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INTEGRAL REPRESENTATIONS AND IDENTITIES ON RANK-1 SYMMETRIC SPACES OF COMPACT TYPE

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ABSTRACT. The Jacobi coefficients $c_j^{\ell}(\alpha,\beta)$ $(1 \leq j \leq \ell; \alpha,\beta > -1)$ associated with the normalised Jacobi polynomials $\mathscr{P}_k^{(\alpha,\beta)}$ $(k = 0,1,2,\dots; \alpha,\beta > -1)$ describe the Maclaurin heat coefficients $b_{2\ell}^N$ $(N,\ell = 1,2,\dots)$ and the associated spectral polynomials $\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}$ of N-dimensional compact rank-1 symmetric spaces. In this paper, apart from constructing a spectral polynomial $\mathscr{R}_\ell^{(\alpha,\beta)}$ associated with the product $\left[\mathscr{P}_k^{(\alpha,\beta)}\right]^2$ we develop integral representations (involving Gegenbauer polynomials and Jacobi polynomials) for $\mathscr{R}_\ell^{(\alpha,\beta)}$ in terms of the spectral sum of integer powers of eigenvalues of the corresponding Gegenbauer and Jacobi operators. These integrals apart from being interesting in their own right lead to integral representations and identities for these eigenvalues and their multiplicities.

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1. INTRODUCTION

Suppose $(\mathscr{X}, g)$ is a compact N-dimensional $(N = 1,2,\dots)$ Riemannian manifold without boundary and let $\Delta = \Delta_{\mathscr{X}}$ denote the (*nonnegative*) Laplace-Beltrami operator on $\mathscr{X}$ acting on smooth functions $f \in C^\infty(\mathscr{X})$ and given in local coordinates by

$$\Delta_{\mathscr{X}} f = -\frac{1}{\sqrt{\det g}} \sum_{j=1}^{N} \partial_j \left(\sum_{k=1}^{N} \sqrt{\det g}\, g^{jk} \partial_k \right) f. \tag{1}$$

By basic spectral theory there exists a complete orthonormal basis $(\varphi_k : k = 0,1,2,\dots)$ consisting of eigenfunctions of $\Delta_{\mathscr{X}}$, in the

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Hilbert space $L^2(\mathscr{X})$, with associated eigenvalues $\lambda_k = \lambda_k(\mathscr{X}), k = 0, 1, 2,$ satisfying $\Delta_{\mathscr{X}}\varphi_k = \lambda_k\varphi_k$. Each λ_k has finite multiplicity and the spectrum can be arranged in ascending order $0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots$ while $\lambda_j \nearrow \infty$. Furthermore by orthogonality, $(\varphi_j, \varphi_k)_{L^2(\mathscr{X})} = 0$ for $0 \le j \ne k$ whilst $||\varphi_j||_{L^2(\mathscr{X})} = 1$ for all $j = 0, 1, 2, \dots$.

The heat semigroup $\big(U(t) := e^{-t\Delta_{\mathscr{X}}} : t > 0\big)$ defined in the usual way admits an integral kernel $K_{\mathscr{X}} = K_{\mathscr{X}}(t, x, y)$, which, for $t > 0$ and $x, y \in \mathscr{X}$, can be expressed by the spectral sum

$$K_{\mathscr{X}}(t, x, y) = \sum_{k=0}^{\infty} e^{-\lambda_k t}\varphi_k(x)\varphi_k(y). \tag{2}$$

One easily sees that the heat kernel $K_{\mathscr{X}}$ is real, symmetric in x and y, i.e., $K_{\mathscr{X}}(t, x, y) = K_{\mathscr{X}}(t, y, x)$ and smooth; indeed $K_{\mathscr{X}} \in C^\infty((0, \infty) \times \mathscr{X} \times \mathscr{X})$. For heat kernels in Riemannian geometry and applications, see the monographs Berger *et al* [1], Chavel [2] and Li [3] and the references therein. See also Bakry *et al* [4]. When $\mathscr{X}$ is a N-dimensional compact rank-1 symmetric space, using the addition formula for the matrix coefficients, the heat kernel can be shown to have the form (see, e.g., Helgason [5, Ch. IV]; see also Awonusika [6, Appendix A.1])

$$K_{\mathscr{X}}(t, \theta) = \sum_{k=0}^{\infty} \frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}\Phi_k^{\mathscr{X}}(\theta)e^{-\lambda_k(\mathscr{X})t}. \tag{3}$$

Here $\lambda_k(\mathscr{X})$ (with $k = 0, 1, 2, \dots$) are the numerically distinct eigenvalues of $\Delta_{\mathscr{X}}$, $M_k(\mathscr{X})$ is the dimension of the eigenspace associated with $\lambda_k(\mathscr{X})$ (i.e., the multiplicity of the eigenvalue $\lambda_k(\mathscr{X})$), $\Phi_k^{\mathscr{X}}(\theta)$ is the spherical function on $\mathscr{X}$ associated with the eigenvalue $\lambda_k(\mathscr{X})$, θ is the geodesic distance between the points $x, y \in \mathscr{X}$ and $\mathrm{Vol}(\mathscr{X})$ is the volume of $\mathscr{X}$. Remarkably in this setting the spherical functions $\Phi_k^{\mathscr{X}}$ can be explicitly expressed as the normalised Jacobi polynomials (see Appendix B) $\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta) := P_k^{(\alpha,\beta)}(\cos\theta)/P_k^{(\alpha,\beta)}(1)$ (with $k = 0, 1, 2, \dots$) and for suitable choice of parameters $\alpha, \beta > -1$ (see TABLE 1).

Examples of rank-1 compact symmetric spaces include the sphere $\mathbb{S}^n = \mathbf{SO}(n+1)/\mathbf{SO}(n)$ (of real dimension $N = n$), the real projective space $\mathbf{P}^n(\mathbb{R}) = \mathbf{SO}(n+1)/\mathbf{O}(n)$ (of real dimension $N = n$), the complex projective space $\mathbf{P}^n(\mathbb{C}) = \mathbf{SU}(n+1)/\mathbf{S}(\mathbf{U}(n) \times \mathbf{U}(1))$ (of real dimension $N = 2n$), the quaternionic projective space $\mathbf{P}^n(\mathbb{H}) = \mathbf{Sp}(n+1)/\mathbf{Sp}(n) \times \mathbf{Sp}(1)$ (of real dimension $N = 4n$) and the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay}) = \mathbf{F}_4/\mathbf{Spin}(9)$ (of real dimension $N = 16$) (see Cahn and Wolf [7], Volchkov and Volchkov [8] and Warner [9]). Here $n = 1, 2, \dots$.

To proceed, let us recall some of the most relevant geometric and spectral data associated with these symmetric spaces that will be needed later on. Indeed these are: the radial part of the Laplacian,

$$\Delta_{\mathscr{X}} = -\frac{\partial^2}{\partial\theta^2} - (a\cot\theta + (1/2)b\cot(\theta/2))\frac{\partial}{\partial\theta}; \tag{4}$$

the Multiplicity (with $k = 0, 1, 2, \dots$)

$$\begin{aligned} M_k(\mathscr{X}) =& \frac{2(k+\varrho)\Gamma(k+2\varrho)\Gamma\left((a+1)/2\right)\Gamma\left(k+N/2\right)}{k!\Gamma(2\varrho+1)\Gamma\left(N/2\right)\Gamma\left(k+(a+1)/2\right)} \\ \varrho =& \left(a+b/2\right)/2,\ \ N = a+b+1, \end{aligned} \tag{5}$$

of the eigenvalues $\lambda_k(\mathscr{X})$ ($k = 0, 1, 2, \dots$) of $\Delta_{\mathscr{X}}$; and the volume

$$\mathrm{Vol}(\mathscr{X}) = 2^N \pi^{\frac{N}{2}} \frac{\Gamma\left((a+1)/2\right)}{\Gamma\left((N+a+1)/2\right)}. \tag{6}$$

TABLE 1 illustrates the parameters $a, b,\ N,\ \alpha$ and β for the symmetric spaces just listed.

