^2} (-4 + \mu - \mu^2) (1 + \varepsilon) ,$$

$$D_1 = -\frac{\sqrt{3}}{2c^2} (1 - 2\mu) (1 + \varepsilon) .$$

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_2 \dot{\alpha} + B_2 \dot{\beta} + C_2 \ddot{\alpha} + D_2 \ddot{\beta},$$

where,

$$A_2 = \frac{\sqrt{3}}{2c^2} (1 - 2\mu) (1 + \varepsilon) ,$$

$$B_2 = \frac{(-4+\mu-\mu^2)}{2c^2} (1 + \varepsilon) ,$$

$$C_2 = \frac{17-2\mu+2\mu^2}{4c^2} + \left\{ \frac{10-\mu+\mu^2}{2c^2} \right\} \varepsilon ,$$

$$D_2 = \frac{\sqrt{3}}{4c^2} (1 - 2\mu) (1 + 2\varepsilon) .$$

$$\frac{d}{dt} \left(\frac{\partial W}{\partial \dot{\eta}} \right)_{\xi=a+\alpha, \eta=b+\beta} = A_3 \dot{\alpha} + B_3 \dot{\beta} + C_3 \ddot{\alpha} + D_3 \ddot{\beta}$$

where,

$$A_3 = \frac{6-5\mu+5\mu^2}{2c^2} (1 + \varepsilon) ,$$

$$B_3 = -\frac{\sqrt{3}}{2c^2} (1 - 2\mu) (1 + \varepsilon) ,$$

$$C_3 = -\frac{\sqrt{3}}{4c^2} (1 - 2\mu) (1 + \varepsilon) ,$$

$$D_3 = \frac{3(5-2\mu+2\mu^2)}{4c^2} + \left\{ \frac{8-5\mu+5\mu^2}{2c^2} \right\} \varepsilon .$$

Thus, the variational equations of motion corresponding to equations (5), on making use of equation (3), can be obtained as

$$\begin{aligned} P_1 \ddot{\alpha} + P_2 \ddot{\beta} + P_3 \dot{\alpha} + P_4 \dot{\beta} + P_5 \alpha + P_6 \beta &= 0, \\ Q_1 \ddot{\alpha} + Q_2 \ddot{\beta} + Q_3 \dot{\alpha} + Q_4 \dot{\beta} + Q_5 \alpha + Q_6 \beta &= 0, \end{aligned} \tag{10}$$

where,

$$\begin{aligned} P_1 &= 1 + C_2, P_2 = D_2, P_3 = A_2 - C, \\ P_4 &= \left\{ B_2 - 2(1 + \varepsilon) \left(1 - \frac{1}{2c^2}(3 - \mu + \mu^2) \right) - D \right\}, P_5 = -A, \\ P_6 &= -B \end{aligned}$$

$$\begin{aligned} Q_1 &= C_3, Q_2 = 1 + D_3, Q_3 = 2(1 + \varepsilon) \left(1 - \frac{1}{2c^2}(3 - \mu + \mu^2) \right) - C_1 \\ &+ A_3, Q_4 = B_3 - D_1, Q_5 = -A_1, Q_6 = -B_1 \end{aligned}$$

Then the corresponding characteristic equation is

$$\begin{aligned} (P_1 Q_2 - P_2 Q_1) \lambda^4 + (P_1 Q_6 + P_5 Q_2 + P_3 Q_4 - P_6 Q_1 - P_2 Q_5 - P_4 Q_3) \lambda^2 \\ + P_5 Q_6 - P_6 Q_5 = 0 \end{aligned} \quad (11)$$

Substituting the values of $P_i, Q_i, i = 1, 2, \dots, 6$ in (11), the characteristic equation (11) after normalization becomes

$$\lambda^4 + b\lambda^2 + d = 0, \quad (12)$$

where,

$$\begin{aligned} b &= \left(1 - \frac{9}{c^2} \right) + \left\{ 8 + \frac{-147+30\mu-30\mu^2}{2c^2} \right\} \varepsilon, \\ d &= \frac{27}{4} \mu(1 - \mu) + \frac{-585\mu+693\mu^2-216\mu^3+108\mu^4}{8c^2} \\ &+ \left\{ \frac{-243\mu+324\mu^2-162\mu^3+81\mu^4}{4c^2} \right\} \varepsilon. \end{aligned}$$

When $\frac{1}{c^2} \rightarrow 0$ and there is no perturbation in the Coriolis force(*i.e.* $\varepsilon = 0$), (12) reduces to its well known classical restricted problem form (see e.g. Szebehely [1]).