TABLE 1. The parameters $a, b,\ N,\ \alpha,\ \beta$ associated with compact rank-1 symmetric spaces $\mathscr{X}$

$\mathscr{X}$	a	b	N	α	β
$\mathbb{S}^n$	$n-1$	0	n	$(n-2)/2$	$(n-2)/2$
$\mathbf{P}^n(\mathbb{R})$	$n-1$	0	n	$(n-2)/2$	$(n-2)/2$
$\mathbf{P}^n(\mathbb{C})$	1	$2(n-1)$	$2n$	$n-1$	0
$\mathbf{P}^n(\mathbb{H})$	3	$4(n-1)$	$4n$	$2n-1$	1
$\mathbf{P}^2(\mathrm{Cay})$	7	8	16	7	3

In a similar way, TABLE 2 lists the geometric and spectral data stated in (4)-(6) for each of the aforementioned compact rank-1 symmetric spaces. [1]

TABLE 2. The compact rank-1 symmetric spaces $\mathscr{X}$

$\mathscr{X}$	$\mathbb{S}^n$	$\mathbf{P}^n(\mathbb{R})$	$\mathbf{P}^n(\mathbb{C})$
$\lambda_k(\mathscr{X})$	$k(k+n-1)$	$2k(2k+n-1)$	$k(k+n)$
$M_k(\mathscr{X})$	$(2k+n-1)\frac{(k+n-2)!}{k!(n-1)!}$	$(4k+n-1)\frac{(2k+n-2)!}{(2k)!(n-1)!}$	$\frac{2k+n}{n}\left[\frac{\Gamma(k+n)}{\Gamma(n)k!}\right]^2$
$\mathrm{Vol}(\mathscr{X})$	$\frac{2\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$	$\frac{\pi^{\frac{n+1}{2}}}{\Gamma\left(\frac{n+1}{2}\right)}$	$\frac{4^n\pi^n}{n!}$

[1]We also write $\lambda_k(\mathscr{X}) = \lambda_k^{(\alpha,\beta)}$.

For the corresponding data for $\mathbf{P}^n(\mathbb{H})$ and $\mathbf{P}^2(\mathrm{Cay})$, see TABLE 3. (See also Volchkov and Volchkov [8], Warner [9], Helgason [10, 11] and Vilenkin [12] for further reference and background on Lie groups and symmetric spaces.)

TABLE 3. Symmetric Spaces $\mathbf{P}^n(\mathbb{H})$ and $\mathbf{P}^2(\mathrm{Cay})$

$\mathscr{X}$	$\mathbf{P}^n(\mathbb{H})$	$\mathbf{P}^2(\mathrm{Cay})$
$\lambda_k(\mathscr{X})$	$k(k+2n+1)$	$k(k+11)$
$M_k(\mathscr{X})$	$\frac{(2k+2n+1)(k+2n)}{(2n)(2n+1)(k+1)}\left[\frac{\Gamma(k+2n)}{k!\Gamma(2n)}\right]^2$	$6(2k+11)\frac{\Gamma(k+8)\Gamma(k+11)}{7!11!k!\Gamma(k+4)}$
$\mathrm{Vol}(\mathscr{X})$	$\frac{(4\pi)^{2n}}{\Gamma(2n+2)}$	$\frac{3!}{11!}(4\pi)^8$

2. THE MACLAURIN SPECTRAL FUNCTIONS ON SPACES $\mathscr{X}$

This section discusses the Maclaurin expansion of the heat kernel $K_{\mathscr{X}}(t,\theta)$ with respect to the θ-variable near the origin $\theta = 0$ (for $t > 0$), and also examines the role of Jacobi coefficients in the description of the Maclaurin heat coefficients. For a more refined analysis and description of the resulting Maclaurin heat coefficients including relationships to other heat invariants, see Awonusika and Taheri [13] (see also Awonusika [6]).

Towards this end, recall that the Maclaurin expansion of $K_{\mathscr{X}}(t,\theta)$ about $\theta = 0$ has the form

$$K_{\mathscr{X}}(t,\theta) = \sum_{j=0}^{\infty} \frac{\theta^{2j}}{(2j)!} \left\{ \frac{\partial^{2j}}{\partial\theta^{2j}} K_{\mathscr{X}}(t,\theta)\bigg|_{\theta=0} \right\}. \tag{7}$$

Note that in view of $K_{\mathscr{X}}(t,\theta)$ being even in the θ-variable [*cf.* (3)] all partial derivatives of odd order vanish at $\theta = 0$ and hence the Maclaurin expansion contains only even terms. Evidently the first term in (7) is given by the usual trace formula, namely,

$$K_{\mathscr{X}}(t,0) = \frac{1}{\mathrm{Vol}(\mathscr{X})} \sum_{k=0}^{\infty} M_k(\mathscr{X}) e^{-\lambda_k(\mathscr{X})t} = \frac{\mathrm{tr}\, e^{-t\Delta_{\mathscr{X}}}}{\mathrm{Vol}(\mathscr{X})}. \tag{8}$$

Proposition 1: (Jacobi coefficients (Awonusika and Taheri [14, 15])) Consider the Jacobi polynomial $\mathscr{P}_k^{(\alpha,\beta)}$ with $k = 0, 1, 2, \ldots; \alpha,$

$\beta > -1$. Then for any integer $\ell \geq 1$ we have

$$\begin{aligned}\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}(\lambda_k) :=& \widetilde{R}_\ell\left(\lambda_k^{(\alpha,\beta)}\right) = \frac{d^{2\ell}}{d\theta^{2\ell}}\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} \\ =& \sum_{j=1}^{\ell} c_j^\ell(\alpha,\beta)[\lambda_k^{(\alpha,\beta)}]^j. \end{aligned} \tag{9}$$

The scalars $(c_j^\ell(\alpha,\beta) : 1 \leq j \leq \ell)$ are called the *Jacobi coefficients*, $\lambda_k^{(\alpha,\beta)}$ $(k = 0, 1, 2, \dots)$ are the eigenvalues of the Jacobi operator, and $\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)} = \widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}(X)$ is a ℓ-degree polynomial in X.

Indeed, by Proposition 1 we have

$$\begin{aligned}\frac{\partial^{2\ell}}{\partial\theta^{2\ell}}K_{\mathscr{X}}(t,\theta)\Big|_{\theta=0} &= \sum_{k=0}^{\infty}\frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}e^{-\lambda_k(\mathscr{X})t}\frac{\partial^{2\ell}}{\partial\theta^{2\ell}}\Phi_k^{\mathscr{X}}(\theta)\Big|_{\theta=0} \\ &= \sum_{k=0}^{\infty}\frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}e^{-\lambda_k^{(\alpha,\beta)}t}\frac{\partial^{2\ell}}{\partial\theta^{2\ell}}\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\Big|_{\theta=0} \\ &= \sum_{k=0}^{\infty}\frac{M_k(\mathscr{X})}{\mathrm{Vol}(\mathscr{X})}e^{-\lambda_k^{(\alpha,\beta)}t}\sum_{j=1}^{\ell}c_j^\ell(\alpha,\beta)\left[\lambda_k^{(\alpha,\beta)}\right]^j \\ &= \sum_{j=1}^{\ell}\frac{c_j^\ell(\alpha,\beta)}{\mathrm{Vol}(\mathscr{X})}\left(-\frac{d}{dt}\right)^j \operatorname{tr} e^{-t\Delta_{\mathscr{X}}}. \end{aligned} \tag{10}$$

Theorem 1: (Maclaurin heat coefficients (Awonusika and Taheri [13])) The Maclaurin heat coefficients $b_{2\ell}^N = b_{2\ell}^N(t)$, $\ell = 1, 2, \dots, t > 0$, associated with the heat kernels $K_{\mathscr{X}}(t,\theta)$ admit the spectral representation

$$\begin{aligned} b_{2\ell}^N(t) =& \sum_{k=0}^{\infty}\frac{M_k(\mathscr{X})e^{-t\lambda_k^{(\alpha,\beta)}}}{\mathrm{Vol}(\mathscr{X})}\sum_{j=1}^{\ell}c_j^\ell(\alpha,\beta)\left[\lambda_k^{(\alpha,\beta)}\right]^j \\ =& \frac{1}{\mathrm{Vol}(\mathscr{X})}\operatorname{tr}\left\{\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}(\Delta_{\mathscr{X}})\mathrm{e}^{-\mathrm{t}\Delta_{\mathscr{X}}}\right\}. \end{aligned} \tag{11}$$

In particular,

$$b_0^N(t) = \operatorname{tr} e^{-t\Delta_{\mathscr{X}}}. \tag{12}$$

The result in Theorem 1 underlines the role of the polynomials $\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}$ and the Jacobi coefficients $c_j^\ell(\alpha,\beta)$ in expressing the Maclaurin coefficients $b_{2\ell}^N$ associated with the heat kernel $K_{\mathscr{X}}$. For more results and discussions on Maclaurin heat coefficients, see Awonusika and Taheri [13], Awonusika [16].

The first few Jacobi coefficients $c_j^\ell(\alpha,\beta)$ are given below.

$$\begin{aligned}
c_1^1(\alpha,\beta) =& -\frac{1}{2(\alpha+1)}\\
c_1^2(\alpha,\beta) =& -\frac{\alpha+3\beta+2}{4(\alpha+1)(\alpha+2)}\\
c_2^2(\alpha,\beta) =& \frac{3}{4(\alpha+1)(\alpha+2)}\\
c_1^3(\alpha,\beta) =& -\frac{4\alpha^2+30\alpha\beta+30\beta^2+20\alpha+60\beta+24}{8(\alpha+1)(\alpha+2)(\alpha+3)}\\
c_2^3(\alpha,\beta) =& \frac{15(\alpha+3\beta+2)}{8(\alpha+1)(\alpha+2)(\alpha+3)}\\
c_3^3(\alpha,\beta) =& -\frac{15}{8(\alpha+1)(\alpha+2)(\alpha+3)}.
\end{aligned} \tag{13}$$

For explicit calculations of these coefficients, see Awonusika and Taheri [15].

3. SPECTRAL POLYNOMIALS ASSOCIATED WITH $\left[\mathscr{P}_k^{(\alpha,\beta)}\right]^2$

In this section, we give a spectral identity relating the differential action on the product $\left[\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right]^2$ to the spectral sum of integer powers of the eigenvalues of the corresponding Jacobi operator. The spectral polynomial $\mathscr{R}_\ell^{(\alpha,\beta)}$ associated with $\left[\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right]^2$ is a generalisation of $\widetilde{\mathscr{R}}_\ell^{(\alpha,\beta)}$ associated with $\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)$ [*cf.* Proposition 1].

As a consequence of Proposition 1, we have the following result:

Proposition 2: (Spectral polynomials) Let $\alpha,\beta>-1$ and $k=0,1,2,\dots$. Then for any integer $\ell\geq 1$ we have

$$\begin{aligned}
\mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k) :=& \frac{d^{2\ell}}{d\theta^{2\ell}}\left[\mathscr{P}_k^{(\alpha,\beta)}(\cos\theta)\right]^2\Big|_{\theta=0}\\
=& \sum_{p=0}^{\ell}\binom{2\ell}{2p}\sum_{i=0}^{\ell-p}\sum_{j=0}^{p} c_i^{\ell-p}(\alpha,\beta)c_j^p(\alpha,\beta)[\lambda_k^{\alpha,\beta}]^{i+j}.
\end{aligned} \tag{14}$$

The scalars $(c_j^\ell(\alpha,\beta): 1\leq j\leq \ell)$ are the usual *Jacobi coefficients*, $\lambda_k^{(\alpha,\beta)}=k(k+\alpha+\beta+1)$ are the eigenvalues of the Jacobi operator and $\mathscr{R}_\ell^{(\alpha,\beta)}=\mathscr{R}_\ell^{(\alpha,\beta)}(X)$ are ℓ-degree polynomials in X.

The first few spectral polynomials $\mathscr{R}_\ell^{(\alpha,\beta)}$ are given below.

- ($\ell = 1$) Here we see that

$$\mathscr{R}_1^{(\alpha,\beta)}(\lambda_k) = e_1^1(\alpha,\beta)\lambda_k^{\alpha,\beta}, \tag{15}$$

where $e_1^1(\alpha,\beta) = 2c_1^1(\alpha,\beta)$.

- ($\ell = 2$) Indeed we have

$$\mathscr{R}_2^{(\alpha,\beta)}(\lambda_k) = e_1^2(\alpha,\beta)\lambda_k^{\alpha,\beta} + e_2^2(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^2, \tag{16}$$

where

$$\begin{aligned} e_1^2(\alpha,\beta) =& 2c_1^2(\alpha,\beta) \\ e_2^2(\alpha,\beta) =& 2\left(3\left[c_1^1(\alpha,\beta)\right]^2 + c_2^2(\alpha,\beta)\right) = \frac{2(2\alpha+3)}{\alpha+1}c_2^2(\alpha,\beta). \end{aligned} \tag{17}$$

- ($\ell = 3$) It is seen here that

$$\mathscr{R}_3^{(\alpha,\beta)}(\lambda_k) = e_1^3(\alpha,\beta)\lambda_k^{\alpha,\beta} + e_2^3(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^2 + e_3^3(\alpha,\beta)\left[\lambda_k^{\alpha,\beta}\right]^3, \tag{18}$$

where

$$\begin{aligned} e_1^3(\alpha,\beta) =& 2c_1^3(\alpha,\beta) \\ e_2^3(\alpha,\beta) =& 2\left(15c_1^1(\alpha,\beta)c_1^2(\alpha,\beta) + c_2^3(\alpha,\beta)\right) = \frac{4(\alpha+2)}{\alpha+1}c_2^3(\alpha,\beta), \\ e_3^3(\alpha,\beta) =& 2\left(15c_1^1(\alpha,\beta)c_2^2(\alpha,\beta) + c_3^3(\alpha,\beta)\right) = \frac{4(2\alpha+5)}{\alpha+1}c_3^3(\alpha,\beta). \end{aligned} \tag{19}$$

4. INTEGRAL REPRESENTATIONS OF SPECTRAL POLYNOMIALS $\mathscr{R}_\ell^{(\alpha,\beta)}$

This section is concerned with the evaluation of certain definite integrals involving Gegenbauer polynomials (see Appendix A) and Jacobi polynomials with product of weight functions. It is interesting to see here that these integrals can be explicitly evaluated in terms of the spectral sum of integer powers of the eigenvalues $\lambda_k(\mathscr{X})$ and their multiplicities $M_k(\mathscr{X})$. Special cases of this identity give integral representations of eigenvalues of the Laplacian on compact rank-1 symmetric spaces. It is important to mention that the results obtained in this section are a generalisation of those in Gradshtejn and Ryzhik [17, Sec. 7.31 (p. 795)].

Our starting point in this direction is the following remarkable product formula for the Jacobi polynomials due to Dijksma and Koornwinder [18].

Proposition 3: (Dijksma and Koornwinder [18]) Let α and β be integers or half-integers greater than or equal to zero. Then

$$\begin{aligned} & P_k^{(\alpha,\beta)}(\cos 2\vartheta) P_k^{(\alpha,\beta)}(\cos 2\phi) \\ =& \chi_k^{\alpha,\beta} \int_{-1}^{1}\int_{-1}^{1} C_{2k}^{\alpha+\beta+1}(x\cos\vartheta\cos\phi + y\sin\vartheta\sin\phi) \\ & \times (1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\, dydx, \end{aligned} \tag{20}$$

where the constant $\chi_k^{\alpha,\beta}$ is given by

$$\chi_k^{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+1)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\pi\Gamma(k+1)\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}. \tag{21}$$

We restate Proposition 3 in the following form.