The discriminant of (12) can be written as

$$\begin{aligned} \Delta &= \left(-\frac{54}{c^2} - \frac{81\varepsilon}{c^2} \right) \mu^4 + \left(\frac{108}{c^2} + \frac{162\varepsilon}{c^2} \right) \mu^3 + \left(27 - \frac{693}{2c^2} - \frac{354\varepsilon}{c^2} \right) \mu^2 \\ &+ \left(-27 + \frac{585}{2c^2} + \frac{273\varepsilon}{c^2} \right) \mu + 16\varepsilon - \frac{18}{c^2} - \frac{291\varepsilon}{c^2} \end{aligned} \quad (13)$$

and its roots are

$$\lambda^2 = \frac{-b \pm \sqrt{\Delta}}{2} \quad (14)$$

where,

$$b = \left(1 - \frac{9}{c^2} \right) + \left\{ 8 + \frac{-147+30\mu-30\mu^2}{2c^2} \right\} \varepsilon,$$

From (13), we have

$$\begin{aligned} \frac{d\Delta}{d\mu} &= 4 \left(-\frac{54}{c^2} - \frac{81\varepsilon}{c^2} \right) \mu^3 + 3 \left(\frac{108}{c^2} + \frac{162\varepsilon}{c^2} \right) \mu^2 \\ &+ 2 \left(27 - \frac{693}{2c^2} - \frac{354\varepsilon}{c^2} \right) \mu + \left(-27 + \frac{585}{2c^2} + \frac{273\varepsilon}{c^2} \right) < 0 \\ &, \forall \mu \in \left(0, \frac{1}{2} \right]. \end{aligned} \quad (15)$$

From (15), it follows that Δ is a decreasing function in $\left(0, \frac{1}{2} \right]$.

But

$$(\Delta)_{\mu=0} = 1 - \frac{18}{c^2} + \left(16 - \frac{291}{c^2}\right) \varepsilon > 0$$

$$\text{and} \tag{16}$$

$$(\Delta)_{\mu=\frac{1}{2}} = -\frac{23}{4} + \frac{207}{4c^2} + \left(16 - \frac{3645}{16c^2}\right) \varepsilon < 0$$

Since $(\Delta)_{\mu=0}$ and $(\Delta)_{\mu=\frac{1}{2}}$ are of opposite signs, and Δ is monotone and continuous, there is one value of μ , e.g. μ_c in the interval $(0, \frac{1}{2}]$ for which Δ vanishes.

Solving the equation $\Delta = 0$, using (13), we obtain critical value of the mass parameter as

$$\mu_c = \frac{1}{18} (9 - \sqrt{69}) - \frac{17\sqrt{\frac{23}{3}}}{162c^2} + \left(\frac{16}{3\sqrt{69}} - \frac{47\sqrt{\frac{23}{3}}}{27c^2} \right) \varepsilon \tag{17}$$

$$= \mu_0 - \frac{17\sqrt{69}}{486c^2} + \left(\frac{16}{3\sqrt{69}} - \frac{47\sqrt{69}}{81c^2} \right) \varepsilon$$

where $\mu_0 = 0.03852\dots$ is the Routh's value

There are three possible cases regarding the sign of the discriminant Δ :

- i. When $0 \leq \mu < \mu_c$, $\Delta > 0$ the values of λ^2 given by (14) are negative and therefore all the four characteristic roots are distinct pure imaginary numbers. Hence, the triangular points are stable.
- ii. When $\mu_c < \mu \leq \frac{1}{2}$, $\Delta < 0$, the real parts of the characteristic roots are positive. Therefore, the triangular points are unstable.
- iii. When $\mu = \mu_c$, $\Delta = 0$, the values of λ^2 given by (14) are the same. Thus the solutions contain secular terms. This induces instability of the triangular points.

Hence the stability region is

$$0 \leq \mu < \mu_0 - \frac{17\sqrt{69}}{486c^2} + \left(\frac{16}{3\sqrt{69}} - \frac{47\sqrt{69}}{81c^2} \right) \varepsilon \tag{18}$$

In the absence of a small perturbation in the Coriolis force (*i.e.* $\varepsilon = 0$), μ_c reduces to the critical mass value of the unperturbed relativistic R3BP. This confirms the result of Douskos and Perdios [16], but disagrees with that of Ahmed *et al* [17].

In the presence of the perturbation ε and in the absence of relativistic term $\frac{1}{c^2}$ (i.e. $\frac{1}{c^2} \rightarrow 0$), μ_c verifies the results of Bhatnagar and Hallan [4] when a small perturbation in centrifugal force is absent and those of Szebehely [3].