Proposition 4: Let α and β be integers or half-integers greater than or equal to zero. Then the following equality holds:

$$\begin{aligned} & \left[P_k^{(\alpha,\beta)}(\cos\theta)\right]^2 \\ =& \eta_k^{\alpha,\beta} \int_{-1}^{1}\int_{-1}^{1} P_k^{(\alpha+\beta+1/2,-1/2)}\left(\frac{1}{2}\left[(x-y)\cos\theta+(x+y)\right]^2-1\right) \\ & \times (1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\, dydx, \end{aligned} \tag{22}$$

where the constant $\eta_k^{\alpha,\beta}$ is given by

$$\begin{aligned} \eta_k^{\alpha,\beta} =& \frac{\Gamma(\alpha+\beta+1)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\pi\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)} \\ & \times \frac{\Gamma\left(2k+2\alpha+2\beta+2\right)\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}{\Gamma(2\alpha+2\beta+2)\Gamma\left(k+\alpha+\beta+\frac{3}{2}\right)\Gamma(2k+1)}. \end{aligned} \tag{23}$$

Proof: By setting $\vartheta=\phi=\theta/2$ and using the identity (79) in (20), we have

$$\begin{aligned} & \left[P_k^{(\alpha,\beta)}(\cos\theta)\right]^2 \\ =& \gamma_k^{\alpha,\beta} \int_{-1}^{1}\int_{-1}^{1} P_{2k}^{(\alpha+\beta+1/2,\alpha+\beta+1/2)}\left(\frac{x-y}{2}\cos\theta+\frac{x+y}{2}\right) \\ & \times (1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\, dydx, \end{aligned} \tag{24}$$

where the constant $\gamma_k^{\alpha,\beta}$ is given by

$$\gamma_k^{\alpha,\beta} = \frac{\Gamma(\alpha+\beta+1)\Gamma(k+\alpha+1)\Gamma(k+\beta+1)}{\pi\Gamma(k+1)\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)} \times \frac{\Gamma\left(\alpha+\beta+\frac{3}{2}\right)\Gamma\left(2k+2\alpha+2\beta+2\right)}{\Gamma\left(2k+\alpha+\beta+\frac{3}{2}\right)\Gamma\left(2\alpha+2\beta+2\right)}. \tag{25}$$

Upon applying the relation (82) gives the desired result.

We now state the main theorem of this paper.

Theorem 2: (Integral-spectral identity) Let α and β be integers or half-integers greater than or equal to zero. Then for any integer $\ell \geq 1$ we have the following integral representation:

$$\frac{H_k^{\alpha,\beta}\mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k)}{[\Gamma(\alpha+1)]^2} = \int_{-1}^{1}\int_{-1}^{1}\sum_{m=1}^{\ell} a_m^\ell \frac{\Gamma(k+\alpha+\beta+m+1)}{\Gamma(k+\alpha+\beta+1)} p_m^\ell(x,y)(x-y)^m \times P_{k-m}^{(\alpha+\beta+m+1/2,-1/2+m)}\left(2x^2-1\right)(1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\,dydx, \tag{26}$$

where a_m^ℓ is a constant coefficient, $p_m^\ell(x,y)$ is a ℓ-degree polynomial in x and y, $\mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k)$ is the spectral polynomial in Proposition 2 and the constant $H_k^{\alpha,\beta}$ is given by

$$H_k^{\alpha,\beta} = \frac{\pi\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{[\Gamma(2k+1)]^{-1}[\Gamma(k+1)]^2\Gamma(\alpha+\beta+1)\Gamma(k+\beta+1)} \times \frac{\Gamma(2\alpha+2\beta+2)\Gamma\left(k+\alpha+\beta+\frac{3}{2}\right)}{\Gamma\left(2k+2\alpha+2\beta+2\right)\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}. \tag{27}$$

Proof: For $\ell \geq 1$, consider the differential relation

$$\frac{1}{\eta_k^{\alpha,\beta}}\frac{d^{2\ell}}{d\theta^{2\ell}}\left\{\left[P_k^{(\alpha,\beta)}(\cos\theta)\right]^2\right\}\Bigg|_{\theta=0} = \int_{-1}^{1}\int_{-1}^{1}\frac{d^{2\ell}}{d\theta^{2\ell}}\left[P_k^{(\rho,-1/2)}\left(\frac{1}{2}\left[(x-y)\cos\theta+(x+y)\right]^2-1\right)\right]\Bigg|_{\theta=0} \times (1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\,dydx,$$

where $\rho = \alpha+\beta+1/2$. Note that the vanishing of the odd terms in the above identity is due to the Jacobi polynomial $P_k^{(\alpha,\beta)}$ being even in the θ-variable. Indeed, from the recursion formula (75) we have

$$\frac{d^{2\ell}}{d\theta^{2\ell}}\left\{\left[P_k^{(\alpha,\beta)}(\cos\theta)\right]^2\right\}\Bigg|_{\theta=0}$$
$$=\eta_k^{\alpha,\beta}\int_{-1}^{1}\int_{-1}^{1}\sum_{m=1}^{\ell}a_m^\ell\frac{\Gamma(k+\alpha+\beta+m+1)}{2^m\Gamma(k+\alpha+\beta+1)}(x-y)^m\,\widetilde{p}_m^\ell(x,y)$$
$$\times P_{k-m}^{\left(\alpha+\beta+m+\frac{1}{2},-\frac{1}{2}+m\right)}\left(2x^2-1\right)(1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\,dydx. \tag{28}$$

On the other hand, by Proposition 2,

$$\frac{d^{2\ell}}{d\theta^{2\ell}}\left\{\left[P_k^{(\alpha,\beta)}(\cos\theta)\right]^2\right\}\Bigg|_{\theta=0}=\left[P_k^{(\alpha,\beta)}(1)\right]^2\mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k). \tag{29}$$

It therefore follows from these two identities the spectral relation

$$\int_{-1}^{1}\int_{-1}^{1}\sum_{m=1}^{\ell}a_m^\ell\frac{\Gamma(k+\alpha+\beta+m+1)}{2^m\Gamma(k+\alpha+\beta+1)}(x-y)^m\,\widetilde{p}_m^\ell(x,y)$$
$$\times P_{k-m}^{(\alpha+\beta+m+1/2,-1/2+m)}\left(2x^2-1\right)(1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\,dydx$$
$$=\frac{\left[P_k^{(\alpha,\beta)}(1)\right]^2}{\eta_k^{\alpha,\beta}}\mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k), \tag{30}$$

and this completes the proof of the theorem.

The first coefficients a_m^ℓ and $p_m^\ell = p_m^\ell(x,y)$ are given in TABLES 4 and 5 respectively.

TABLE 4. The first coefficients a_m^ℓ.

a_1^1	a_1^2	a_2^2	a_1^3	a_2^3	a_3^3	a_1^4	a_2^4	a_3^4	a_4^4
-1	1	3	-1	-15	-15	1	63	210	105

TABLE 5. The first coefficients $p_m^\ell = p_m^\ell(x,y)$.

p_1^1	p_1^2	p_2^2	p_1^3	p_2^3
x	$\frac{5x}{2}-\frac{3y}{2}$	x^2	$\frac{17x}{2}-\frac{15y}{2}$	$\frac{5x^2}{2}-\frac{3xy}{2}$
p_3^3	p_1^4	p_2^4	p_3^4	p_4^4
x^3	$\frac{65x}{2}-\frac{63y}{2}$	$\frac{29x^2}{4}-\frac{15xy}{2}+\frac{5y^2}{4}$	$\frac{5x^3}{2}-\frac{3x^2y}{2}$	x^4

It is observed that a_m^ℓ $(1\le m\le\ell)$ are integer coefficients and $p_\ell^\ell(x,y)=x^\ell, \ell\ge 1$.

4.1. INTEGRAL REPRESENTATIONS OF $\lambda_k(\mathscr{X})$ and $M_k(\mathscr{X})$

This subsection describes the eigenvalues $\lambda_k(\mathscr{X})$ and the multiplicities $M_k(\mathscr{X})$ as integrals involving Gegenbauer polynomials C_j^ν. The identities established here are novel in the context of special functions.

Towards this end, we restate Theorem 2 in the following form.
Theorem 3: (Integral-spectral identity) Let $\alpha, \beta > -1/2$ and $k = 0, 1, 2, \ldots$. Then for any integer $\ell \geq 1$ we have the following integral formula:

$$\begin{aligned} &\frac{I_k^{\alpha,\beta} \mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k)}{[\Gamma(\alpha+1)]^2} \\ &= \int_{-1}^1 \int_{-1}^1 \sum_{m=1}^{\ell} a_m^\ell \frac{\Gamma(\alpha+\beta+m+1)}{\Gamma(\alpha+\beta+1)} C_{2k-m}^{\alpha+\beta+m+1}(x) \\ &\quad \times (x-y)^m (1-x^2)^{\alpha-\frac{1}{2}} (1-y^2)^{\beta-\frac{1}{2}}\, dy dx, \end{aligned} \tag{31}$$

where a_m^ℓ, $\mathscr{R}_\ell^{(\alpha,\beta)}(\lambda_k)$ are as defined in Theorem 2 and the constant $I_k^{\alpha,\beta}$ is given by

$$I_k^{\alpha,\beta} = \frac{\pi \Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(k+1)\Gamma(\alpha+\beta+1)\Gamma(k+\beta+1)}. \tag{32}$$

A special case of Theorem 3 is given in the following theorem.
Theorem 4: (Integral representations of $\lambda_k^{(\alpha,\beta)}$) Let α and β be integers or half-integers greater than or equal to zero. Then the eigenvalue $\lambda_k^{(\alpha,\beta)}$ admits the integral representation

$$\widetilde{I}_k^{\alpha,\beta} \lambda_k^{(\alpha,\beta)} = \int_{-1}^1 \int_{-1}^1 C_{2k-1}^{\alpha+\beta+2}(x)(x-y)(1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\, dy dx, \tag{33}$$

where

$$\widetilde{I}_k^{\alpha,\beta} := \frac{\pi \Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(\alpha+1)\Gamma(\alpha+2)\Gamma(\alpha+\beta+2)\Gamma(k+1)\Gamma(k+\beta+1)}. \tag{34}$$

In what follows we present integral representations of the eigenvalues $\lambda_k(\mathscr{X})$ with multiplicities $M_k(\mathscr{X})$ according to whether $\mathscr{X}$ is the sphere $\mathbb{S}^n$, the complex projective space $\mathbf{P}^n(\mathbb{C})$, the quaternionic projective space $\mathbf{P}^n(\mathbb{H})$, or the Cayley projective plane $\mathbf{P}^2(\mathrm{Cay})$.
Theorem 5: The following integral representations hold on symmetric spaces $\mathscr{X}$:

(1) $(\mathscr{X} = \mathbb{S}^n : \alpha = \beta = (n-2)/2)$

$$M_k(\mathbb{S}^n)\,\lambda_k(\mathbb{S}^n) = \left[\frac{\omega_1^{n-2}}{\omega_1^{n-1}}\right]^2 \int_{-1}^{1}\int_{-1}^{1} \mathsf{C}_k^{\mathbb{S}^n}(x)(x-y)(1-x^2)^{\frac{n-3}{2}}(1-y^2)^{\frac{n-3}{2}}\,dydx, \tag{35}$$

where

$$\mathsf{C}_k^{\mathbb{S}^n}(x) := \left[n\left(k+\frac{n-1}{2}\right)\right] C_{2k-1}^{n}(x). \tag{36}$$

(2) $(\mathscr{X} = \mathbf{P}^n(\mathbb{C}) : \alpha = n-1, \beta = 0)$

$$M_k(\mathbf{P}^n(\mathbb{C}))\,\lambda_k(\mathbf{P}^n(\mathbb{C})) = \frac{\omega_3^{n-3/2}}{\omega_3^{n}} \int_{-1}^{1}\int_{-1}^{1} \mathsf{C}_k^{\mathbf{P}^n(\mathbb{C})}(x)(x-y)(1-x^2)^{n-\frac{3}{2}}(1-y^2)^{-\frac{1}{2}}\,dydx, \tag{37}$$

where

$$\mathsf{C}_k^{\mathbf{P}^n(\mathbb{C})}(x) := 8(2k+n)C_{2k-1}^{n+1}(x). \tag{38}$$

(3) $(\mathscr{X} = \mathbf{P}^n(\mathbb{H}) : \alpha = 2n-1, \beta = 1)$

$$M_k(\mathbf{P}^n(\mathbb{H}))\,\lambda_k(\mathbf{P}^n(\mathbb{H})) = \frac{\omega_4^{n-5/4}}{\omega_4^{n-1/2}} \int_{-1}^{1}\int_{-1}^{1} \mathsf{C}_k^{\mathbf{P}^n(\mathbb{H})}(x)(x-y)(1-x^2)^{2n-\frac{3}{2}}(1-y^2)^{\frac{1}{2}}\,dydx, \tag{39}$$

where

$$\mathsf{C}_k^{\mathbf{P}^n(\mathbb{H})}(x) := 16(2k+2n+1)C_{2k-1}^{2n+2}(x). \tag{40}$$

(4) $(\mathscr{X} = \mathbf{P}^2(\mathrm{Cay}) : \alpha = 7, \beta = 3)$

$$M_k\big(\mathbf{P}^2(\mathrm{Cay})\big)\,\lambda_k\big(\mathbf{P}^2(\mathrm{Cay})\big) = \int_{-1}^{1}\int_{-1}^{1} \mathsf{C}_k^{\mathbf{P}^2(\mathrm{Cay})}(x)(x-y)(1-x^2)^{\frac{13}{2}}(1-y^2)^{\frac{5}{2}}\,dydx, \tag{41}$$

where

$$\mathsf{C}_k^{\mathbf{P}^2(\mathrm{Cay})}(x) := \frac{262144}{2145\pi^2}(2k+11)C_{2k-1}^{12}(x). \tag{42}$$

Here $\omega_1^n = \mathrm{Vol}(\mathbb{S}^n)$, $\omega_3^n = \mathrm{Vol}(\mathbf{P}^n(\mathbb{C}))$ and $\omega_4^n = \mathrm{Vol}(\mathbf{P}^n(\mathbb{H}))$.
Proof: From (33) and (34) we see that

(1) ($\mathscr{X} = \mathbb{S}^n : \alpha = \beta = (n-2)/2$)

$$\begin{aligned}
&\widetilde{I}_k^{\frac{n-2}{2},\frac{n-2}{2}} \lambda_k^{\left(\frac{n-2}{2},\frac{n-2}{2}\right)} \\
&= \int_{-1}^{1}\int_{-1}^{1} C_{2k-1}^{n}(x)(x-y)(1-x^2)^{\frac{n-3}{2}}(1-y^2)^{\frac{n-3}{2}}\,dydx, \qquad (43)
\end{aligned}$$

where

$$\begin{aligned}
\widetilde{I}_k^{\frac{n-2}{2},\frac{n-2}{2}} &= \frac{2\pi\left[\Gamma\left(\frac{n-1}{2}\right)\right]^2 \Gamma(k+n-1)}{\left[\Gamma\left(\frac{n}{2}\right)\right]^2 \Gamma(n+1)\Gamma(k+1)} \\
&= \frac{2M_k\left(\mathbb{S}^n\right)}{n(2k+n-1)}\left[\frac{\omega_1^{n-1}}{\omega_1^{n-2}}\right]^2, \qquad (44)
\end{aligned}$$

and we have used $\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n-1}{2}\right)/\Gamma\left(\frac{n}{2}\right) = \omega_1^{n-1}/\omega_1^{n-2}$;