It is noticed from (18) that Coriolis force has stabilizing influence. This agrees with the result of Singh [9] and those of AbdulRaheem and Singh [5].

5. DISCUSSION

Equations (5) and (6) describe the motion of a test particle in the relativistic R3BP with a small perturbation ε in the Coriolis force. Equations (9) determine the positions of triangular points which are not affected by the perturbation in the Coriolis force but are affected by the relativistic factor. These positions correspond to those of Bhatnagar and Hallan [14], Douskos and Perdios [16] and Ahmed *et al.* [17]. It is noticed from equation (9) that the triangular points form isosceles triangles with the two primaries bodies contrary to the classical case in which they form equilateral triangles. We also in (9) observe that both ξ, η coordinates are affected by the mass ratio μ , contrary to the classical case where only the abscissa is affected. Equation (17) gives the critical value of the mass parameter which depends upon the small perturbation in the Coriolis force and relativistic factor. It is noticed from equation (18) that the Coriolis' perturbation expands when $\varepsilon > 0$, contracts when $\varepsilon < 0$ and the relativistic factor reduces the size of the region of stability separately, but their joint effect keeps contraction when $\varepsilon > 0$ and expansion when $\varepsilon < 0$ in that size of stability.

In the absence of perturbation in the Coriolis force (i.e. $\varepsilon = 0$), the stability results obtained in this study are in agreement with those of Douskos and Perdios [16] and disagree with those of Ahmed *et al.* [17] and Bhatnagar and Hallan [14]. In the absence of relativistic terms, the present results coincide with those, of AbdulRaheem and Singh [5] when the primaries are spherical dark bodies and the small perturbation in the centrifugal force is absent. The result also coincides with that of Szebehely [3], and Bhatnagar and Hallan [14] when the centrifugal force is absent.

6. NUMERICAL RESULTS

The values of $\mu_{critical}$ obtained from (17) using the Sun-Earth system for various values of the parameter ε are tabulated in Table 1.

Table1. Critical mass ($\mu = 0.000003003500$, $c = 10064.84$)

Perturbation parameter ε	μ_0	μ_c (with $\varepsilon = 0$)	μ_c (Eq.17)
0.050000	0.0385208965	0.0385208946	0.0706237871
-0.050000	0.0385208965	0.0385208946	0.0064180020
0.010000	0.0385208965	0.0385208946	0.0449414731
-0.010000	0.0385208965	0.0385208946	0.0321003161
0.005000	0.0385208965	0.0385208946	0.0417311838
-0.005000	0.0385208965	0.0385208946	0.0353106053
0.001000	0.0385208965	0.0385208946	0.0391629524
-0.001000	0.0385208965	0.0385208946	0.0378788367
0.000500	0.0385208965	0.0385208946	0.0388419235
-0.000500	0.0385208965	0.0385208946	0.0381998656
0.000100	0.0385208965	0.0385208946	0.0385851004
-0.000100	0.0385208965	0.0385208946	0.0384566888
0.000050	0.0385208965	0.0385208946	0.0385529974
-0.000050	0.0385208965	0.0385208946	0.0384887917
0.000010	0.0385208965	0.0385208945	0.0385273152
-0.000010	0.0385208965	0.0385208945	0.0385144739
0.000005	0.0385208965	0.0385208945	0.0385241018
-0.000005	0.0385208965	0.0385208945	0.0385176843

7. CONCLUSION

By considering a small perturbation in the Coriolis force in the relativistic CR3BP, the positions of the triangular points have been determined and their linear stability has been examined. It has also been observed that their positions are not affected by a small change in the Coriolis force but are affected by the relativistic factor. It is also noticed that the general stabilizing characteristic of the Coriolis force remains unaltered and the region of stability is affected by both the relativistic factor and perturbation in the Coriolis force.

It is seen from Table 1, that the relativistic factor reduces the size of the region of stability separately. It is also observed that $\mu_c > \mu_0$ for $\varepsilon > 0$ and $\mu_c < \mu_0$ for $\varepsilon < 0$, establishing that the Coriolis force has a stabilizing characteristic behavior.

It is noticed that expressions for A, D, A_2, C_2 in Bhatnagar and Hallan [14] differ from those of the unperturbed case of the present study. Consequently, the expressions for P_1, P_3, P_4, P_5 and the characteristic equation are also different. This led Bhatnagar and Hallan [14] to infer that triangular points are unstable, contrary to Douskos and Perdios [16] and the present result.

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