(2) ($\mathscr{X} = \mathbf{P}^n(\mathbb{C}) : \alpha = n-1, \beta = 0$)

$$\widetilde{I}_k^{n-1,0}\lambda_k^{(n-1,0)} = \int_{-1}^{1}\int_{-1}^{1} C_{2k-1}^{n+1}(x)(x-y)(1-x^2)^{n-\frac{3}{2}}(1-y^2)^{-\frac{1}{2}}\,dydx, \qquad (45)$$

where

$$\begin{aligned}
\widetilde{I}_k^{n-1,0} &= \frac{\pi^{\frac{3}{2}}\left[\Gamma(k+n)\right]^2 \Gamma\left(n-\frac{1}{2}\right)}{\Gamma(n)\left[\Gamma(n+1)\right]^2\left[\Gamma(k+1)\right]^2} \\
&= \frac{\pi^{\frac{3}{2}}\Gamma\left(n-\frac{1}{2}\right)}{n\Gamma(n+1)}\left[\frac{\Gamma(k+n)}{\Gamma(n)\Gamma(k+1)}\right]^2 \\
&= \frac{\pi^{\frac{3}{2}}\Gamma\left(n-\frac{1}{2}\right)M_k\left(\mathbf{P}^n(\mathbb{C})\right)}{\Gamma(n+1)(2k+n)} \\
&= \frac{\omega_3^n}{\omega_3^{n-3/2}}\frac{M_k\left(\mathbf{P}^n(\mathbb{C})\right)}{8(2k+n)}; \qquad (46)
\end{aligned}$$

(3) ($\mathscr{X} = \mathbf{P}^n(\mathbb{H}) : \alpha = 2n-1, \beta = 1$)

$$\widetilde{I}_k^{2n-1,1}\lambda_k^{(2n-1,1)} = \int_{-1}^{1}\int_{-1}^{1} C_{2k-1}^{2n+2}(x)(x-y)(1-x^2)^{2n-\frac{3}{2}}(1-y^2)^{\frac{1}{2}}\,dydx, \qquad (47)$$

where

$$\begin{aligned}
\widetilde{I}_k^{2n-1,1} &= \frac{\pi^{\frac{3}{2}}\Gamma(k+2n)\Gamma(k+2n+1)\Gamma\left(2n-\frac{1}{2}\right)}{2\Gamma(2n)\Gamma(2n+1)\Gamma(2n+2)\Gamma(k+1)\Gamma(k+2)} \\
&= \frac{\pi^{\frac{3}{2}}(k+2n)\Gamma\left(2n-\frac{1}{2}\right)}{2(2n)\Gamma(2n+2)(k+1)}\left[\frac{\Gamma(k+2n)}{\Gamma(2n)\Gamma(k+1)}\right]^2 \\
&= \frac{\Gamma\left(2n-\frac{1}{2}\right)}{\Gamma(2n+1)}\frac{\pi^{\frac{3}{2}}M_k\left(\mathbf{P}^n(\mathbb{H})\right)}{2(2k+2n+1)} \\
&= \frac{\omega_4^{n-1/2}}{\omega_4^{n-5/4}}\frac{M_k\left(\mathbf{P}^n(\mathbb{H})\right)}{16(2k+2n+1)}; \qquad (48)
\end{aligned}$$

(4) $(\mathscr{X}=\mathbf{P}^2(\mathrm{Cay}):\alpha=7,\beta=3)$

$$\widetilde{I}_k^{7,3}\lambda_k^{(7,3)} = \int_{-1}^{1}\int_{-1}^{1} C_{2k-1}^{12}(x)(x-y)(1-x^2)^{\frac{13}{2}}(1-y^2)^{\frac{5}{2}}\,dydx, \qquad (49)$$

where

$$\begin{aligned}
\widetilde{I}_k^{7,3} &= \frac{225\cdot 9009\cdot\pi^2\cdot\Gamma(k+8)\Gamma(k+11)}{16\cdot 64\cdot 8!\cdot 7!\cdot 11!\cdot\Gamma(k+1)\Gamma(k+4)} \\
&= \frac{225\cdot 9009\cdot\pi^2 M_k\left(\mathbf{P}^2(\mathrm{Cay})\right)}{16\cdot 64\cdot 8!\cdot 6(2k+11)}. \qquad (50)
\end{aligned}$$

4.2. EXPLICIT VALUES OF SOME DEFINITE INTEGRALS

Here we give some special cases of Theorem 2 which are novel in the context of special functions and integral transforms.

Theorem 6: Let α and β be integers or half-integers greater than or equal to zero. Then the following formula holds:

$$\begin{aligned}
&\int_{-1}^{1}\int_{-1}^{1} x(x-y)P_{k-1}^{\left(\alpha+\beta+\frac{3}{2},\frac{1}{2}\right)}\left(2x^2-1\right)(1-x^2)^{\alpha-\frac{1}{2}}(1-y^2)^{\beta-\frac{1}{2}}\,dydx \\
=&\frac{\pi k\Gamma(k+\alpha+1)\Gamma(k+\alpha+\beta+1)\Gamma\left(\alpha+\frac{1}{2}\right)\Gamma\left(\beta+\frac{1}{2}\right)}{\Gamma(\alpha+1)\Gamma(\alpha+2)[\Gamma(k+1)]^2\Gamma(\alpha+\beta+1)\Gamma(k+\beta+1)} \\
&\times\frac{\Gamma(2\alpha+2\beta+2)\Gamma\left(k+\alpha+\beta+\frac{3}{2}\right)\Gamma(2k+1)}{\Gamma\left(2k+2\alpha+2\beta+2\right)\Gamma\left(\alpha+\beta+\frac{3}{2}\right)}. \qquad (51)
\end{aligned}$$

Proposition 4: For the special values of $\alpha, \beta = 0$, the following integral formula holds:

$$\int_{-1}^{1}\int_{-1}^{1} x(x-y)P_{k-1}^{\left(\frac{3}{2},\frac{1}{2}\right)}\left(2x^2-1\right)(1-x^2)^{-\frac{1}{2}}(1-y^2)^{-\frac{1}{2}}\,dydx$$
$$=\frac{\Gamma\left(k+\frac{1}{2}\right)}{\pi^{-\frac{3}{2}}\Gamma(k)}. \tag{52}$$

Theorem 7: The following integral representations hold on symmetric spaces $\mathscr{X}$:

(1) The Sphere $\mathscr{X} = \mathbb{S}^n$. For $\alpha = \beta = (n-2)/2$, we have

$$M_k\left(\mathbb{S}^n\right)$$
$$=\Omega_1^n \int_{-1}^{1}\int_{-1}^{1} x(x-y)\mathsf{P}_k^{\mathbb{S}^n}\left(2x^2-1\right)(1-x^2)^{\frac{n-3}{2}}(1-y^2)^{\frac{n-3}{2}}\,dydx, \tag{53}$$

where

$$\Omega_1^n = \frac{n\omega_1^{n-2}\omega_1^n}{4\pi\left[\omega_1^{n-1}\right]^2} \tag{54}$$

$$\mathsf{P}_k^{\mathbb{S}^n}\left(2x^2-1\right) := \frac{(2k+n-1)(2n-2)_{2k}}{\left(n-\frac{1}{2}\right)_k (k)_{k+1}} P_{k-1}^{\left(n-\frac{1}{2},\frac{1}{2}\right)}\left(2x^2-1\right). \tag{55}$$

(2) The Complex Projective Space $\mathscr{X} = \mathbf{P}^n(\mathbb{C})$. For $\alpha = n-1, \beta = 0$, we have the following identity:

$$M_k\left(\mathbf{P}^n(\mathbb{C})\right)$$
$$=\Omega_3^n \int_{-1}^{1}\int_{-1}^{1} x(x-y)\mathsf{P}_k^{\mathbf{P}^n(\mathbb{C})}\left(2x^2-1\right)(1-x^2)^{n-\frac{3}{2}}(1-y^2)^{-\frac{1}{2}}\,dydx, \tag{56}$$

where

$$\Omega_3^n = \frac{\omega_3^{n-3/2}}{\pi\omega_3^{n-1}} \tag{57}$$

$$\mathsf{P}_k^{\mathbf{P}^n(\mathbb{C})}\left(2x^2-1\right) := \frac{(2k+n)(2n)_{2k}}{\left(n+\frac{1}{2}\right)_k (k)_{k+1}} P_{k-1}^{\left(n+\frac{1}{2},\frac{1}{2}\right)}\left(2x^2-1\right). \tag{58}$$

(3) The Quaternionic Projective Space $\mathscr{X} = \mathbf{P}^n(\mathbb{H})$. For $\alpha = 2n-1, \beta = 1$, the following formula holds:

$$M_k\left(\mathbf{P}^n(\mathbb{H})\right) = \Omega_4^n \int_{-1}^{1}\int_{-1}^{1} x(x-y)\mathsf{P}_k^{\mathbf{P}^n(\mathbb{H})}\left(2x^2-1\right)(1-x^2)^{2n-\frac{3}{2}}(1-y^2)^{\frac{1}{2}}\,dydx, \tag{59}$$

where

$$\Omega_4^n = \frac{16\omega_4^{n-5/4}}{\omega_4^{n-1/2}} \tag{60}$$

$$\mathsf{P}_k^{\mathbf{P}^n(\mathbb{H})}\left(2x^2-1\right) := \frac{(2k+2n+1)(4n+2)_{2k}}{(2n+1)\left(2n+\frac{3}{2}\right)_k (k)_{k+1}} P_{k-1}^{\left(2n+\frac{3}{2},\frac{1}{2}\right)}\left(2x^2-1\right). \tag{61}$$

(4) The Cayley Projective Plane $\mathscr{X} = \mathbf{P}^2(\mathrm{Cay})$. For $\alpha = 7, \beta = 3$, we have

$$M_k\left(\mathbf{P}^2(\mathrm{Cay})\right) = \Omega \int_{-1}^{1}\int_{-1}^{1} x(x-y)\mathsf{P}_k^{\mathbf{P}^2(\mathrm{Cay})}\left(2x^2-1\right)(1-x^2)^{\frac{13}{2}}(1-y^2)^{\frac{5}{2}}\,dydx, \tag{62}$$

where

$$\Omega = \frac{\pi^{-2}6144\cdot 8!}{2475\cdot 9009} \tag{63}$$

$$\mathsf{P}_k^{\mathbf{P}^2(\mathrm{Cay})}\left(2x^2-1\right) := \frac{(2k+11)\,(22)_{2k}}{\left(\frac{23}{2}\right)_k (k)_{k+1}} P_{k-1}^{\left(\frac{23}{2},\frac{1}{2}\right)}\left(2x^2-1\right). \tag{64}$$

For the analysis of weighted inequalities and estimates for fractional integrals on compact rank-1 symmetric spaces, see Ciaurri *et al* [19].

A. GEGENBAUER POLYNOMIALS C_k^ν

The Gegenbauer polynomial $C_k^\nu = C_k^\nu(t)$ $(k = 0, 1, 2, \ldots, \nu > -1/2)$ is a natural generalisation of the Legendre polynomial $P_k(t)$ (coincides when $\nu = 1/2$) and is defined by the coefficient of α^k in the generating function

$$\left(1-2t\alpha+\alpha^2\right)^{-\nu} = \sum_{k=0}^{\infty} C_k^\nu(t)\alpha^k. \tag{65}$$

For $\nu > -1/2$ the Gegenbauer polynomial $C_k^\nu(t)$ has a nice truncated series representation resulting from the series solution to the Gegenbauer differential equation (*see* (68)) in the form

$$C_k^\nu(t) = \sum_{0 \le l \le \frac{k}{2}} (-1)^l \frac{\Gamma(k-l+\nu)}{\Gamma(\nu) l!(k-2l)!}(2t)^{k-2l}, \tag{66}$$

with the derivatives satisfying the recursive relation

$$\frac{d^m}{dt^m} C_k^\nu(t) = 2^m \frac{\Gamma(\nu+m)}{\Gamma(\nu)} C_{k-m}^{\nu+m}(t). \tag{67}$$

The Gegenbauer polynomial $y = C_k^\nu(t)$ satisfies the second-order homogeneous differential equation

$$\left(1-t^2\right)\frac{d^2y}{dt^2} - (2\nu+1)t\frac{dy}{dt} + k(k+2\nu)y = 0. \tag{68}$$

The pair form a so-called *regular* Sturm-Liouville system with the corresponding Gegenbauer operator a second-order differential operator in the weighted space $L^2[-1,1;(1-t^2)^{\nu-1/2}dt]$ having the discrete spectrum $(\lambda_k = k(k+2\nu) : k = 0,1,2,\dots)$ and associated eigenfunctions $y = C_k^\nu(t)$. In particular, we have the orthogonality relations $(k, m = 0,1,2,\dots)$

$$\begin{aligned}(C_k^\nu, C_m^\nu)_{L^2[-1,1;(1-t^2)^{\nu-1/2}dt]} &= \int_{-1}^{1} C_k^\nu(t)\, C_m^\nu(t) \left(1-t^2\right)^{\nu-\frac{1}{2}} dt \\ &= \frac{\pi 2^{1-2\nu}\Gamma(2\nu+m)}{m!(m+\nu)\Gamma(\nu)^2}\delta_{km},\end{aligned} \tag{69}$$

where δ_{km} is the usual Kronecker delta, that is, $\delta_{km} = 0$ when $k \neq m$ and $\delta_{km} = 1$ when $k = m$. The Gegenbauer polynomial can be expressed by the so-called Rodrigues' formula

$$C_k^\nu(t) = \frac{(-1)^k \Gamma(\frac{1+2\nu}{2})\Gamma(k+2\nu)\,(1-t^2)^{\frac{1}{2}-\nu}}{2^k k! \Gamma(2\nu)\Gamma\left(\frac{2\nu+1}{2}+k\right)} \frac{d^k}{dt^k}\left(1-t^2\right)^{k+\nu-\frac{1}{2}}, \tag{70}$$

and satisfies the pointwise value identities

$$C_k^\nu(1) = \frac{(2\nu)_k}{k!}, \qquad C_k^\nu(-t) = (-1)^k C_k^\nu(t) \tag{71}$$

where $(x)_k = \Gamma(x+k)/\Gamma(x)$. The normalised form $\mathscr{C}_k^\nu(t)$ is defined by

$$\mathscr{C}_k^\nu(t) = \frac{C_k^\nu(t)}{C_k^\nu(1)}, \tag{72}$$

and as a result $\mathscr{C}_k^\nu(1) = 1$.

B. JACOBI POLYNOMIALS $P_k^{(\alpha,\beta)}$

The Jacobi polynomial $P_k^{(\alpha,\beta)} = P_k^{(\alpha,\beta)}(t)$ ($k = 0, 1, 2, \dots;\ \alpha, \beta > -1$) which is a natural generalisation of the Gegenbauer polynomial C_k^ν is defined by the coefficient of z^k in the generating function relation

$$2^{\alpha+\beta} R^{-1} (1 - z + R)^{-\alpha} (1 + z + R)^{-\beta} = \sum_{k=0}^{\infty} P_k^{(\alpha,\beta)}(t) z^k,$$
$$R = \sqrt{1 - 2tz + z^2},\ |z| < 1. \tag{73}$$

It is not difficult to see that the Jacobi polynomial satisfies

$$P_k^{(\alpha,\beta)}(-t) = (-1)^k P_k^{(\beta,\alpha)}(t), \quad P_k^{(\alpha,\beta)}(1) = \frac{(\alpha+1)_k}{k!}, \tag{74}$$

and the differential recursion formula ($m = 1, 2, \dots$)

$$\frac{d^m}{dt^m} P_k^{(\alpha,\beta)}(t) = \frac{1}{2^m} \frac{\Gamma(k+m+\alpha+\beta+1)}{\Gamma(k+\alpha+\beta+1)} P_{k-m}^{(\alpha+m,\beta+m)}(t). \tag{75}$$

As a result the Jacobi polynomial $y = P_k^{(\alpha,\beta)}(t)$ satisfies the second-order differential equation

$$\left(1 - t^2\right) y'' - (\alpha - \beta + (\alpha+\beta+2)t) y' + k(k+\alpha+\beta+1) y = 0. \tag{76}$$

The spectrum here is purely discrete and given by the sequence of eigenvalues and eigenfunctions

$$\lambda_k^{(\alpha,\beta)} = k(k+\alpha+\beta+1), \qquad y = P_k^{(\alpha,\beta)}(t), \qquad k = 0, 1, 2, \dots. \tag{77}$$

As an orthogonal polynomial the Jacobi polynomial satisfies the orthogonality relation ($k, m = 0, 1, 2, \dots$)

$$\int_{-1}^{1} P_k^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t) (1-t)^\alpha (1+t)^\beta \, dt = \mathsf{c}_k^{\alpha,\beta} \delta_{km}, \tag{78}$$

where the scalars $\mathsf{c}_k^{\alpha,\beta}$ on the right are given by

$$\mathsf{c}_k^{\alpha,\beta} = \frac{(\alpha+1)_k (\beta+1)_k (\alpha+\beta+k+1)}{k! (\alpha+\beta+2)_k (\beta+\alpha+2k+1)} \frac{2^{\alpha+\beta+1} \Gamma(\alpha+1) \Gamma(\beta+1)}{\Gamma(\alpha+\beta+2)}.$$

The Jacobi and Gegenbauer polynomials are related to one-another through the identity

$$C_k^\nu(t) = \frac{(2\nu)_k}{\left(\nu + \frac{1}{2}\right)_k} P_k^{\left(\nu-\frac{1}{2}, \nu-\frac{1}{2}\right)}(t), \qquad \nu > -1/2, \tag{79}$$

while the Legendre polynomial is linked to the latter by $P_k^{(0,0)}(t) = C_k^{\frac{1}{2}}(t) = P_k(t)$. In terms of the Gauss hypergeometric function, the Jacobi polynomial is given by

$$\mathscr{P}_k^{(\alpha,\beta)}(t) := \frac{k! P_k^{(\alpha,\beta)}(t)}{(\alpha+1)_k} = F\left(-k, k+\alpha+\beta+1; \alpha+1; \frac{1-t}{2}\right), \tag{80}$$

and that for the sake of future reference we often use the normalised form of the Jacobi polynomial

$$\mathscr{P}_k^{(\alpha,\beta)}(t) = \frac{P_k^{(\alpha,\beta)}(t)}{P_k^{(\alpha,\beta)}(1)} = \frac{k!}{(\alpha+1)_k} P_k^{(\alpha,\beta)}(t), \tag{81}$$

with $\mathscr{P}_k^{(\alpha,\beta)}(1) = 1$.

The following formula also holds:

$$P_{2k}^{(\nu,\nu)}(t) = \frac{\Gamma(2k+\nu+1)\Gamma(k+1)}{\Gamma(k+\nu+1)\Gamma(2k+1)} P_k^{(\nu,-\frac{1}{2})}(2t^2-1), \quad \nu > -1. \tag{82}$$

For more information on these orthogonal polynomials the interested reader is referred to Gradshtejn and Ryzhik [17], Askey [20], Szegö [21], Koornwinder [22, 23].

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REFERENCES

[1] M. Berger, P. Gauduchon, E. Mazet, *Le spectre dúne variétè Riemannienne*, Springer, 1971.
[2] I. Chavel, *Eigenvalues in Riemannian Geometry*, Academic Press, 1984.
[3] P. Li, *Geometric Analysis*, Cambridge Studies in Advanced Mathematics, Vol. **134**, Cambridge University Press, 2012.
[4] D. Bakry, I. Gentil, M. Ledoux, *Analysis and Geometry of Markov Diffusion Operators*, Grundlehren der Mathematischen Wissenschaften, Vol. **348**, Springer, 2008.
[5] S. Helgason, *Groups and Geometric Analysis: Radon Transforms, Invariant Differential Operators and Spherical Functions*, Academic Press, 1984.
[6] R.O. Awonusika, Generalised heat coefficients and associated spectral zeta functions on complex projective spaces $\mathbf{P}^n(\mathbb{C})$, *Complex Variables and Elliptic Equations* **65** 588-620, 2020.
[7] R.S. Cahn, J.A. Wolf, Zeta functions and their asymptotic expansions for compact symmetric spaces of rank one, *Comm. Math. Helv.* **51** 1-21, 1976.
[8] V.V. Volchkov, V.V. Volchkov, *Harmonic Analysis of Mean Periodic Functions on Symmetric Spaces and the Heisenberg Group*, Springer Monographs in Mathematics, Springer, 2009.

[9] G. Warner, *Harmonic Analysis on Semisimple Lie Groups*, Vols **I** & **II**, Springer, 1972.

[10] S. Helgason, Eigenspaces of the Laplacian; integral representations and irreducibility, *J. Funct. Anal.* **17** 328-353, 1974.

[11] S. Helgason, *Topics in Harmonic Analysis on Homogeneous Spaces*, Birkhäuser, 1981.

[12] N.J. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs, Vol. **22**, American Mathematical Society, 1968.

[13] R. O. Awonusika, A. Taheri, On Jacobi polynomials $(\mathscr{P}_k^{(\alpha,\beta)} : \alpha, \beta > -1)$ and Maclaurin spectral functions on rank one symmetric spaces, *J. Anal.* **25** 139-166, 2017.

[14] R. O. Awonusika, A. Taheri, On Gegenbauer polynomials and coefficients $c_j^{\ell}(\nu)$ $(1 \le j \le \ell,\, \nu > -1/2)$, *Results. Math.* **72** 1359-1367, 2017.

[15] R. O. Awonusika, A. Taheri, A spectral identity on Jacobi polynomials and its analytic implications, *Canad. Math. Bull.* **61** 473-482, 2018.

[16] R.O. Awonusika, On Jacobi polynomials and fractional spectral functions on compact symmetric spaces, *J. Anal.* **29** 987-1024, 2021.

[17] I.S. Gradshtejn, I. M. Ryzhik, *Table of Integrals, Series and Products*, Academic Press, 2007.

[18] A. Dijksma, T.H. Koornwinder, Spherical harmonics and the product of two Jacobi polynomials, *Indag. Math.* **33** 191-196, 1971.

[19] Ó. Ciaurri, L. Roncal, P.R. Stinga, Fractional integrals on compact Riemannian symmetric spaces of rank one, *Adv. Math.* **235** 627-647, 2013.

[20] R. Askey, *Orthogonal Polynomials and Special Functions,* Regional Conference Series in Applied Mathematics, Society for Industrial and Applied Mathematics, 1975.

[21] G. Szegö, *Orthogonal Polynomials,* fourth edition, American Mathematical Society, Colloquium Publications **XXIII**, American Mathematical Society, Providence, 1975.

[22] T.H. Koornwinder, The addition formula for Jacobi polynomials: **I** Summary of results, *Indag. Math.* **34** 188-191, 1974.

[23] T.H. Koornwinder, A new proof of a Paley-Wiener type theorem for the Jacobi transform, *Ark. Matematik* **13** 145-159, 1975.

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